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VOL. XXXV.

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PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

VOL. XXXV.

THIRTY-EIGHTH SESSION, 1901–1902

(since the Formation of the Society, January 16th, 1865).

Thursday, May 8th, 1902.

Dr. HOBSON, F.R.S., President, in the Chair.

Fourteen members present.

Mr. J. E. Wright, B.A., Scholar of Trinity College, Cambridge, was elected a member.

Dr. G. Prasad was admitted into the Society.

Dr. G. Prasad read a paper "On the use of Fourier's Series in the Theory of Conduction of Heat." The President, Dr. Larmor, and Prof. Love took part in a discussion of the paper.

Mr. A. E. Western communicated a note on "Fermat's Theorem on Binary Powers," pointing out an exception to the theorem. Lt.-Col. Cunningham gave an account of exceptions which had been found previously.

Dr. F. S. Macaulay read a paper on "Some Formulæ in Elimination."

The following papers were communicated from the Chair:—

On Groups in which every two Conjugate Operations are Permutable: Prof. W. Burnside.

The Application of Contour Integration to the Solution of general Problems in the Conduction of Heat and to the Expansion of an Arbitrary Function in Series: Mr. H. S. Carslaw.

The following presents were made to the Library of the Society :—

Klein, F.—"Anwendung der Differential- und Integralrechnung auf Geometrie," 4to ; Leipzig, 1902.

Forsyth, A. R.—"Theory of Differential Equations," Vol. IV., Pt. 3 : "Ordinary Linear Equations," 8vo ; Cambridge, 1902.

Pierantonii, L. F.—"Dimostrazione del Postulato di Euclide," 8vo ; Chieti, 1901.

"Wiadomosci Matematyczne," Tom VI., Zeszyt 1–3 ; Warsaw, 1902.

"University Quarterly," Vol. II., No. 7, July, 1901 ; Kansas, 1902.

"L'Enseignement Mathématique," Ann. IV., No. 2 ; 1902.

"Educational Times," May, 1902.

"Indian Engineering," Vol. XXXI., Nos. 12–15, March 22–April 12, 1902.

The following exchanges were received :—

"Periodico di Matematica," Serie 2, Vol. IV., Fasc. 5 ; Livorno, 1902.

"Supplemento al Periodico di Matematica," Anno V., Fasc. 6 ; Livorno, 1902.

"Jahresbericht der Deutschen Mathematiker-Vereinigung," Bd. XI., Hefte 1–4 ; Leipzig, 1902.

"Proceedings of the American Philosophical Society," Vol. XL., No. 167 ; Philadelphia, 1901.

"Annals of Mathematics," Series 2, Vol. III., No. 3 ; Salem, 1902.

"American Journal of Mathematics," Vol. XXIV., No. 2 ; Baltimore, 1902.

"Bulletin of the American Mathematical Society," Vol. VIII., No. 7 ; New York, 1902.

"Jornal de Sciencias Mathematicas e Astronomicas," Vol. XIV., No. 6 ; Coimbra, 1901.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Vol. VIII., Fasc. 3 ; Napoli, 1902.

"Archives Néerlandaises," Tome VII., Liv. 1 ; La Haye, 1902.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. XI., Fasc. 6, 7 ; Roma, 1902.

"Journal of the Institute of Actuaries," Vol. XXXVI., Pts. 5, 6 ; 1902.

"Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich," Jahrgang XLVI., Hefte 3, 4 ; 1902.

"Proceedings of the Physical Society," Vol. XVIII., Pt. 1 ; 1902.

"Proceedings of the Cambridge Philosophical Society," Vol. XI., Pt. 5 ; 1902.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Klasse, Heft 1 ; 1902.

Some Formulæ in Elimination. *By* F. S. Macaulay.

Received and read May 8th, 1902.

1. The object of the following paper is to investigate the properties of the determinants which arise in the theory of elimination when conducted according to the methods of Bezout, and, in particular, to find a simple expression for the resultant. The equations are supposed homogeneous, of different orders, and *general*, that is, complete in all their terms with unconnected literal coefficients.

Cayley* has given (without proof) an extremely general expression for the resultant of n equations in the form $D/D_1/\dots/D_{n-2}$, or $DD_2\dots/D_1D_3\dots$, where D is any non-vanishing determinant of the complete matrix corresponding to the function $C_1S_1+\dots+C_nS_n$ of order t_n (*cf.* § 3 below), and $D_1, D_2, \dots, D_{n-2}$ are other determinants. The simpler, but less general, expression for the resultant found below is D/Δ, where D is a determinant selected arbitrarily in accordance with a certain rule (§ 6*a*) from the same matrix, and Δ is a minor of D.

For three equations it can be verified that the two results D/D_1 and D/Δ are the same; D_1 and Δ are not, however, composed entirely of the same elements for the same D, but each is independent of the elements in which they differ. To verify the identity of the two results for more than three equations would be difficult, and of little use. The advantage of the simpler form D/Δ lies in the fact that Δ can be at once written down from D, whereas $D_1, D_2, \dots$ are only obtained by a complicated process, which Cayley does not fully explain.

The theory suggested by Cayley has been developed in considerable detail by K. Bes.† He discusses at length the case of three equations, from which he infers the result for n equations. He does not prove that $D, D_1, \dots$ can be so chosen that no one of them vanishes identically; and he is scarcely justified in describing his method as a new process, since it does not appear to differ in any essential feature from that of Cayley.

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* The method is described generally in the *Camb. and Dub. Math. Jour.*, Vol. III., 1848, p. 116, and is explained more in detail in Salmon's *Higher Algebra* (4th edition, 1885), p. 87.

† "Théorie générale de l'Elimination, d'après la méthode Bezout, suivant un nouveau procédé," *Verhandelingen der Koninklijke Akademie van Wetenschappen te Amsterdam* (Sectie 1), Deel VI., No. 7, 1899, 8vo, pp. 1–121.

H. Laurent* has also given a supposed explicit expression for the resultant, but an incorrect one.

The resultant of n *general* equations may be defined as an integral function of the coefficients, without repeated factors, whose vanishing is the necessary and sufficient condition that the equations should have a common solution. In the case of n equations containing

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* "L'Elimination," *Scientia, Phys.-Math.*, No. 7, 1900, pp. 1–75. This monograph, although curious and interesting, is rendered practically valueless in what relates to equations in several unknowns by its unreliable methods and conclusions. The resultant is nowhere defined and is regarded as an indefinite fractional expression. The following are some of the principal omissions and errors :—

(1) The proof (§ 15) of the theorem that, when two (non-homogeneous) polynomials in two variables are given as moduli, one variable can be expressed as an integral function of the other is incomplete, since two general assumptions are made proofs of which are not supplied.

(2) The proof (§ 16) that the z-eliminant C of three equations in x, y, z of orders m, n, p is of order mnp in z is faulty, since the author's method for expressing C leads to a fraction instead of an integral function of z. The same error appears still more prominently in § 17.

(3) The proof (§ 18) of Bezout's reduced form of a given polynomial with respect to other given polynomials as moduli completely fails when it passes beyond reduction in one variable.

(4) The statement (§ 20) that $\Omega^2/\Pi J$ [the author uses D for J; *cf.* § 10 (14) of this paper] is a determinant with all its elements to the left of the diagonal zero is an error, but an unimportant one, since it breaks up into a product of determinants in the diagonal. The statement that $\Omega^2/\Pi J$ depends only on the coefficients of the terms of highest order in the several equations is correct, but the proof is lacking. The author's proof of the same result in the *Nouv. Ann. de Math.*, Series 3, Vol. II., 1883, p. 147, is not valid. In the same place, p. 149, he is in error in stating that Ω cannot vanish unless the equations have a double solution, from which he deduces incorrect conclusions. Again, in § 20 of the monograph, the author states that $\Omega^2/\Pi J$ is independent of the roots of the equations. He does not explain what the statement means; but it is certainly untrue. If it were true, then the ratio of Ω to any other expression Ω' formed in like manner would also be independent of the roots, which can easily be tested and found incorrect for the case of a linear and a quadratic equation in two unknowns. Netto, in referring to Laurent, says that $\Omega^2/\Pi J$ is a *constant*, without further explanation (*Encyklopädie d. Math. Wiss.*, Teil I., Band I., Heft 3, 1899, p. 274). It would seem that both writers have been misled by an assumed, but false, analogy with an equation in a single unknown.

(5) In § 23 is contained the so-called explicit expression for the resultant referred to above; but the author is in error in supposing this expression "indépendant des a_{ij}," and in supposing it to be the resultant, or to contain the resultant as a factor.

(6) In § 26 the author implies that in order to calculate the resultant of n homogeneous equations in n unknowns it is of advantage to make the orders equal by multiplying the equations of inferior order by powers of the same unknown, overlooking the fact that, if two of the equations have a common factor, the resultant vanishes identically. Multiplying by powers of different unknowns is also of no advantage.

In contrast with the above we may mention § 19, which gives a proof of Jacobi's theorem, and § 22, which proves that, if the vanishing points (or solutions) of n given polynomials f in n variables are distinct, finite, and complete, then any polynomial which vanishes at all these points is of the form $\Sigma\phi f$, *i.e.*, vanishes identically with respect to the f's as moduli.

more than n unknowns the resultant of elimination of $(n-1)$ of the unknowns is called the eliminant in the remaining unknowns. The most usual form of expression for the resultant is by means of the Poisson product. Select any one of the equations $C = 0$; solve the remaining $(n-1)$ equations, after putting one of the unknowns x equal to 1; substitute all the solutions in C, and take the product ΠC; then the numerator R of ΠC (when reduced to its lowest terms) is the resultant of the equations. R is an integral function of the coefficients, being the numerator of a symmetric function of the roots of the $(n-1)$ equations; and the vanishing of R is a necessary and sufficient condition that the n equations have a common solution. The degree of R in the coefficients of any one of the equations is equal to the product of the orders of the remaining equations, and the weight of every term in R is equal to the product of the orders of all the equations. The denominator of the Poisson product is the m-th power of the resultant of the $(n-1)$ equations when $x = 0$, m being the order of C. R is independent of the particular choice of the unknown x and the equation $C = 0$;* this also follows from § 4 and (18) of § 10 in this paper. R is non-factorisable.† Thus R satisfies all the conditions required by the definition of the resultant. In this paper the resultant is regarded from a different point of view, viz., as a factor of a determinant; but it is identified with R by means of its properties, and also actually identified in § 10 (18).

2. Notation.—Let $C_1^{(n)}$, $C_2^{(n)}$, ..., $C_n^{(n)}$, or C_1, C_2, ..., C_n, be n given homogeneous *general* polynomials in n variables $x_1, x_2, \ldots, x_n$, of orders $m_1, m_2, \ldots, m_n$ respectively.

Let $C_r^{(l)}$ denote the value of $C_r^{(n)}$ $(r = 1, 2, \ldots, n)$ when $x_{l+1}, x_{l+2}, \ldots, x_n$ are all zero; so that $C_r^{(l)}$ is a homogeneous polynomial in l variables $x_1, x_2, \ldots, x_l$.

We imagine a correspondence to exist between the variables $x_1, x_2, \ldots, x_n$ and the polynomials $C_1, C_2, \ldots, C_n$ respectively. Thus we may, if we like, regard C_r as a polynomial in x_r whose coefficients are polynomials in $x_1, \ldots, x_{r-1}, x_{r+1}, \ldots, x_n$.

A polynomial containing no arguments $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ divisible by $x_1^{m_1}$ is said to be reduced in x_1; if, further, it contains no arguments

* Hadamard, "Mémoire sur l'Elimination," *Acta Math.*, Vol. xx., 1897, p. 201.
† Netto, *Algebra*, Bd. I., 1896, p. 169, and Bd. II., 1898, p. 79.

divisible by $x_2^{m_2}$, it is said to be reduced in x_1 and x_2; and so on. When the variables are not specified, a reduced argument or polynomial means one which is reduced in all the variables.

In the function $C_1 S^{(0)} + C_2 S^{(1)} + C_3 S^{(2)} + \ldots$ of order t, it is to be understood that $S^{(0)}$ is a general polynomial of order $t - m_1$, with all its coefficients at disposal, $S^{(1)}$ a general polynomial of order $t - m_2$ reduced in x_1, $S^{(2)}$ a general polynomial of order $t - m_3$ reduced in x_1, x_2, and so on. In the function $C_p S^{(0)} + C_q S^{(1)} + C_r S^{(2)} + \ldots$, where C_p, C_q, C_r, ... are chosen from C_1, C_2, ..., C_n, it is supposed that $S^{(1)}$ is reduced in x_p, $S^{(2)}$ in x_p, x_q, and so on; so that the significance of $S^{(1)}$, $S^{(2)}$, ... depends on the order in which C_1, C_2, ... appear in the constituent terms of the function.

Theorem.—It is a known theorem* that any polynomial C of order t can be expressed uniquely in the form

$$C_p S^{(0)} + C_q S^{(1)} + \ldots + C_r S^{(l-1)} + S^{(l)},$$

where l ($\leqslant n$) is the number of the given *general* polynomials $C_p, \ldots, C_r$.

In order to prove this, we have to show that $S^{(0)}, S^{(1)}, \ldots, S^{(l)}$ can be chosen in one and only one way so as to satisfy the identity

$$C_p S^{(0)} + C_q S^{(1)} + \ldots + C_r S^{(l-1)} + S^{(l)} = C.$$

Equate coefficients of the arguments on the two sides of the identity. The number of equations is equal to the number of arguments of order t; this is equal to the number of the unknowns, viz., the coefficients of $S^{(0)}, S^{(1)}, \ldots, S^{(l)}$, as may be seen by considering the polynomial

$$x_p^{m_p} S^{(0)} + x_q^{m_q} S^{(1)} + \ldots + x_r^{m_r} S^{(l-1)} + S^{(l)},$$

in which each argument of order t comes in once and once only. Again, the determinant of the coefficients of the unknowns in the equations is not zero; for, if it were, then the identity

$$C_p S^{(0)} + C_q S^{(1)} + \ldots + C_r S^{(l-1)} + S^{(l)} = 0$$

could be satisfied without $S^{(0)}, S^{(1)}, \ldots, S^{(l)}$ all vanishing identically;

---

* This theorem is a fundamental one in Bezout's method, and is probably contained in his *Théorie générale des Equations Algébriques* (Paris, 1779, 4to, 471 pp.), which I have not been able to consult.

and this is not possible, since the identity

$$x_p^{m_p} S^{(0)} + x_q^{m_q} S^{(1)} + \ldots + x_r^{m_r} S^{(l-1)} + S^{(l)} = 0$$

cannot be so satisfied. Hence the theorem is proved. In the last step we make use of the fundamental hypothesis that $C_1, C_2, \ldots, C_n$ are *general*, from which we are entitled to assume that, since a certain function of the coefficients of $C_1, C_2, \ldots$ does not vanish when $C_1, C_2, \ldots$ have the particular values $x_1^{m_1}, x_2^{m_2}, \ldots$, it cannot vanish for the actual values of $C_1, C_2, \ldots$.

From this theorem follows another of special importance, viz., that a homogeneous polynomial of the form $C_p S_1 + C_q S_2 + \ldots + C_r S_l$ can be expressed uniquely in any one of the standard forms

$$C_{p'} S^{(0)} + C_{q'} S^{(1)} + \ldots + C_{r'} S^{(l-1)},$$

where $C_{p'}, C_{q'}, \ldots, C_{r'}$ are the l polynomials $C_p, C_q, \ldots, C_r$ taken in any order we please. For $C_p S_1 + C_q S_2 + \ldots + C_r S_l$ can clearly be written in the form

$$C_{p'} S_1' + C_{q'} S_2' + \ldots + C_{r'} (S_l' + C_{p'} S^{(0)} + C_{q'} S^{(1)} + \ldots),$$

which, by a proper choice of $S^{(0)}, S^{(1)}, \ldots, S^{(l-2)}$, can be made of the form

$$C_{p'} S_1' + C_{q'} S_2' + \ldots + C_{r'} S^{(l-1)};$$

and this can clearly be brought step by step to the form

$$C_{p'} S^{(0)} + C_{q'} S^{(1)} + \ldots + C_{r'} S^{(l-1)}.$$

Similar reasoning leads to the theorem that, if $C_1 S_1 + \ldots + C_l S_l$, of order t, vanishes identically, then

$$S_q = \sum_{p=1}^{p=l} C_p S_{pq} \quad (q = 1, 2, \ldots, l),$$

where the polynomials S_{pq} are of assignable orders, and satisfy the relations $S_{pq} = -S_{qp}$ and $S_{pp} = 0$. The same result holds if $C_1, \ldots, C_l$ are *any* given polynomials, provided that certain functions of the coefficients do not vanish.

3. Notation.—The matrix corresponding to a homogeneous integral function $C_1 S_1 + C_2 S_2 + \ldots + C_n S_n$ of order t (also called a matrix of the coefficients of $C_1, C_2, \ldots, C_n$) is formed as follows. Write down horizontally all the arguments $\omega_1, \omega_2, \ldots, \omega_\mu$ of order t. Multiply C_p by any argument ω of S_p, and write the coefficients of ωC_p under their corresponding arguments $\omega_1, \omega_2, \ldots, \omega_\mu$, thus giving a row of the matrix. Write to the left of the row the coefficient of ω in S_p. If

$\lambda_1, \lambda_2, \ldots, \lambda_\rho$ are all the coefficients of $S_1, S_2, \ldots, S_n$, we thus obtain a matrix with μ columns and ρ rows, viz.,

	ω_1	ω_2	...	ω_μ
λ_1	a_{11}	a_{12}	...	$a_{1\mu}$
λ_2	a_{21}	a_{22}	...	$a_{2\mu}$
...	...	...	...	...
λ_ρ	$a_{\rho 1}$	$a_{\rho 2}$	...	$a_{\rho\mu}$

This (bordered) matrix is a diagrammatic representation of the function $C_1 S_1 + \ldots + C_n S_n$, viz., the function is the sum of the products of every element of the matrix by the two corresponding elements in the border. It represents also the whole coefficient of each argument in the function, viz., the coefficient of ω_q is $\sum_{p=1}^{p=\rho} \lambda_p a_{pq}$.

$D(n, t)$ denotes the determinant whose vanishing is the condition that the identity

$$C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)} = S^{(n)},$$

of order t, can be satisfied. Thus $D(n, t)$ is obtained from the matrix corresponding to $C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)}$ by omitting the columns corresponding to arguments contained in $S^{(n)}$, that is, all columns corresponding to reduced arguments. We take $D(n, t) = 1$ when t is less than the least of $m_1, m_2, \ldots, m_n$.

$R(n, t)$ denotes the H.C.F. of the $n!$ determinants formed in a similar way to $D(n, t)$ when $C_1, C_2, \ldots, C_n$ are arranged in any order. $R(n, t) = D(n, t)$ when t is less than the sum of the least two of $m_1, m_2, \ldots, m_n$; otherwise $R(n, t) < D(n, t)$.

$D(l, t)$, $l \leqslant n$, is the determinant whose vanishing is the condition that the identity

$$C_1^{(l)} S^{(0)} + C_2^{(l)} S^{(1} + \ldots + C_l^{(l)} S^{(l-1)} = S^{(l)},$$

of order t, can be satisfied. Here l is the number of the variables (§ 2), and also the number of the given polynomials C. $R(l, t)$ denotes the H.C.F. of the $l!$ determinants like $D(l, t)$.

We take t_n to stand for $m_1 + m_2 + \ldots + m_n - n + 1$, and t_l for $m_1 + \ldots + m_l - l + 1$.

THEOREM.—The resultant of $C_1, C_2, \ldots, C_n$ is $R(n, t_n)$.

The matrix from which $D(n, t_n)$ is obtained, viz.,

	ω_1	ω_2	...	ω_μ
λ_1	a_{11}	a_{12}	...	$a_{1\mu}$
λ_2	a_{21}	a_{22}	...	$a_{2\mu}$
...	...	...	...	...
λ_μ	$a_{\mu 1}$	$a_{\mu 2}$	...	$a_{\mu\mu}$

is the determinant $D(n, t_n)$ itself; for there are no reduced arguments of order t_n (since the reduced argument of highest order is $x_1^{m_1-1} x_2^{m_2-1} \ldots x_n^{m_n-1}$ which is of order t_n-1), and consequently there are no columns to be omitted. It is then evident that $D(n, t_n)$ vanishes if the equations

$$a_{11}\omega_1 + a_{12}\omega_2 + \ldots + a_{1\mu}\omega_\mu = 0,$$
$$a_{21}\omega_1 + a_{22}\omega_2 + \ldots + a_{2\mu}\omega_\mu = 0,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{\mu 1}\omega_1 + a_{\mu 2}\omega_2 + \ldots + a_{\mu\mu}\omega_\mu = 0$$

can be satisfied. But these equations are satisfied, when the resultant vanishes, by giving to $\omega_1, \omega_2, \ldots, \omega_\mu$ the values which they have for the common solution of the equations $C_1 = 0$, $C_2 = 0$, ..., $C_n = 0$. Hence the resultant is a factor of $D(n, t_n)$, and of all the $n!$ determinants like $D(n, t_n)$; therefore it is a factor of their H.C.F., viz., $R(n, t_n)$. Also $R(n, t_n)$ is of the same degree as the resultant in the coefficients of each of the polynomials $C_1, C_2, \ldots C_n$ (proved in § 4). Hence $R(n, t_n)$ is the resultant.

4. THEOREM.—The degree of $R(n, t)$ in the coefficients of C_r is equal to the number of arguments of order $t-m_r$ which are reduced in all the variables except x_r, *i.e.*, it is equal to the coefficient of x^t in $\dfrac{x^{m_r}}{1-x^{m_r}} \prod_{p=1}^{p=n} \dfrac{1-x^{m_p}}{1-x}$.

Let $D'(n, t)$ be the determinant like $D(n, t)$ for a different order of the polynomials, viz., for the order $C_l, \ldots, C_p, C_q, C_r$.

$D'(n, t)$ is a determinant of the matrix corresponding to

$$C_l S^{(0)} + \ldots + C_q S^{(n-2)} + C_r S^{(n-1)}.$$

Hence the degree of $D'(n, t)$ in the coefficients of C_r is equal to the

number of arguments in $S^{(n-1)}$ of order $t-m_r$. But $R(n, t)$ is a factor of $D'(n, t)$, and therefore cannot contain the coefficients of C_r to a higher degree than the number of arguments of order $t-m_r$ reduced in all the variables except x_r.

Now the polynomial $C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)}$ is represented by

	ω_1	ω_2	...	ω_μ
λ_1	a_{11}	a_{12}	...	$a_{1\mu}$
λ_2	a_{21}	a_{22}	...	$a_{2\mu}$
...	...	...		...
λ_ρ	$a_{\rho 1}$	$a_{\rho 2}$	...	$a_{\rho\mu}$

and (§ 2) this can be brought identically to the form

$$C_l S'^{(0)} + \ldots + C_q S'^{(n-2)} + C_r S'^{(n-1)},$$

which is represented by

	ω_1	ω_2	...	ω_μ
λ'_1	a'_{11}	a'_{12}	...	$a'_{1\mu}$
λ'_2	a'_{21}	a'_{22}	...	$a'_{2\mu}$
...	...	...		...
λ'_ρ	$a'_{\rho 1}$	$a'_{\rho 2}$	...	$a'_{\rho\mu}$

Hence $$\sum_{p=1}^{p=\rho} \lambda_p a_{pr} = \sum_{p=1}^{p=\rho} \lambda'_p a'_{pr} \quad (r = 1, 2, \ldots, \mu).$$

Taking any ρ values for r, it follows that the ratio of any determinant in the first matrix to the corresponding determinant in the second is equal to the transformation determinant $\begin{pmatrix}\lambda'\\ \lambda\end{pmatrix}$ derived from the identical expression of $\lambda'_1, \lambda'_2, \ldots, \lambda'_\rho$ as linear functions of $\lambda_1, \lambda_2, \ldots, \lambda_\rho$. In particular, we have

$$\frac{D(n, t)}{D'(n, t)} = \begin{pmatrix}\lambda'\\ \lambda\end{pmatrix}.$$

We examine then how the quantities λ' are expressed as linear functions of the quantities λ, or, what comes to the same thing, how

I. $$C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)}$$

is changed to the form

II. $$C_l S'^{(0)} + \ldots + C_q S'^{(n-2)} + C_r S'^{(n-1)}.$$

The first step is to write I. in the form

$$\text{III.} \quad C_l(S^{(l-1)}+C_rS''^{(0)})+\ldots+C_q(S^{(q-1)}+C_rS''^{(n-2)}) \\ +C_r(S^{(r-1)}-C_lS''^{(0)}-\ldots-C_qS''^{(n-2)}),$$

and equate the co-factors of C_r in II. and III. We thus have

$$C_lS''^{(0)}+\ldots+C_qS''^{(n-2)}+S'^{(n-1)}=S^{(r-1)}.$$

This identity determines uniquely all the coefficients of $S''^{(0)}, \ldots, S''^{(n-2)}$, $S'^{(n-1)}$ in terms of the coefficients λ in $S^{(r-1)}$, by § 2. The coefficients λ' in $S'^{(n-1)}$ are therefore linear functions of the coefficients λ in $S^{(r-1)}$, and the coefficients of the expressions for these quantities λ' in terms of the quantities λ are fractional functions of the coefficients of $C_l, \ldots, C_p, C_q$, *i.e., they are independent of the coefficients of* C_r.

The next step is to change the co-factor of C_q in III., keeping the co-factor of C_r unchanged, and requires the identity

$$C_lS'''^{(0)}+\ldots+C_pS'''^{(n-3)}+S'^{(n-2)}=S^{(q-1)}+C_rS''^{(n-2)}$$

to be satisfied, $S''^{(n-2)}$ being already determined from the first step. From this we see that the coefficients λ' in $S'^{(n-2)}$ are linear functions of the coefficients λ in $S^{(q-1)}$ and $S^{(r-1)}$, and that *the denominators of the coefficients of the expressions for these quantities* λ' *in terms of the quantities* λ *are independent of the coefficients of* C_r. This last property clearly holds for all the quantities λ' when expressed in terms of the quantities λ. Hence the denominator of $\binom{\lambda'}{\lambda}$ is independent of the coefficients of C_r.

$D(n, t)$ is therefore divisible by all the factors of $D'(n, t)$ which contain the coefficients of C_r; and similarly each one of the $n!$ determinants like $D(n, t)$ is divisible by the same factors. Hence $R(n, t)$ is divisible by the same factors, and therefore $R(n, t)$ is of the degree stated above in the coefficients of C_r. The degree of $R(n, t)$ in all the coefficients combined is equal to the coefficient of x^t in $\Sigma\dfrac{x^{m_p}}{1-x^{m_p}}\times\Pi\dfrac{1-x^{m_p}}{1-x}$.

It easily follows that the degree of $R(n, t_n)$ in the coefficients of C_r is $\Pi m/m_r$. This completes the proof that $R(n, t_n)$ is the resultant.

The continued ratio of the determinants of the first matrix above is the same as for any one of the $n!$ matrices formed in a similar way when $C_1, C_2, \ldots, C_n$ are arranged in any order, and is equal to the continued ratio of the H.C.F.'s of the sets of $n!$ corresponding de-

terminants. Hence, since $R(n, t)$ is the H.C.F. of the set corresponding to $D(n, t)$, it follows that all the determinants of the first matrix are divisible by $D(n, t)/R(n, t)$. Also $C_1 S^{(0)} + \ldots + C_n S^{(n-1)}$ can be changed identically to $C_1 S_1 + \ldots + C_n S_n$, where $S_1, \ldots, S_n$ contain any the same number ρ of arguments in all as $S^{(0)}, \ldots, S^{(n-1)}$, provided only that the determinants of the matrix corresponding to $C_1 S_1 + \ldots + C_n S_n$ do not all vanish identically. Hence, since corresponding determinants will still remain proportional, it follows that the determinants of the matrix corresponding to $C_1 S_1 + \ldots + C_n S_n$ will have a common factor of the same degree as $D(n, t)/R(n, t)$ in the coefficients of all the polynomials $C_1, C_2, \ldots, C_n$ combined. Similar results hold for the matrices corresponding to

$$C_1 S^{(0)} + \ldots + C_l S^{(l-1)} \quad \text{and} \quad C_1 S_1 + \ldots + C_l S_l.$$

5. Theorem.—To prove that, neglecting sign,

$$\frac{D(n, t)}{R(n, t)} = \frac{D(n-1, t)}{R(n-1, t)} \frac{D(n-1, t-1)}{R(n-1, t-1)} \cdots \frac{D(n-1, t-m_n+1)}{R(n-1, t-m_n+1)}$$
$$\times D(n-1, t-m_n)\, D(n-1, t-m_n-1) \ldots D(n-1, 1).$$

$R(n, t)$ is a factor of $D(n, t)$, and the remaining factors of $D(n, t)$ are independent of the coefficients of C_n (§ 4). Let a_n be the coefficient of $x_n^{m_n}$ in C_n, and r the number of arguments in $S^{(n-1)}$ (of order $t-m_n$) which are severally used as multipliers of C_n in forming the r rows which correspond to C_n in $D(n, t)$. The element a_n appears in all these r rows of $D(n, t)$, and occupies the columns corresponding to the arguments of $x_n^{m_n} S^{(n-1)}$, the only columns absent from $D(n, t)$ being those which correspond to reduced arguments (§ 3), or arguments comprised in $S^{(n)}$. The remaining columns of $D(n, t)$ are those corresponding to all arguments of order t which are not comprised in $x_n^{m_n} S^{(n-1)} + S^{(n)}$, *i.e.*, $S^{(n-1)}$. Hence the coefficient of a_n^r in the expansion of $D(n, t)$ is the determinant whose vanishing is the condition that the identity

$$C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_{n-1} S^{(n-2)} = S^{(n-1)},$$

of order t, can be satisfied. To find this determinant, assume the identity satisfied, and put $x_n = 0$; then C_q becomes $C_q^{(n-1)}$, and if $S^{(q)}$ becomes $S'^{(q)}$, we have the identity

$$C_1^{(n-1)} S'^{(0)} + C_2^{(n-1)} S'^{(1)} + \ldots + C_{n-1}^{(n-1)} S'^{(n-2)} = S'^{(n-1)},$$

of order t. Hence either $D(n-1, t) = 0$, or $S'^{(0)}, S'^{(1)}, \ldots, S'^{(n-1)}$ all vanish identically. In the latter case, $S^{(0)}, S^{(1)}, \ldots, S^{(n-1)}$ are all divisible by x_n, and on dividing it out we have the identity

$$C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_{n-1} S^{(n-2)} = S^{(n-1)},$$

of order $t-1$. Hence $D(n-1, t-1) = 0$, or a similar identity holds of order $t-2$. It follows that the determinant sought is

$$\prod_{p=0}^{p=t-1} D(n-1, t-p).$$

Hence
$$D(n, t) = a_n^r \prod_{p=0}^{p=t-1} D(n-1, t-p) + \ldots .$$

We next find the coefficient of a_n^r in $R(n, t)$. Consider the determinant like $D(n, t)$ when the order of $C_1, C_2, \ldots, C_n$ is changed to $C_n, C_1, \ldots, C_{n-1}$, that is, the determinant whose vanishing is the condition that the identity

$$C_n S^{(0)} + C_1 S^{(1)} + \ldots + C_{n-1} S^{(n-1)} = S^{(n)},$$

of order t, can be satisfied. The element a_n appears in all the r' rows corresponding to C_n, and occupies the columns corresponding to the arguments of $x^{m_n} S^{(0)}$; hence, on expanding, the coefficient of $a_n^{r'}$ is the determinant whose vanishing is the condition that the identity

$$C_1 S^{(1)} + \ldots + C_{n-1} S^{(n-1)} = S^{(n)} + x_n^{m_n} S^{(0)},$$

of order t, can be satisfied. The coefficient of $a_n^{r'}$ is therefore

$$\prod_{p=0}^{p=m_n-1} D(n-1, t-p),$$

by a similar proof to the above. Keeping now C_n fixed, while the order of $C_1, C_2, \ldots, C_{n-1}$ is altered in all possible ways, the H.C.F. of the coefficients of $a_n^{r'}$ in the several expansions is

$$\prod_{p=0}^{p=m_n-1} R(n-1, t-p).$$

This is the coefficient of a_n^r in $R(n, t)$; for it is easily seen to be of the same degree as $R(n, t)$ in the coefficients of $C_1, \ldots, C_{n-1}$. Hence

$$R(n, t) = a_n^r \prod_{p=0}^{p=m_n-1} R(n-1, t-p) + \ldots .$$

Also the ratio of $D(n, t)$ to $R(n, t)$ is equal to the ratio of their first terms when expanded in this way; thus we have the theorem. When $t \geq t_n$,

$$R(n, t) = R(n, t_n) = a_n^{m_1 \ldots m_{n-1}} R(n-1, t_{n-1})^{m_n} + \ldots .$$

6. Theorem.—$R(n, t)$ is the quotient of $D(n, t)$ by the minor of $D(n, t)$ obtained by omitting the columns corresponding to all arguments reduced in $(n-1)$ of the variables $x_1, x_2, \ldots, x_n$, and the rows corresponding to C_r $(r = 1, 2, \ldots, n-1)$ for all multipliers reduced in $x_{r+1}, \ldots, x_n$.*

The resultant $R(n, t_n)$ of $C_1, C_2, \ldots, C_n$ is consequently the quotient of $D(n, t_n)$ by the corresponding minor of $D(n, t_n)$.

Let $\Delta(n, t)$ denote the minor of $D(n, t)$ mentioned above. To prove the theorem, viz., to prove that $\Delta(n, t) = D(n, t)/R(n, t)$, it will be sufficient (§ 5) to show that

$$\Delta(n, t) = \prod_{p=0}^{p=m_n-1} \Delta(n-1, t-p) \prod_{p=m_n}^{p=t-1} D(n-1, t-p),$$

and to verify that

$$\Delta(2, t) = D(2, t)/R(2, t).$$

Now $\Delta(n, t)$ is the determinant whose vanishing is the condition that the identity

I. $$C_1\Sigma^{(0)} + C_2\Sigma^{(1)} + \ldots + C_{n-1}\Sigma^{(n-2)} = \Sigma,$$

of order t, can be satisfied; where $\Sigma^{(0)}$ is a polynomial whose arguments are non-reduced in at least one of the variables $x_2, x_3, \ldots, x_n$; $\Sigma^{(1)}$ a polynomial whose arguments are reduced in x_1, but non-reduced in at least one of the variables $x_3, \ldots, x_n$; and similarly for $\Sigma^{(2)}, \ldots, \Sigma^{(n-2)}$ (*the last consequently divisible by* $x_n^{m_n}$); and finally Σ a polynomial whose arguments are reduced in at least $n-1$ of the variables. The number of coefficients in $\Sigma^{(0)}, \Sigma^{(1)}, \ldots, \Sigma^{(n-2)}$ is equal to the number of the equations they have to satisfy, and $\Delta(n, t)$ is not identically zero. This is seen by considering the polynomial

$$x_1^{m_1}\Sigma^{(0)} + x_2^{m_2}\Sigma^{(1)} + \ldots + x_{n-1}^{m_{n-1}}\Sigma^{(n-2)} + \Sigma,$$

in which every argument of order t occurs once and once only (*cf.* § 2).

Putting $x_n = 0$, and writing Σ' for the value each Σ then takes, we see that identity I. becomes

$$C_1^{(n-1)}\Sigma'^{(0)} + C_2^{(n-1)}\Sigma'^{(1)} + \ldots + C_{n-2}^{(n-1)}\Sigma'^{(n-3)} = \Sigma',$$

---

* It is to be remembered that the multipliers of C_r are also reduced in $x_1, \ldots, x_{r-1}$. The columns to be omitted are those which contain the elements $a_1, a_2, \ldots, a_n$ in the omitted rows, where a_r is the coefficient of $x_r^{m_r}$ in C_r.

of order t. Hence $\Delta(n-1, t) = 0$, or $\Sigma'^{(0)}, \Sigma'^{(1)}, \ldots, \Sigma'^{(n-3)}, \Sigma'$ all vanish identically. In the latter case x_n divides out of each Σ in I., and we have

II. $$C_1\Sigma^{(0)} + C_2\Sigma^{(1)} + \ldots + C_{n-1}\Sigma^{(n-2)} = \Sigma,$$

of order $t-1$. Since II. is obtained by dividing x_n out of I., the part played by $x_n^{m_n}$ in I. is now taken by $x_n^{m_n-1}$; so that in II., for any argument to be reduced or non-reduced in x_n means that it is non-divisible or divisible by $x_n^{m_n-1}$.

Putting $x_n = 0$ in II., we see that $\Delta(n-1, t-1) = 0$, or else identity I. still holds with t, $x_n^{m_n}$ changed to $t-2$, $x_n^{m_n-2}$. Proceeding in this way, we find that

$$\Delta(n, t) = \Delta(n-1, t)\,\Delta(n-1, t-1) \ldots \Delta(n-1, t-m_n+1)\,\Delta',$$

where Δ' is the determinant whose vanishing is the condition that identity I. holds when t, $x_n^{m_n}$ are changed to $t-m_n$, x_n^0. Thus each Σ is now necessarily non-reduced in x_n, and consequently $\Sigma^{(r)}$ takes the form $S^{(r)}$; while Σ is reduced in $x_1, x_2, \ldots, x_{n-1}$, and takes the form $S^{(n-1)}$. Hence identity I. becomes

$$C_1S^{(0)} + C_2S^{(1)} + \ldots + C_{n-1}S^{(n-2)} = S^{(n-1)}, \ldots$$

of order $t-m_n$. Hence (§ 5)

$$\Delta' = \prod_{p=m_n}^{p=t-1} D(n-1, t-p),$$

which proves the theorem.

Thus definite expressions have now been found for $R(n, t)$, $R(n, t_n)$, viz., $R(n, t) = D(n, t)/\Delta(n, t)$, $R(n, t_n) = D(n, t_n)/\Delta(n, t_n)$.

Another way of expressing the rule for obtaining $\Delta(n, t)$ from $D(n, t)$ is the following:—$\Delta(n, t)$ is the determinant formed by the elements of $D(n, t)$ occurring in all the columns corresponding to arguments non-reduced in two or more variables, and all the rows corresponding to C_r $(r = 1, 2, \ldots, n-1)$ for multipliers non-reduced in one or more variables leaving x_r out of account.

*6*a*. A more general way of forming an expression for the resultant of $C_1, C_2, \ldots, C_n$ is the following: — Form the complete matrix

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* This section was added in a revision of the paper, June 17th, 1902. It supplies a further proof of the theorem of § 6.

(having more rows than columns) corresponding to the function $C_1 S_1 + C_2 S_2 + \ldots + C_n S_n$ of order $t_n = 1 + \Sigma(m-1)$. Call he rows which make up $D(n, t_n)$, that is, the rows corresponding to

$$C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)},$$

the primary rows, and the rest the supplementary rows. Select any $\Sigma\mu$ of the primary rows, of which μ_r correspond to C_r, where

$$\mu_r m_r = \Pi m \quad (r = 1, 2, \ldots, n);$$

and add to them any supplementary rows, so as to form a determinant of the complete matrix which does not vanish identically. Then the resultant is the quotient of this determinant by the minor obtained from it by omitting the $\Sigma\mu$ rows, and the columns which contain the elements $a_1, a_2, \ldots, a_n$ in the $\Sigma\mu$ rows, where a_r is the coefficient of $x_r^{m_r}$ in C_r. Observe that there is one element $a_1, a_2, \ldots, a_n$ in each row and each column of the complete set of primary rows.

The theorem is also true for $R(n, t)$, when $\mu_1, \mu_2, \ldots, \mu_n$ are given their proper values; but for this case the proof given below requires amplification in one or two details.

Let $\rho_1, \rho_2, \ldots, \rho_n$ denote the numbers of the arguments of order t_n of the form $x_1^{m_1}\omega^{(0)}$, $x_2^{m_2}\omega^{(1)}$, $\ldots$, $x_n^{m_n}\omega^{(n-1)}$ respectively, so that $\Sigma\rho$ is the number of columns of the matrix, and $\rho_n = \mu_n$. Let $S_{(q)}^{(p)}$ denote a polynomial, and $\omega_{(q)}^{(p)}$ an argument, which is reduced in the first p and last q of the variables $x_1, x_2, \ldots, x_n$.

The complete set of supplementary rows forms a matrix of rank $\Sigma(\rho-\mu)$, which is the number of supplementary rows in any one of the determinants above; for the matrix corresponds to

$$C_2(x_1^{m_1}S) + C_3(x_1^{m_1}S + x_2^{m_2}S) + \ldots + C_n(x_1^{m_1}S + x_2^{m_2}S + \ldots + x_{n-1}^{m_{n-1}}S)$$

$$= x_1^{m_1}(C_2 S_{(n-2)} + C_3 S_{(n-3)} + \ldots + C_n S_{(0)}) + x_2^{m_2}(C_3 S_{(n-3)} + \ldots + C_n S_{(0)}) + \ldots$$

$$\ldots + x_{n-1}^{m_{n-1}}(C_n S_{(0)})$$

$$= C_2(x_1^{m_1} S_{(n-2)}^{(0)}) + C_3(x_1^{m_1} S_{(n-3)}^{(0)} + x_2^{m_2} S_{(n-3)}^{(1)}) + \ldots$$

$$\ldots + C_n(x_1^{m_1} S_{(0)}^{(0)} + x_2^{m_2} S_{(0)}^{(1)} + \ldots + x_{n-1}^{m_{n-1}} S_{(0)}^{(n-2)})$$

$$= x_1^{m_1}(C_2 S_{(n-2)}^{(0)} + C_3 S_{(n-3)}^{(0)} + \ldots + C_n S_{(0)}^{(0)})$$

$$+ x_2^{m_2}(C_3 S_{(n-3)}^{(1)} + C_4 S_{(n-4)}^{(1)} + \ldots + C_n S_{(0)}^{(1)}) + \ldots + x_{n-1}^{m_{n-1}}(C_n S_{(0)}^{(n-2)}),$$

which contains $\Sigma(\rho-\mu)$ parameters only. The number of para-

meters, for example, in the second bracket of the last line equals the number of arguments in the function

$$x_3^{m_3} S^{(1)}_{(n-3)} + x_4^{m_4} S^{(1)}_{(n-4)} + \ldots + x_n^{m_n} S^{(1)}_{(0)}$$

of order $t_n - m_2$, that is, the total number of arguments $\omega^{(1)}$ of order $t_n - m_2$ less the number in $x_2^{m_2} S^{(1)}_{(n-2)} + S^{(n)}_{(0)}$, or $S^{(1)}_{(n-2)}$, which equals $\rho_2 - \mu_2$.

Hence, if two determinants are chosen having the same primary rows, but different supplementary rows, then the determinants in one set of supplementary rows will be proportional to those in the other (§ 4), and the two original determinants will be in the same proportion. The theorem is therefore true for any set of supplementary rows, if it is true for one set.

The proof of the remaining part of the theorem will be sufficiently indicated by taking $n = 5$. Consider the determinant D arranged in columns and rows as in the following diagram :—

	ρ_1 columns $x_1^{m_1}\omega$	ρ_2 columns $x_2^{m_2}\omega^{(1)}$	ρ_3 columns $x_3^{m_3}\omega^{(2)}$	ρ_4 columns $x_4^{m_4}\omega^{(3)}$	$\rho_5 (=\mu_5)$ cols. $x_5^{m_5}\omega^{(4)}$
μ_1 rows ωC_1					
μ_2 rows $\omega^{(1)} C_2$		D_1			D_1
μ_3 rows $\omega^{(2)} C_3$			D_2		D_2
μ_4 rows $\omega^{(3)} C_4$				D_3	D_3
$\mu_5 (=\rho_5)$ rows $\omega^{(4)} C_5$					D_4 D_4 D_4 D_4
$\rho_4 - \mu_4$ rows $x_4^{m_4}\omega^{(3)}_{(0)} C_5$				Δ_4 D_3	D_3
$\rho_3 - \mu_3$ rows $x_3^{m_3}(\omega^{(2)}_{(1)} C_4, \omega^{(2)}_{(0)} C_5)$			Δ_3 D_2		D_2
$\rho_2 - \mu_2$ rows $x_2^{m_2}(\omega^{(1)}_{(2)} C_3, \omega^{(1)}_{(1)} C_4, \omega^{(1)}_{(0)} C_5)$		Δ_2 D_1			D_1
$\rho_1 - \mu_1$ rows $x_1^{m_1}(\omega^{0}_{(3)} C_2, \omega^{(0)}_{(2)} C_3, \omega^{(0)}_{(1)} C_4, \omega^{(0)}_{(0)} C_5)$	Δ_1				

The $\Sigma\mu$ primary rows correspond to $C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_5 S^{(4)}$ where $S^{(r-1)}$ $(r = 1, 2, 3, 4, 5)$ is an incomplete polynomial containing any μ_r arguments $\boldsymbol{\omega}^{(r-1)}$. The $\Sigma(\rho - \mu)$ supplementary rows are chosen in a particular way, each of the arguments $\boldsymbol{\omega}^{(p)}_{(q)}$ in the diagram being given all the values of which it is capable. The elements to the right of the dotted line are all zeros. D_r $(r = 1, 2, 3, 4)$ is the determinant marked in the diagram by writing D_r at the four corners. Δ_r $(r = 1, 2, 3, 4)$ is the determinant marked in the diagram on the supposition that the μ_r columns containing the element a_r in the primary rows are omitted. Let R be the resultant of $C_1, C_2, \ldots, C_5$, and R_r $(r = 1, 2, 3, 4)$ the resultant of $C_{r+1}, \ldots, C_5$ when $x_1, \ldots, x_r$ are made zero, so that $R_4 = a_5$. Then D_r is divisible by $R_r^{m_1 \ldots m_r}$. We shall prove this for the case $r = 2$, by the method of §§ 4, 5.

Let $C_3 S^{(2)}_3 + C_4 S^{(2)}_4 + C_5 S^{(2)}_5$ be the function whose matrix has D_2 for a determinant, $S^{(2)}_3$, $S^{(2)}_4$, $S^{(2)}_5$ being the sums of the multipliers of C_3, C_4, C_5 in D_2 affected with arbitrary coefficients λ. We shall be able to compare D_2 with a standard determinant of a similar type by bringing $C_3 S^{(2)}_3 + C_4 S^{(2)}_4 + C_5 S^{(2)}_5$ to a standard form, viz., the function $C_3 S^{(2)}_{(2)} + C_4 S^{(2)}_{(1)} + C_5 S^{(2)}_{(0)}$, with respect to the arguments to which the columns of D_2 correspond. These are the arguments of the type $x_3^{m_3}\boldsymbol{\omega}^{(2)}$, $x_4^{m_4}\boldsymbol{\omega}^{(3)}$, $x_5^{m_5}\boldsymbol{\omega}^{(4)}$, or $\boldsymbol{\omega}^{(2)}$, or $x_3^{m_3}\boldsymbol{\omega}^{(2)}_{(2)}$, $x_4^{m_4}\boldsymbol{\omega}^{(2)}_{(1)}$, $x_5^{m_5}\boldsymbol{\omega}^{(2)}_{(0)}$, from which we see that the standard function contains just the necessary number of parameters. Now the following identities, regarding the functions on the left hand as the unknowns,

$$(1)\quad S^{(2)}_{(2)} + C_4 S'^{(2)}_{(1)} + C_5 S'^{(2)}_{(0)} + (x_1^{m_1} S_1 + x_2^{m_2} S_2^{(1)}) = S^{(2)}_3,$$

$$(2)\quad S^{(2)}_{(1)} + C_5 S''^{(2)}_{(0)} + (x_1^{m_1} S'_1 + x_2^{m_2} S'^{(1)}_1) = S^{(2)}_4 + C_3 S'^{(2)}_{(1)},$$

$$(3)\quad S^{(2)}_{(0)} + (x_1^{m_1} S''_1 + x_2^{m_2} S''^{(1)}_2) = S^{(2)}_5 + C_3 S'^{(2)}_{(0)} + C_4 S''^{(2)}_{(0)},$$

can be satisfied in succession uniquely, and on multiplying by C_3, C_4, C_5, and adding, it is seen that the required transformation has been effected.

From (1), (2), (3) we see that the parameters λ' of $S^{(2)}_{(2)}$, $S^{(2)}_{(1)}$, $S^{(2)}_{(0)}$, when expressed in terms of the arbitrary parameters λ of $S^{(2)}_3$, $S^{(2)}_4$, $S^{(2)}_5$, contain the coefficients of C_3 only in the numerator. Hence the determinant D_2 contains all the factors of the standard determinant corresponding to $C_3 S^{(2)}_{(2)} + C_4 S^{(2)}_{(1)} + C_5 S^{(2)}_{(0)}$ which involve the coefficients of C_3 (§ 4). Now the standard determinant, which comes from equating to zero all terms in $C_3 S^{(2)}_{(2)} + C_4 S^{(2)}_{(1)} + C_5 S^{(2)}_{(0)}$ containing

$x_1^{\alpha_1} x_2^{\alpha_2}$ ($\alpha_1 = 0, 1, \ldots, m_1 - 1$; $\alpha_2 = 0, 1, \ldots, m_2 - 1$), breaks up into $m_1 m_2$ determinant factors of the type $D(3, t)$, each of which is divisible by R_2. The determinant D_2 is therefore divisible by $R_2^{m_1 m_2}$; and similarly D_r is divisible by $R_r^{m_1 \ldots m_r}$.

Again, D_2 and $R_2^{m_1 m_2}$ are each of degree μ_3 in the coefficients of C_3, while the coefficient of $a_3^{\mu_3}$ in D_2 is $\Delta_3 D_3$, and in $R_2^{m_1 m_2 m_3}$ is $R_3^{m_1 m_2 m_3}$ (§ 5) ; hence

$$\frac{D_2}{R_2^{m_1 m_2}} = \frac{\Delta_3 D_3}{R_3^{m_1 m_2 m_3}}.$$

From this, and similar results, we have

$$\frac{D}{R} = \Delta_1 \frac{D_1}{R_1^{m_1}} = \Delta_1 \Delta_2 \frac{D_2}{R_2^{m_1 m_2}} = \Delta_1 \Delta_2 \Delta_3 \frac{D_3}{R_3^{m_1 m_2 m_3}} = \Delta_1 \Delta_2 \Delta_3 \Delta_4 \frac{D_4}{R_4^{m_1 m_2 m_3 m_4}}$$

$$= \Delta_1 \Delta_2 \Delta_3 \Delta_4 ;$$

therefore $$R = D / \Delta_1 \Delta_2 \Delta_3 \Delta_4.$$

This proves the theorem for the case $n = 5$, and in a similar way it follows generally.

7. The problem of dividing out the extraneous factor $\Delta(n, t_n)$ from $D(n, t_n)$, so as to bring the resultant $R(n, t_n)$ to an integral form, appears to be a more difficult one than that of merely finding the extraneous factor. Any series of operations for finding the integral form of $R(n, t_n)$ would probably be very long and complicated.

A slight reduction in the magnitude of the extraneous factor is obtained as follows. Taking $m_1 \leqslant m_2 \ldots \leqslant m_n$, the coefficient a_1 of $x_1^{m_1}$ in C_1 raised to the power $(m_2 + \ldots + m_{n-1})_{n-1} / (n-1)!$ divides out of both $D(n, t_n)$ and $\Delta(n, t_n)$ at sight. This is not, however, the whole power of a_1 that divides $D(n, t_n)$ unless $m_1, m_2, \ldots, m_n$ are all equal. Among the multipliers of C_1 we may omit all those divisible by $x_1^{m_n}$. This will result in a diminution of $(m_2 + \ldots + m_{n-1})_{n-1} / (n-1)!$ in the number of rows of $D(n, t_n)$, and the same diminution in the number of columns, viz., the columns corresponding to all arguments divisible by $x_1^{m_1 + m_n}$. The extraneous factor in the reduced determinant D' is a minor of D'. To obtain it we omit all the rows and columns in D' which had to be omitted in $D(n, t_n)$ and which appear in D'. Those not appearing in D' are the rows corresponding to C_1 for multipliers $x_1^{m_n} \omega$, and the columns corresponding to arguments $x_1^{m_1 + m_n} \omega$, where ω is any argument of order $t_n - m_1 - m_n$ reduced in $x_2, x_3, \ldots, x_n$. We must then omit some other rows and columns of D' in place of those

which have disappeared. They may be chosen in several ways. We may, for example, choose the rows corresponding to C_1 for multipliers $x_1^{m_n-m_2}x_2^{m_2}\omega'$, where ω' is the value that ω takes when x_1 and x_2 are interchanged; for it can be proved that each of these rows simply supplies a factor a_1 to the minor $\Delta(n, t_n)$ of $D(n, t_n)$. Since ω is not divisible by $x_2^{m_2}$, ω' is not divisible by $x_1^{m_2}$, and $x_1^{m_n-m_2}x_2^{m_2}\omega'$ is not divisible by $x_1^{m_n}$, and so is a multiplier for C_1 in D'. The extra columns to be omitted from D' are those corresponding to the arguments $x_1^{m_1+m_n-m_2}x_2^{m_2}\omega'$.

A greater reduction in the magnitude of the extraneous factor is suggested by a method of Sylvester's.* Taking $m_1 \leqslant m_2 \ldots \leqslant m_n$ and

$$(a_1-1)+(a_2-1)+\ldots+(a_n-1)=m_1-1,$$

then any argument in C_1 or C_2 ... or C_n is of higher order than $x_1^{a_1-1}x_2^{a_2-1}\ldots x_n^{a_n-1}$, and therefore divisible by $x_1^{a_1}$ or $x_2^{a_2}$ ... or $x_n^{a_n}$; so that we can write

$$C_r = x_1^{a_1}A_r+x_2^{a_2}B_r+\ldots+x_n^{a_n}K_r \quad (r=1, 2, \ldots, n),$$

where $A_r, \ldots, K_r$ are polynomials. The number of solutions of the equation $\Sigma(a-1)=m_1-1$ in positive integral values of $a_1, \ldots, a_n$, excluding zeros, is equal to the number of arguments of order m_1-1; which is also the number of reduced arguments of order t_n-m_1, as may be seen by dividing any argument of order m_1-1 into $x_1^{m_1-1}x_2^{m_2-1}\ldots x_n^{m_n-1}$. There are therefore the same number of polynomials $\Sigma \pm ABC \ldots K$ of order t_n-m_1 as of reduced arguments of order t_n-m_1, taking only one set of polynomials $A, B, \ldots, K$ for each solution of the equation $\Sigma(a-1)=m_1-1$. The determinant D' corresponding to these polynomials and the function

$$C_1S^{(0)}+C_2S^{(1)}+\ldots+C_nS^{(n-1)},$$

of order t_n-m_1, will have the resultant $R(n, t_n)$ as a factor. The proof that D' does not vanish identically, provided that only one polynomial $\Sigma \pm AB \ldots K$ is chosen for each solution of

$$\Sigma(a-1)=m_1-1,$$

is somewhat complicated, and we omit it. It is clear that D' is divisible by the common factor of the determinants of the matrix

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* Salmon's *Higher Algebra*, 1885, p. 86, and *Camb. and Dub. Math. Jour.*, Vol. VII., 1852, p. 68.

corresponding to $C_1 S^{(0)} + \ldots + C_n S^{(n-1)}$; hence (end of § 4) D' is divisible by $\Delta(n, t_n - m_1)$. Also the quotient is of the same degree as $R(n, t_n)$ in the coefficients of $C_1, C_2, \ldots, C_n$, and is therefore identical with $R(n, t_n)$. Thus the extraneous factor in D' is $\Delta(n, t_n - m_1)$, which is obtainable from D' by the same rule (end of § 6) as $\Delta(n, t_n)$ from $D(n, t_n)$.

This method has the greatest effect in reducing the extraneous factor when $m_1, m_2, \ldots, m_n$ are all equal. When $n = 3$ and $m_1 = m_2 = m_3$, it gets rid of the extraneous factor altogether; and when $n = 3$ and $m_1 \leqslant m_2 \leqslant m_3$, it reduces the extraneous factor to a_1 raised to the power

$$\tfrac{1}{2}(m_2 - m_1)(m_2 - m_1 - 1) + \tfrac{1}{2}(m_3 - m_1)(m_3 - m_1 - 1),$$

of which a_1 to the power $\tfrac{1}{2}(m_2 - m_1)(m_2 - m_1 - 1)$ can be divided out, leaving a_1 to the power $\tfrac{1}{2}(m_3 - m_1)(m_3 - m_1 - 1)$ as an extraneous factor which does not divide out at sight.

8. We add a further list of formulæ without entering into details of proof. The formulæ of the present article are proved by methods that have been already employed. In §§ 9, 10 some indications are given as to how the results are obtained.

(1) $$D(n, t) = D(n-1, t)\, D(n, t-1), \text{ when } t > t_n,$$

(2) $$D(n, t) = \prod_{l=1}^{l=n} D(l, t_l)^{p_l}, \text{ when } t > t_n,$$

where

$$p_l = -\begin{vmatrix} \binom{t_n - t_{l+1}}{1} & 1 & 0 & \ldots\ldots & 0 \\ \binom{t_n - t_{l+2}}{2} & \binom{t_n - t_{l+2}}{1} & 1 & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \binom{t_n - t_{n-1}}{n-l-1} & \binom{t_n - t_{n-1}}{n-l-2} & \binom{t_n - t_{n-1}}{n-l-3} & \ldots\ldots & 1 \\ \binom{t_n - t}{n-l} & \binom{t_n - t}{n-l-1} & \binom{t_n - t}{n-l-2} & \ldots\ldots & \binom{t_n - t}{1} \end{vmatrix},$$

$\binom{t}{r}$ denoting the coefficient of x^r in $(1+x)^t$; $t_n - t$ in the last row of the determinant is a negative number.

(3) $$R(n, t) = R(n, t_n), \text{ when } t > t_n.$$

$R(n, t_n)$ is a prime, *i.e.*, non-factorisable (Netto, *l.c.*, p. 5). It is also

probable that $R(n, t)$ is a prime for all values of t when $m_1, m_2, \ldots, m_n$ are all equal. When $m_1, m_2, \ldots, m_n$ are not all equal it is probable that $R(n, t)$ has only one factor containing the coefficients of all the C's. In this case, if m_n is not less than any other m, the factor is the numerator of

$$R(n, t) \Big/ \prod_{p=0}^{p=m_n-1} R(n-1, t-p)$$

when reduced to its lowest terms.

If m_1 is less than any other m, then the coefficient of $x_1^{m_1}$ in C_1 is a factor of $R(n, t)$ when

$$t = m_1 + (m_2 - m_1 - 1) + \ldots + (m_n - m_1 - 1) = t_n - (n-1)\, m_1 ;$$

for $C_1 \times x_2^{m_2-m_1-1} \ldots x_n^{m_n-m_1-1}$ is of the form $S^{(n)}$ when this coefficient of C_1 vanishes.

If $m_1, \ldots, m_r$ are equal and less than any other m, then the determinant of the coefficients of the highest powers of $x_1, x_2, \ldots, x_r$ in $C_1, C_2, \ldots, C_r$ is a factor of $R(n, t)$ when

$$t = m_1 + (m_{r+1} - m_1 - 1) + \ldots + (m_n - m_1 - 1) = t_n - (n-1)\, m_1 + (r-1) ;$$

for $(\lambda_1 C_1 + \ldots + \lambda_r C_r)\, x_{r+1}^{m_{r+1}-m_1-1} \ldots x_n^{m_n-m_1-1}$ can be made of the form $S^{(n)}$ when this determinant vanishes.

Let $D(l, t)^{(n)}$ denote the determinant whose vanishing is the condition that $C_1^{(n)} S^{(0)} + C_2^{(n)} S^{(1)} + \ldots + C_l^{(n)} S^{(l-1)} + S^{(l)}$ can be made identically zero; so that $D(n, t)^{(n)}$ is the same as $D(n, t)$. Then $(n > l)$

$$(4) \qquad D(l, t)^{(n)} = D(l, t)^{(n-1)}\, D(l, t-1)^{(n)}$$

$$= \prod_{p=0}^{p=t-1} D(l, t-p)^{p+n-l-1\,!\,/\,p\,!\,n-l-1\,!}.$$

Thus $D(l, t)^{(n)}$ is independent of the coefficients of all terms in $C_1, C_2, \ldots, C_l$ which contain any variable other than $x_1, x_2, \ldots, x_l$.

Let $R(l, t)^{(n)}$ be the H.C.F. of the $l!$ determinants similar to $D(l, t)^{(n)}$ for the $l!$ different permutations of $C_1, C_2, \ldots, C_l$. Then $(n > l)$

$$(5) \qquad R(l, t)^{(n)} = R(l, t)^{(n-1)}\, R(l, t-1)^{(n)}$$

$$= \prod_{p=0}^{p=t-1} R(l, t-p)^{(n-1)}$$

$$= \prod_{p=0}^{p=t-1} R(l, t-p)^{p+n-l-1\,!\,/\,p\,!\,n-l-1\,!}.$$

By induction we have $(n \geq l)$

$$(6) \qquad \frac{D(l, t)^{(n)}}{R(l, t)^{(n)}} = \frac{D(l-1, t)^{(n)}}{R(l-1, t)^{(n)}} R(l-1, t-m_l)^{(n)},$$

the case for $n = l$ having been already proved (§ 5).

From (6) we have

$$(7) \quad D(n, t) = R(n, t) R(n-1, t-m_n)^{(n)} R(n-2, t-m_{n-1})^{(n)} \ldots \\ \ldots R(1, t-m_2)^{(n)},$$

from which $D(n, t)$ is expressed in prime factors when the m's are all equal, by (5) and (3).

The number of different prime factors involved in all the expressions $D(l, t)^{(n)}$ is $1+t_1+t_2+\ldots+t_{n-1}$ when $m_1, m_2, \ldots, m_n$ are in ascending order of magnitude; in other cases it is equal to or less than this number.

9. From identities of the type $\omega C_l = \omega C_l$ of order t, by writing C_l in full on the left hand, and solving for the arguments, we obtain

$$(8) \quad D(n, t)\,\omega_q = D_{q1}\omega_1 + \ldots + D_{qr}\omega_r + C_1 S^{(0)} + C_2 S^{(1)} + \ldots + C_n S^{(n-1)},$$

where $\omega_1, \ldots, \omega_r$ are the arguments of order t reduced in all the variables, and ω_q $(q > r)$ is any non-reduced argument. D_{qp} is a contraction for $D(n, t)_{qp}$, and may be regarded as being obtained from $D(n, t)$ by replacing the column corresponding to ω_q by the column corresponding to ω_p out of the same matrix, and then altering the sign. Identity (8) is written

$$(9) \qquad D(n, t)\,\omega_q \equiv \sum_{p=1}^{p=r} D_{qp}\omega_p \pmod{C_1, C_2, \ldots, C_n}.$$

Dividing out the common factor $D(n, t)/R(n, t)$, we have

$$(10) \qquad R(n, t)\,\omega_q \equiv \sum_{p=1}^{p=r} R_{qp}\omega_p \pmod{C_1, C_2, \ldots, C_n},$$

where $R_{q1}, R_{q2}, \ldots, R_{qr}$ are integral functions of the coefficients. Formulæ (9), (10) are unique, since there is only one expression for ω_q of the kind given.

In a similar way we have

$$(11) \qquad D(l, t)^{(n)}\,\omega_q \equiv \sum_{p=1}^{p=r} D_{qp}\omega_p \pmod{C_1, C_2, \ldots, C_l},$$

where $\omega_1, \omega_2, \ldots, \omega_r$ are *all* the arguments of order t reduced in $x_1, x_2, \ldots, x_l$, and ω_q is any non-reduced argument involving *only* $x_1, x_2, \ldots, x_l$. This equation may be multiplied by any argument

involving only x_{l+1}, ..., x_n, leaving the R's unchanged, but increasing the order of the arguments.

10. Notation.—We proceed to give some formulæ involving the roots of C_1, C_2, ..., C_{n-1}. For this purpose it is convenient to take $x_n = 1$, and regard C_1, C_2, ..., C_{n-1} as non-homogeneous polynomials in x_1, x_2, ..., x_{n-1}. We can reinsert x_n whenever we please. Let μ be the total number of arguments ω_1, ω_2, ..., ω_μ of order $\leqslant t$ (or of order t when x_n is reinserted), r the number of arguments reduced (in x_1, x_2, ..., x_{n-1}) of order $\leqslant t$, viz., ω_1, ..., ω_r in ascending order (so that $\omega_1 = 1$), r' the number of arguments reduced of order $\leqslant t - m_n$, viz., ω_1, ω_2, ..., $\omega_{r'}$, and ρ $(= m_1 \ldots m_{n-1})$ the number of vanishing points common to C_1, C_2, ..., C_{n-1}. When $t < t_{n-1} - 1$ then $r < \rho$, and when $t \geqslant t_{n-1} - 1$ then $r = \rho$. Select the ρ points in any order, calling them the 1st, 2nd, ..., ρ-th points. Let C_{ni} denote the value of C_n and ω_{pi} the value of ω_p, at the i-th point ($p = 1, 2, \ldots, \mu$; $i = 1, 2, \ldots, \rho$).

The matrix corresponding to $C_1 S_1 + \ldots + C_{n-1} S_{n-1}$, where S_1, ..., S_{n-1} are complete (or incomplete) polynomials of orders $t - m_1$, $t - m_2$, &c., is of *rank* $\mu - r$, *i.e.*, any sub-determinant of the matrix containing more than $\mu - r$ rows and columns is identically zero. For the identical vanishing of $C_1 S_1 + \ldots + C_{n-1} S_{n-1}$ requires μ equations of which only $\mu - r$ are independent, since $C_1 S_1 + \ldots + C_{n-1} S_{n-1}$ can be brought to a form $C_1 S^{(0)} + \ldots + C_{n-1} S^{(n-2)}$ involving only $\mu - r$ parameters.

Similarly, the matrix corresponding to $C_1 S_1 + \ldots + C_n S_n$ is of rank $\mu - r + r'$; $r - r'$ is the number of arguments of order t reduced in all the variables when x_n is reinserted.

The matrix of the roots, viz.,

$$\begin{vmatrix} \omega_{11}, & \omega_{21}, & \ldots, & \omega_{\mu 1} \\ \omega_{12}, & \omega_{22}, & \ldots, & \omega_{\mu 2} \\ \ldots & \ldots & \ldots & \ldots \\ \omega_{1\rho}, & \omega_{2\rho}, & \ldots, & \omega_{\mu\rho} \end{vmatrix}$$

is of rank r when $r < \rho$, or $t < t_{n-1} - 1$; for $S^{(0)}$, $S^{(1)}$, ..., $S^{(n-2)}$ can be so chosen that $C_1 S^{(0)} + \ldots + C_{n-1} S^{(n-2)}$ reduces to a polynomial containing only $r+1$ arguments arbitrarily assigned, and this polynomial vanishes at any $r+1$ of the ρ points.

Select $\mu - r$ rows of the matrix corresponding to $C_1 S_1 + \ldots + C_{n-1} S_{n-1}$ so as to form a matrix of the coefficients the determinants of which

do not all vanish identically. These $\mu - r$ rows cannot be chosen arbitrarily. Also select (arbitrarily) r rows of the matrix of the roots, viz., the first r rows; for these correspond to any r of the ρ points. The two resulting matrices are "corresponding," *i.e.*, the sum of the products of corresponding elements of any row in the first and any row in the second vanishes. Hence (neglecting sign) the ratio of any determinant of the first to the corresponding or complementary determinant of the second is constant.* This is expressed in the form

$$(12) \qquad \frac{D(\omega_{p_1}, \omega_{p_2}, \ldots, \omega_{p_r})}{\Sigma \pm \omega_{p_1 1} \omega_{p_2 2} \ldots \omega_{p_r r}} = \frac{D(\omega_{q_1}, \omega_{q_2}, \ldots, \omega_{q_r})}{\Sigma \pm \omega_{q_1 1} \omega_{q_2 2} \ldots \omega_{q_r r}},$$

where $D(\omega_{p_1}, \omega_{p_2}, \ldots, \omega_{p_r})$ is the determinant obtained from the matrix of the coefficients containing $\mu - r$ rows by omitting the columns corresponding to $\omega_{p_1}, \ldots, \omega_{p_r}$.

Notation.—Let Ω stand for the determinant $\Sigma \pm \omega_{11} \omega_{22} \ldots \omega_{\rho\rho}$ (the i-th row $\omega_{1i}, \ldots, \omega_{\rho i}$ corresponding to the i-th point), where $\omega_1, \ldots, \omega_\rho$ are all the arguments reduced (in $x_1, x_2, \ldots, x_{n-1}$), and $\omega_1 = 1$, $\omega_\rho = x_1^{m_1 - 1} \ldots x_{n-1}^{m_{n-1} - 1}$; also let M_i be the co-factor of $\omega_{\rho i}$ in Ω, J the Jacobian of $C_1, C_2, \ldots, C_{n-1}$, and J_i the value of J at the i-th of the ρ points. Then $\sum_{i=1}^{i=\rho} \frac{\omega_{\rho i}}{J_i} = 0$ when ω_ρ is of less order than J, viz., $t_{n-1} - 1$ (*Jacobi's theorem*†). Hence it follows that

$$(13) \qquad M_1 J_1 = \ldots = M_\rho J_\rho = \frac{\Omega}{\Sigma \frac{\omega_{\rho i}}{J_i}} = \frac{\Omega R(n-1, t_{n-1})}{R(n-1, t_{n-1}-1)},$$

omitting a numerical factor in the right-hand expression.

Dividing each element of the i-th row in Ω by J_i $(i = 1, 2, \ldots, \rho)$, and multiplying by Ω, the product $\Omega^2/\Pi J$ breaks up into t_{n-1} determinant factors in the diagonal [*cf.* p. 4, footnote (4)]. These can be evaluated and the p-th, counting from either end, is found to be

$$\frac{R(n-1, t_{n-1}-p)\, R(n-1, p-1)}{R(n-1, t_{n-1})},$$

* Gordan-Kerschensteiner, *Vorlesungen über Invariantentheorie*, Bd. I., 1885, pp. 95 and 110. The result also follows by cross-multiplication.

† Jacobi proves the theorem for two equations, *Crelle, Jour. f. Math.*, Vol. XIV., 1835, p. 281, and states it to be true for three equations, Vol. XV., p. 306; also *Werke*, 1881, &c., Vol. III., pp. 285, 352. Clebsch proves the general theorem, *Crelle, Jour. f. Math.*, Vol. LXIII., 1864, p. 224, and also Laurent, "L'Elimination," *Scientia, Phys.-Math.*, No. 7, 1900, p. 38.

except for a numerical factor. Hence

$$\frac{\Omega^2}{\Pi J} = \frac{[R(n-1,\ t_{n-1}-1)^{(n)}]^2}{R(n-1,\ t_{n-1})^{t_{n-1}}}, \tag{14}$$

omitting a numerical factor.

In a similar way a more general result may be obtained. Let any r' of the ρ points be chosen, where r' is the number of reduced arguments of order $\leqslant t'$. The number of the remaining r'' points is equal to the number of reduced arguments of order $\leqslant t''$, where $t'+t'' = t_{n-1}-2$. Let Ω', Ω'' be the values of the determinant $\Sigma \pm \omega_{11}\omega_{22}\dots\omega_{rr}$ when t is taken equal to t' and t'', and the points selected are the r' and r'' points respectively, and J', J'' the products of the J's for the r' and r'' points. Then, omitting a numerical factor,

$$\frac{\Omega'}{R(n-1,\ t')^{(n)}} \Big/ \left[\frac{J'}{R(n-1,\ t_{n-1})^{t'}}\right]^{\frac{1}{2}} = \frac{\Omega''}{R(n-1,\ t'')^{(n)}} \Big/ \left[\frac{J''}{R(n-1,\ t_{n-1})^{t''}}\right]^{\frac{1}{2}}. \tag{15}$$

This includes (13) and (14).

If $D(n,\ t)$ is multiplied by $\Sigma \pm \omega_{11}\omega_{22}\dots\omega_{rr}$, the product is found to be equal to $D(n-1,\ t)^{(n)}$ multiplied by the determinant

$$\begin{vmatrix} \omega_{11}C_{n1}\dots\omega_{r'1}C_{n1} & \omega_{r'+11}\dots\omega_{r1} \\ \omega_{12}C_{n2}\dots\omega_{r'2}C_{n2} & \omega_{r'+12}\dots\omega_{r2} \\ \dots\quad\dots & \dots\quad\dots \\ \omega_{1r}C_{nr}\dots\omega_{r'r}C_{nr} & \omega_{r'+1r}\dots\omega_{rr} \end{vmatrix}.$$

Hence

$$\frac{D(n,\ t)}{D(n-1,\ t)^{(n)}} = \frac{R(n,\ t)}{\prod_{p=0}^{p=m_n-1} R(n-1,\ t-p)} = \frac{\Sigma \pm \omega_{11}C_{n1}\dots\omega_{r'r'}C_{nr'}\omega_{r'+1r'+1}\dots\omega_{rr}}{\Sigma \pm \omega_{11}\omega_{22}\dots\omega_{rr}}. \tag{16}$$

This may also be generalized. Let D be the matrix formed from any $\mu-r+r'$ rows of the matrix corresponding to $C_1S_1+\dots+C_nS_n$, there being only r' rows containing the coefficients of C_n, viz., those obtained by multiplying C_n by $\omega_{p_1}, \omega_{p_2}, \dots, \omega_{p_{r'}}$. Let $D(\omega_{p_{r'+1}}, \dots, \omega_{p_r})$ be the determinant formed from D by omitting the columns corre-

sponding to $\omega_{p_{r'+1}}, \ldots, \omega_{p_r}$, and D' the matrix formed from D by omitting the r' rows corresponding to C_n. Then

$$(17) \qquad \frac{D(\omega_{p_{r'+1}}, \ldots, \omega_{p_r})}{D'(\omega_{q_1}, \ldots, \omega_{q_r})} = \frac{\Sigma \pm \omega_{p_1 1} C_{n1} \ldots \omega_{p_{r'} r'} C_{nr'} \omega_{p_{r'+1} r'+1} \ldots \omega_{p_r r}}{\Sigma \pm \omega_{q_1 1} \omega_{q_2 2} \ldots \omega_{q_r r}},$$

with the sufficient condition that neither numerator nor denominator on the left hand vanishes.

From (16) and (17) we have, when $t \geq t_n$, so that $r = r' = \rho$,

$$(18) \qquad \frac{D(n, t)}{D(n-1, t)^{(n)}} = \frac{R(n, t_n)}{R(n-1, t_{n-1})^{m_n}} = \prod_{i=1}^{i=\rho} C_{ni},$$

and

$$\frac{D}{D'(\omega_{q_1}, \ldots, \omega_{q_\rho})} = \frac{\Sigma \pm \omega_{p_1 1} \ldots \omega_{p_\rho \rho}}{\Sigma \pm \omega_{q_1 1} \ldots \omega_{q_\rho \rho}} \prod_{i=1}^{i=\rho} C_{ni}.$$

[To express any integral symmetric function of the roots in terms of the coefficients the following method may be adopted. Let ω_p be any argument, of order t. Eliminate $x_1, x_2, \ldots, x_{n-1}$ from the $n-1$ given equations and the additional equation $C_n = \alpha - \omega_p = 0$, obtaining the determinant form $D(n, t_n) = 0$, where $m_n = t$, $a_n = \alpha$, $t_n = t_{n-1} + t - 1$. The multipliers of C_n in $D(n, t_n)$ are the reduced arguments ω_i $(i = 1, 2, \ldots, \rho)$. Expand $D(n, t_n)$ in powers of α (§ 5). The result to two terms is

$$\alpha^\rho D(n-1, t_n)^{(n)} - \alpha^{\rho-1} \sum_{i=1}^{i=\rho} D_{pi} + \ldots = 0,$$

where D_{pi} is a determinant of the matrix to which $D(n-1, t_n)^{(n)}$ belongs, obtained by replacing the column corresponding to $\omega_p \omega_i$ in $D(n-1, t_n)^{(n)}$ by the column corresponding to ω_i and changing the sign. D_{pi} is zero if there is no column $\omega_p \omega_i$ in $D(n-1, t_n)^{(n)}$, *i.e.*, if $\omega_p \omega_i$ is a reduced argument. The roots of the equation in α are the ρ values of ω_p; hence we have ...

$$(19) \qquad D(n-1, t_n)^{(n)} \Sigma \omega_{pi} = \Sigma D_{pi}.$$

When the factor common to both sides of (19) is divided out, $R(n-1, t_{n-1})^t \Sigma \omega_{pi}$ is expressed as an integral function of the coefficients (§ 5). The other symmetric functions are expressible in terms of the functions $\Sigma \omega_{pi}$; *e.g.*, $\Sigma \omega_{pi} \omega_{qj} = \Sigma \omega_{pi} \times \Sigma \omega_{qi} - \Sigma (\omega_p \omega_q)_i$, and $R(n-1, t_{n-1})^t \Sigma \omega_{pi} \omega_{qj}$ is integral in the coefficients if t is the order of the higher of the two arguments ω_p, ω_q.—*October 8th*, 1902.]

On Groups in which every two Conjugate Operations are Permutable. *By* W. Burnside. Received and read May 8th, 1902.

In a paper published in the *Quarterly Journal of Mathematics* (1902), "On an Unsettled Question in the Theory of Discontinuous Groups," I have determined the order of a group with given generating operations when subject to the condition that the order of every operation shall be 3. If P and Q are any two operations of such a group, the relations

$$P^3 = 1, \quad Q^3 = 1, \quad (PQ)^3 = 1, \quad (PQ^2)^3 = 1$$

lead at once to $$P \,.\, QPQ^{-1} \,.\, Q^2PQ^{-2} = 1$$

and $$P \,.\, Q^2PQ^{-2} \,.\, QPQ^{-1} = 1;$$

so that P and QPQ^{-1} are permutable. The condition that every operation is of order 3 involves therefore that every two conjugate operations are permutable.

In the present paper I have considered the general problem thus presented; viz., the nature of a group generated by a finite number of operations when every two conjugate operations of the group are permutable. It will be seen that the general problem is closely connected with the more special one above referred to. When no further limitation is imposed on the operations, it is found that every operation of the group is given once and only once by a form

$$P^xQ^y \ldots R^z,$$

where P, Q, ..., R are a finite number of operations belonging to the group; and of the indices x, y, z, ... a certain number take all values from $-\infty$ to $+\infty$, while the remainder take the values 0, 1, 2.

The sufficient and necessary conditions that the group shall be of finite order are that the generating operations be of finite order. When this is the case, the group is the direct product of groups whose orders are powers of primes. In general for such a group the commutator of any two operations is a self-conjugate operation; but the case in which the order is a power of 3 is, as might be expected, exceptional.

1. In dealing with groups in which every two conjugate operations are permutable, the following notation will be used.

If $$P_a,\ P_b,\ P_c,\ \ldots$$

are any operations of such a group, the result of transforming P_a by P_b will be written $P_a P_{ab}$, so that

$$P_a^{-1} P_b^{-1} P_a P_b = P_{ab}.$$

Similarly, $$P_{ab}^{-1} P_c^{-1} P_{ab} P_c = P_{abc},$$

$$P_{abc}^{-1} P_d^{-1} P_{abc} P_d = P_{abcd},$$

and so on.

Further, the notation will be extended so that

$$P_{abc}^{-1} P_{de}^{-1} P_{abc} P_{de} = P_{(abc)(de)}.$$

The use of brackets in the suffixes will prevent any ambiguity; thus $P_{(abcd)e}$ is the same as P_{abcde}; but these are not necessarily the same as $P_{a(bcde)}$ or $P_{(ab)(cde)}$. From the definition of P_{ab} it follows that

$$P_{ba} = P_b^{-1} P_a^{-1} P_b P_a = P_{ab}^{-1}.$$

The operation P_{ab} may be regarded as the product of P_a^{-1} and $P_b^{-1} P_a P_b$; or as that of $P_a^{-1} P_b^{-1} P_a$ and P_b. Hence, since every operation is permutable with its conjugates, P_{ab} is permutable with both P_a and P_b.

Now, from $P_b^{-1} P_a P_b = P_a P_{ab}$ and $P_b^{-1} P_{ab} P_b = P_{ab}$,

it follows that $$P_b^{-y} P_a P_b^y = P_a P_{ab}^y,$$

and $$P_b^{-y} P_a^x P_b^y = P_a^x P_{ab}^{xy},$$

or $$P_a^{-x} P_b^{-y} P_a^x P_b^y = P_{ab}^{xy}.$$

Since P_{ab} is the product of P_a^{-1} and $P_b^{-1} P_a P_b$, it follows that P_{abc}, P_{abcd}, ... are products of powers of operations which are conjugate to P_a. Similarly, P_{abcd} may be expressed as products of powers of operations which are conjugate to P_b (or to P_c, or to P_d). Hence

$$P_{abcda} = 1,$$

$$P_{abcdb} = 1,$$

&c.

Again, P_{abc} and P_{ad} can both be represented as products of powers of

operations which are conjugate to P_a. They are therefore permutable, and

$$P_{(abe)(ad)} = 1.$$

Hence, generally, $P_{(abe\ldots)(de\ldots)(gh\ldots)\ldots} = 1,$

if in the multiple suffix any simple suffix occurs more than once.

Since P_{ab} is permutable with both P_a and P_b, every substitution of the sub-group generated by P_a and P_b can be represented in the form

$$P_a^x P_b^y P_{ab}^z.$$

In fact
$$\begin{aligned} P_a^x P_b^y P_{ab}^z P_a^{x'} P_b^{y'} P_{ab}^{z'} &= P_a^{x+x'} P_a^{-x'} P_b^y P_a^{x'} P_b^{-y'} P_b^{y+y'} P_{ab}^{z+z'} \\ &= P_a^{x+x'} P_{ab}^{-x'y} P_b^{y+y'} P_{ab}^{z+z'} \\ &= P_a^{x+x'} P_b^{y+y'} P_{ab}^{z+z'-x'y}. \end{aligned}$$

Let P_c be any operation which does not belong to this sub-group. Then

$$P_a P_c P_a^{-1} = P_c P_{ca}^{-1},$$

$$P_b P_a P_c P_a^{-1} P_b^{-1} = P_c P_{cb}^{-1} P_{ca}^{-1} P_{cab},$$

$$P_a^{-1} P_b P_a P_c P_a^{-1} P_b^{-1} P_a = P_c P_{cb}^{-1} P_{cba}^{-1} P_{cab};$$

and therefore
$$P_{ab}^{-1} P_c P_{ab} = P_c P_{cba}^{-1} P_{cab},$$

or
$$P_{c(ab)} = P_{cba}^{-1} P_{cab}.$$

Now
$$P_{c(ab)} = P_{abc}^{-1};$$

hence
$$P_{abc} P_{cba}^{-1} P_{cab} = 1,$$

or, since
$$P_{cba} = P_{bca}^{-1},$$

$$P_{abc} P_{bca} P_{cab} = 1. \qquad \text{(i)}$$

Again, since $P_a P_b$ and $P_c^{-1} P_a P_b P_c$ $(= P_a P_{ac} P_b P_{bc})$ are conjugate, they are permutable. Hence

$$\begin{aligned} P_a P_{ac} P_b P_{bc} &= P_b^{-1} P_a^{-1} (P_a P_{ac} P_b P_{bc}) P_a P_b \\ &= P_b^{-1} (P_a P_{ac} P_b P_{ba} P_{bc} P_{bca}) P_b \\ &= P_a P_{ab} P_{ac} P_{acb} P_b P_{ba} P_{bc} P_{bca}. \end{aligned}$$

But P_a, P_{ab}, P_{ac}, P_{acb} are all permutable, as also are P_b, P_{ba}, P_{bc}, P_{bca}.

Hence
$$P_a P_{ac} P_b P_{bc} = P_a P_{ac} P_{acb} P_{ab} P_{ba} P_{bca} P_b P_{bc},$$

or
$$P_{acb} P_{bca} = 1. \qquad \text{(ii)}$$

If P_c belongs to the sub-group generated by P_a and P_b, the relations

(i) and (ii) become identities. They are therefore true in any case, and for any permutation of the suffixes. Now, (ii) may be written

$$P_{bca} = P_{cab},$$

or

$$P_{acb} = P_{cba}.$$

Hence from (i) and (ii) together it follows that

$$P_{abc} = P_{bca} = P_{cab} = P_{acb}^{-1} = P_{cba}^{-1} = P_{bac}^{-1}; \qquad \text{(iii)}$$

and

$$P_{abc}^{3} = 1. \qquad \text{(iv)}$$

The relations (iii) are equivalent to the statement that when any permutation of the suffixes is effected in the symbol P_{abc} the operation represented is unaltered or changed into its inverse, according as the permutation is an even one or an odd one. Since

$$P_{ab\,\ldots\,def} = P_{(ab\,\ldots\,d)ef} = P_{(ab\,\ldots\,d)fe}^{-1} = P_{ab\,\ldots\,dfe}^{-1},$$

this statement may clearly be extended at once to any such symbol as $P_{ab\,\ldots\,def}$. Again,

$$P_{a(ij\,\ldots\,k)} = P_{(ij\,\ldots\,k)a}^{-1} = P_{ij\,\ldots\,ka}^{-1} = P_{aij\,\ldots\,k}^{(-1)^{r+1}},$$

where r is the number of suffixes in the set $i, j, \ldots, k$. Hence

$$P_{l\,\ldots\,m)(ij\,\ldots\,k)} = P_{l\,\ldots\,mij\,\ldots\,k}^{(-1)^{r+1}};$$

and thus any symbol $P_{(\ )(\ )\ldots}$ can be replaced by one in which there are no brackets in the suffix.

From the relation (iv) it follows that any symbol with three or more letters in its suffix is an operation of order 3, or else is the identical operation. Further, since

$$P_{abc}^{3} = 1$$

may be written

$$P_{ab}^{-3} P_{c}^{-1} P_{ab}^{3} P_{c} = 1,$$

the cube of every P with two letters in the suffix (*i.e.*, the cube of the commutator of any two operations) is a self-conjugate operation of the group. Again,

$$P_{abc}^{3} = 1$$

may be written

$$P_{ab}^{-1} P_{c}^{-3} P_{ab} P_{c}^{3} = 1.$$

Hence the cube of every operation of the group is permutable with

every operation whose suffix contains two or more letters; *i.e.*, with every operation of the derived group.

2. Let $$P_1, P_2, \ldots, P_n$$

be n independent operations which generate a group G, and suppose that the only conditions to which they are subject are that every two conjugate operations of G are permutable.

The product of any two operations of the form

$$P_1^a P_2^b \ldots P_n^e P_{12}^d P_{13}^e \ldots P_{n-1,n}^f P_{123}^g \ldots P_{123\ldots n}^h, \qquad (\mathrm{v})$$

where every P with a multiple suffix occurs once, while the P's are written in a definite sequence, is another operation of the same form. In fact, from the preceding paragraph,

$$P^{-y}_{a_1 a_2 \ldots a_r} P^{x}_{b_1 b_2 \ldots b_s} P^{y}_{a_1 a_2 \ldots a_r} = P^{x}_{b_1 b_2 \ldots b_s} P^{(-1)^{r+1} xy}_{b_1 b_2 \ldots, b_s a_1 a_2 \ldots a_r};$$

so that the multiplication can be actually carried out, and in the result the P's can be re-arranged in the original sequence. Hence with suitably chosen indices every operation of G can be represented in the form (v.).

To specify all distinct operations of the group it remains to show under what conditions a symbol of the form (v) represents the identical operation. As the basis of an induction it will be assumed that when there are $n-1$ generating operations the conditions are that (α) the index of each P with a single or double suffix is zero, and (β) the index of each P with a triple or higher suffix is zero or a multiple of 3.

If to the conditions defining G we add

$$P_1 = 1,$$

a new group is defined, which is simply isomorphic with G/H, where H is the self-conjugate sub-group of G generated by P_1 and its conjugate operations. The latter is an Abelian group, and cannot therefore be identical with G.

Now $$P_1 = 1$$

involves $$P_{12} = 1, \; P_{13} = 1, \; \ldots,$$

$$P_{123} = 1, \; \ldots .$$

Hence G/H is simply isomorphic with the group generated by $P_2, P_3, \ldots, P_n$; and this sub-group of G can therefore have no

operation, except identity, in common with H. Suppose now that

$$P_1^a P_2^b \dots P_n^c P_{12}^d P_{13}^e P_{23}^f \dots P_{n-1,n}^g P_{123}^h \dots P_{234}^i \dots = 1.$$

By preceding processes the factors on the left may be rearranged so that all the P's containing 1 in the suffix come at the end, the indices of the remaining P's being unaltered. Hence

$$P_2^b \dots P_n^c P_{23}^f \dots P_{n-1,n}^g P_{234}^i \dots = P_1^a P_{12}^{d'} P_{13}^{e'} \dots P_{123}^{h'} \dots .$$

Now the operation on the right belongs to H, and that on the left to $\{P_2, P_3, \dots, P_n\}$. Hence each must be the identical operation, and therefore by the assumption made

$$b = \dots = c = f = \dots = g = 0,$$

$$i = \dots = 0, \text{ or a multiple of 3.}$$

Similar series of results follow by considering the self-conjugate sub-groups arising respectively from the suppositions

$$P_2 = 1, \text{ or } P_3 = 1, \dots .$$

Hence an operation of the form (v) can only represent the identical operation, in a group generated by n operations, when the indices of *all* the P's with less than n symbols in the suffix satisfy the assumed conditions. But when this is so the operation reduces to $P_{123\dots n}^m$. If m is neither zero nor a multiple of 3,

$$P_{123\dots n}^m = 1, \quad \text{and} \quad P_{123\dots n}^3 = 1$$

involve

$$P_{123\dots n} = 1,$$

which would constitute an additional limitation on the group, not contained in the original conditions. Hence m must be zero or a multiple of 3; and the induction is completed.

Finally, therefore, every operation of G is given once, and once only, by the form (v), if the indices of the P's with a single or double suffix take all values from $-\infty$ to $+\infty$, while the indices of the P's with a triple or higher suffix take all values from 0 to 2.

3. So far it has been supposed that the only limitation on the group considered is that every two conjugate operations are permutable. The group under these conditions necessarily contains operations whose order is not finite. It will still have this property under a variety of further limitations. For instance, the condition

$$P_{123} = 1$$

implies that every P with a multiple suffix in which 1, 2, and 3 occur is the identical operation. The generality of the group is thus reduced; but it still contains operations whose order is not finite.

From the form giving the operations of the group it is clear that, if the order of every operation of the group is finite, the group is one of finite order. Moreover, the necessary and sufficient conditions for this are that each one of the generating operations should be of finite order. That these conditions are necessary is clear from the form (v). To show that they are sufficient, it is only necessary to notice that, if P_1 is of finite order m, P_{i2} is of finite order, equal to or a factor of m.

Suppose now that G is of finite order, and consider the operation $S^{-1}T^{-1}ST$ of G. As the product of S^{-1} and $T^{-1}ST$ its order must be equal to or a factor of that of S. Similarly its order is equal to or a factor of that of T. Hence, if the orders of S and T are relatively prime,

$$S^{-1}T^{-1}ST = 1,$$

or S and T are permutable. Since any two operations of G whose orders are relatively prime are permutable, G must be the direct product of a number of groups, the order of each of which is the power of a prime. In dealing with groups of finite order with the property considered, it is therefore sufficient to suppose the order to be p^α, where p is a prime.

The case $p = 3$ evidently stands by itself. Suppose first that p is not 3. Then P_{123} is an operation whose order is a power of p. But in any case it has been seen that

$$P_{123}^3 = 1.$$

Hence

$$P_{123} = 1,$$

or the commutator of any two operations of the group is a self-conjugate operation.

Every operation of the derived group is therefore a self-conjugate operation. That this condition is sufficient to ensure that in a group of order p^α every two conjugate operations are permutable is obvious.

If

$$P_1,\ P_2,\ \ldots,\ P_n,$$

of orders

$$p^{\alpha_1},\ p^{\alpha_2},\ \ldots,\ p^{\alpha_n}$$

$$(\alpha_1 \geq \alpha_2 \geq \alpha_3 \ldots \geq \alpha_n),$$

are the generating operations of such a group, and if the only conditions beyond the given orders of the generating operations are that

every two conjugate operations are permutable, the order of P_{rs} $(r<s)$ is p^{α_s}. Every operation of the group will then be given once by

$$\Pi P_r^{\gamma_r} \cdot \Pi P_{rs}^{\gamma_{rs}}$$

$$(r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, n;\ r<s),$$

where x_r takes all values from 0 to $p^{\alpha_r}-1$ and x_{rs} from 0 to $p^{\alpha_s}-1$. Hence the order of the group is

$$p^{\Sigma r\alpha_r}$$

When p is equal to 3, the only condition that P with a triple or higher suffix is subjected to is

$$P_{abc\ldots}^3 = 1,$$

and it is no longer the case that the commutator of any two operations is a self-conjugate operation. If the group is generated by

$$P_1,\ P_2,\ \ldots,\ P_n,$$

of orders $3^{\alpha_1},\ 3^{\alpha_2},\ \ldots,\ 3^{\alpha_n}$

$$(\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n),$$

every operation of the group will be represented once in the form (v), where the index of P_r lies between 0 and $3^{\alpha_r}-1$, the index of P_{rs} $(r<s)$ lies between 0 and $3^{\alpha_s}-1$, and all the other indices lie between 0 and 2; both limits included. The order of the group is therefore

$$3^{\Sigma r\alpha_r + 2^n - 1 - \frac{1}{2}n(n+1)}.$$

In particular, if the order of each of the generating operations is 3, the order of the group is

$$3^{2^n-1}.$$

This is the case already referred to in the introduction.

4. Groups of finite order p^a, of the kind here considered, possess two general properties in common with Abelian groups. First, the totality of the operations of the group whose orders are equal to or are less than p^r, where r is a given integer, constitute a sub-group. If P_1 and P_2 are any two operations of the group, of orders p^{α_1} and p^{α_2} $(\alpha_1 \geqslant \alpha_2)$, the order of P_{12} is equal to or is less than p^{α_2}. Now, by a repeated use of the formula

$$P_1^{-a} P_2^{-b} P_1^a P_2^b = P_{12}^{ab},$$

it follows that $$(P_1P_2)^x = P_1^x P_2^x P_{12}^{-\frac{1}{2}x(x-1)}.$$

Hence, if $$P_1^x = 1, \quad P_2^x = 1, \quad P_{12}^{\frac{1}{2}x(x-1)} = 1,$$

then $$(P_1P_2)^x = 1.$$

Now, if $p > 2$, the condition

$$P_{12}^{\frac{1}{2}x(x-1)} = 1$$

may be replaced by $$P_{12}^{x(x-1)} = 1.$$

It follows that, if $p > 2$, the order of P_1P_2 is equal to or is less than the order of P_1. The sub-group generated by all the operations whose orders are equal to or less than p^r consists therefore of these operations and of no others; for the product of any two of these operations is an operation whose order does not exceed p^r.

The case $p = 2$ clearly forms an exception to the general theorem; for, if P_1, P_2 are two non-permutable operations of the group of order 2, P_{12} is an operation of order 2. But

$$P_{12} = (P_1P_2)^2,$$

and P_1P_2 is therefore an operation of order 4.

Secondly, the totality of the distinct operations which arise, when every operation of the group is raised to the power p^r, where r is a given integer, constitute a sub-group. Consider the case in which $p > 3$, and the group is generated by two operations P_1 and P_2. The operations of the sub-group generated by P_1^p, P_2^p, and P_{12}^p are given by the form

$$P_1^{px} P_2^{py} P_{12}^{pz}.$$

The p-th power of any operation $P_1^a P_2^b P_{12}^c$ is given by the relation, obtained as above,

$$(P_1^a P_2^b P_{12}^c)^p = P_1^{pa} P_2^{pb} P_{12}^{pc - \frac{1}{2}abp(p-1)}.$$

Now $\frac{1}{2}(p-1)$ is an integer. Hence

$$P_1^{px} P_2^{py} P_{12}^{pz} = (P_1^x P_2^y P_{12}^{z + \frac{1}{2}(p-1)xy})^p;$$

i.e., every operation of $\{P_1^p, P_2^p, P_{12}^p\}$ is the p-th power of an operation of the group, and the p-th power of every operation of the group is contained in $\{P_1^p, P_2^p, P_{12}^p\}$. For more than two generating operations the proof will proceed on precisely similar lines. The theorem is therefore true when $r = 1$. But it must also be true for the resulting sub-group; so that it is true generally. The case $p = 2$ forms again an exception, since $\frac{1}{2}(p-1)$ is not then an integer. In fact, in this case $\{P_1^2, P_2^2, P_{12}^2\}$ will not contain P_{12}, which is the square of P_1P_2.

The case $p = 3$ requires separate treatment. It may be easily shown that in this case the cube of any operation of the group when expressed in the form (v) contains no P with a triple or higher suffix ; and from this the truth of the theorem immediately follows.

Thursday, June 12th, 1902.

Dr. E. W. HOBSON, F.R.S., President, in the Chair.

Fifteen members present.

The President announced that the "De Morgan medal" for 1902 had been awarded to Prof. A. G. Greenhill.

Mr. A. C. Porter was admitted into the Society.

The following paper was communicated by Prof. Love :—

Prof. A. W. Conway : "The Principle of Huygens in a Uniaxal Crystal."

Lieut.-Col. A. Cunningham gave an account of "Some Investigations concerning the repetition of the Sum-Factor Operation."

The following papers were communicated from the Chair :—

Prof. E. Picard : "Sur un théorème fondamental dans la théorie des équations différentielles."

Mr. G. H. Hardy : "Some Arithmetical Theorems."

Prof. M. J. M. Hill : "On a Geometrical Proposition connected with the Continuation of Power Series."

Mr. J. H. Grace : "Types of Perpetuants."

The following presents were made to the Library :—

"Educational Times," June, 1902.

"Indian Engineering," Vol. xxxi., Nos. 16–20 ; 1902.

"Queen's College, Galway—Calendar for 1901–1902."

Penfield, S. L.—"The Stereographic Projection and its Possibilities from a Graphical Standpoint," 1901.

Penfield, S. L.—"On the Use of the Stereographic Projection for Geographical Maps and Sailing Charts," 1902.

"Periodico di Matematica," Serie 2, Vol. iv., Fasc. 6 ; Livorno, 1902.

"Supplemento al Periodico di Matematica," Anno v., Fasc. 7 ; Livorno, 1902.

"L'Enseignement Mathématique," Année IV., No. 3; 1902.

"Mathematical Gazette," Vol. II., No. 33, 1902.

Guldberg, A.—"Ueber Integralinvarianten und Integralparameter bei Berührungs-Transformationsgruppen," 1902.

"Journal de l'Ecole Polytechnique—Hommage rendu à M. le Colonel Mannheim," 1902.

De Morgan, A.—"Theory of Probabilities" (extract from *Encyclopædia Metropolitana*). From Mr. R. Tucker.

Carvallo, E.—"L'Electricité" (*Scientia*, No. 19).

"Mathematical Questions and Solutions from the *Educational Times*," New Series, Vol. I.; 1902.

D. Ocagne, M.—"Sur quelques Travaux récents relatifs à la Nomographie."

The following exchanges were received :—

Académie Royale de Belgique :—

"Annuaire, 1902," Bruxelles.

"Bulletin de la Classe des Sciences," Nos. 1–3; Bruxelles, 1901–1902.

"Mémoires Couronnés," Bruxelles, 1901–1902.

"Mémoires," Tome LIV., Fasc. 1–4; Bruxelles, 1900–1901.

"Mémoires Couronnés et Mémoires des Savants Etrangers," Tome LIX., Fasc. 1, 2; Bruxelles, 1901.

"Proceedings of the Royal Society," Vol. LXX., Nos. 459, 460; 1902.

"Reports to the Evolution Committee of the Royal Society"; 1902.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXVI., No. 5; Leipzig, 1902.

"Bulletin de la Société Mathématique de France," Tome XXX., Fasc. 1; Paris, 1902.

"Bulletin of the American Mathematical Society," Vol. VIII., No. 8; "Transactions," Vol. III., No. 2; New York, 1902.

"Bulletin des Sciences Mathématiques," Tome XXV., "Contents," 1901; Tome XXVI., Mars, 1902; Paris.

"Annali di Matematica," Tomo VII., Fasc. 2, 3; Milano, 1902.

"Atti della Reale Accademia dei Lincei—Rendiconti," Sem. 1, Vol. XI., Fasc. 8, 9, 10; Roma, 1902.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," Nos. 1–22; 1902.

Sur un théorème fondamental dans la théorie des équations différentielles. *Par* M. Emile Picard. Received May 29th, 1902. Read June 12th, 1902.

Je viens de lire dans les *Proceedings* l'intéressant article de M. W. H. Young sur le théorème fondamental de la théorie des équations différentielles. Etant considérée l'équation

$$\frac{dy}{dx} = f(x, y),$$

où f est holomorphe dans le voisinage de $x = a$, $y = b$, il existe une intégrale prenant la valeur b pour $x = a$, et *il n'en existe qu'une seule.*

La première partie du théorème est due à Cauchy, et notre savant collègue veut bien rappeler que j'ai donné dans mon *Traité d'Analyse* une démonstration très simple de la seconde partie du théorème. Récemment M. Painlevé a donné une autre démonstration que commente M. Young. Celui-ci fait ensuite quelques remarques au sujet d'objections non fondées adressées à la seconde partie du théorème. Je me permettrai de rappeler que ces objections faites pour la première fois par M. Fuchs m'étaient bien connues dès 1893, date de l'apparition du tome II de mon *Traité*, et je les discute d'une manière explicite dans mon ouvrage.

Il suffira de transcrire deux passages. On lit tout d'abord (p. 314) :

Par système d'intégrales $y_1, y_2, \ldots, y_n$, prenant pour $x = x_0$ les valeurs $y_1^0, y_2^0, \ldots, y_n^0$, nous entendons un système jouissant de la propriété suivante. On imagine qu'autour de x_0, $y_1^0, \ldots, y_n^0$ on décrive des cercles de rayons très petits, et l'on suppose que, x restant à l'intérieur du premier en suivant un arc de courbe C aboutissant au point x_0, les valeurs correspondantes des y restent respectivement à l'intérieur des autres cercles ; de plus, quand, x restant sur C tend vers x_0, les y tendent respectivement vers les y_0.

Plus loin (p. 317) on lit encore :

J'ai insisté sur l'existence unique du système d'intégrales dans le cas général où les équations ne présentent aucune circonstance singulière ; c'est en effet une proposition tout à fait fondamentale et la base même de l'emploi des équations différentielles dans toutes les applications. Il semble de plus, à lire certaines phrases d'un mémoire de M. Fuchs (*Sitzungsberichte der Berliner Akademie*, 1886) que ce théorème puisse être mis en doute. Je ne puis partager le scepticisme de l'illustre géomètre. Sans doute, on peut faire une légère critique à la démonstration

de Briot et Bouquet, mais la proposition elle-même n'en subsiste pas moins, comme je viens de le montrer, si l'on précise bien l'énoncé. Prenons, en le simplifiant, un exemple cité par M. Fuchs : Soit l'équation

$$\frac{dy}{dx} = -\frac{y^2}{x},$$

dont l'intégrale générale est visiblement

$$y = \frac{1}{\log x + C},$$

C étant une constante arbitraire. Soit $x_0 \neq 0$; lorsque x part du voisinage de x_0 et y revient après avoir tourné un très grand nombre de fois autour de l'origine, une intégrale quelconque y a une valeur de plus en plus petite, mais il est bien clair que l'on ne peut pas dire qu'on a là une intégrale s'annulant pour $x = x_0$, au sens que nous avons spécifié plus haut.

Il semble donc bien que dès 1893 la question était épuisée. J'ajoute seulement une remarque relative à la première partie du théorème. La démonstration de Cauchy montre, avec les notations usuelles, que l'intégrale holomorphe a au moins comme rayon de convergence l'expression

$$a\,(1 - e^{-b/2aM}) ;$$

j'ai montré (*Bulletin des Sciences Mathématiques*, 1888, et *Traité d'Analyse*, tome II, p. 313) que ce rayon est au moins la plus petite des deux quantités

$$a \quad \text{et} \quad \frac{b}{M}.$$

J'en ai donné une seconde démonstration dans le tome III de mon *Traité* (p. 90, en note).

The Repetition of the Sum-Factor Operation. Abstract of an informal communication made by Lieut.-Col. A. Cunningham. June 12th, 1902.

Let $\sigma(N)$ denote the sum of the sub-factors of N (including 1, but excluding N). It was found that, with most numbers, $\sigma^n N = 1$, when the operation (σ) is repeated often enough. There is a small class for which $\sigma^n N = P$ (a *perfect* number), and then repeats; another small class for which $\sigma^n N = A$, $\sigma^{n+1} N = B$, where A, B are *amicable* numbers, and then repeats (A, B alternately); another small class for which (even when N is *small*, < 1000) $\sigma^n N$ increases beyond the practical power of calculation.

On a Geometrical Proposition connected with the Continuation of Power-Series.

By M. J. M. Hill, Professor of Mathematics, University College, London. Received June 11th, 1902. Read June 12th, 1902.

1. The following notation will be employed in this paper.

The circle of convergence of the power series $P(x-x_0)$ will be called C_0 and its radius R_0.

If a new series is derived from $P(x-x_0)$ by Taylor's theorem, say $P(x-x_0 \mid x_n)$, then its circle of convergence will be denoted by C_n, and the radius of that circle by R_n.

Also the circle described with centre x_n so as to touch the circle C_s internally will be called C_{ns}.

2. Suppose that by means of Taylor's theorem a new power series is obtained from $P(x-x_0)$, say $P(x-x_0 \mid x_1)$. Then

$$P(x-x_0) = P(x-x_0 \mid x_1) \qquad \text{(I.)}$$

throughout the circles C_0 and C_1.

The object of this paper is to draw attention to a geometrical proposition arising out of the mode adopted in Harkness and Morley's *Introduction to the Theory of Analytic Functions* for proving this result, which is perhaps of some interest. It is shown in Art. 89 of the book referred to that, if x_2 be a point in the region for which the relation (I.) has been proved, and if with x_2 as centre circles be described so as to touch the circles C_0 and C_1 internally (viz., the circles C_{20} and C_{21}), then the relation (I.) also holds throughout the smaller of these two circles, so that the region in which (I.) is known to be valid is extended by so much of the smaller of the two circles C_{20}, C_{21} as may be external to that region.

3. In order to demonstrate the validity of the relation (I.) throughout the whole of the space common to C_0 and C_1, it is necessary to prove that it is possible to choose successive positions of x_2 so that every part of the region common to C_0 and C_1 shall be interior to one at least of the circles C_{20}, C_{21} for which the relation (I.) can be

proved to hold. This is the geometrical proposition to which reference is made in the heading of the paper.

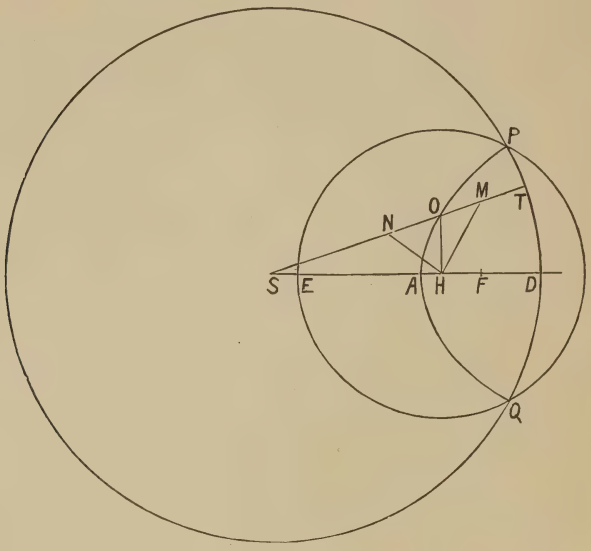

Fig. 1.

Let S (Fig. 1) be the point x_0, let H be the point x_1.

Let P and Q be the intersections of the circles C_0, C_1 (the figure is drawn for the case $R_0 > R_1$).

Let the arc of the hyperbola whose foci are S and H and which passes through P and Q be drawn.

Let A be that vertex of the hyperbola which lies on this arc.

Let SH cut the circle C_0 at D, and let F be the middle point of AD.

4. Let x_2 be a point somewhere in the region common to C_0 and C_1. It will be proved that—

(a) If x_2 lie on the arc PQ, then the circles C_{20} and C_{21} coincide.

Let x_2 be at O (Fig. 1).

Then $$SO-HO = SP-HP = R_0-R_1;$$

therefore $$|x_2-x_0|-|x_2-x_1| = R_0-R_1;$$

therefore $$R_0-|x_2-x_0| = R_1-|x_2-x_1|;$$

therefore the radius of C_{20} is equal to the radius of C_{21}.

(*b*) If x_2 be separated by the hyperbolic arc from x_0, then the circle C_{20} is smaller than the circle C_{21}.

Take x_2 at M.

Let SM meet the hyperbolic arc PQ at O.

Join HO, HM.

Then
$$\begin{aligned} SM-HM &= SO+OM-HM \\ &= SO-HO+(HO+OM-HM) \\ &> SO-HO ; \end{aligned}$$

therefore
$$|x_2-x_0|-|x_2-x_1| > R_0-R_1 ;$$

therefore
$$R_0-|x_2-x_0| < R_1-|x_2-x_1| ;$$

therefore the radius of C_{20} is less than the radius of C_{21}.

(*c*) If x_2 be separated by the hyperbolic arc from x_1, then the circle C_{21} is smaller than the circle C_{20}.

Take x_2 at N.

Let SN meet the hyperbolic arc PQ at O.

Join HO, HN.

Then
$$\begin{aligned} SN-HN &= SO-ON-HN \\ &= SO-HO-(ON+HN-HO) \\ &< SO-HO ; \end{aligned}$$

therefore
$$|x_2-x_0|-|x_2-x_1| < R_0-R_1 ;$$

therefore
$$R_1-|x_2-x_1| < R_0-|x_2-x_0| ;$$

therefore the radius of C_{21} is less than the radius of C_{20}.

5. It is necessary to find out whether the vertex of the hyperbolic arc, viz., the point A, is inside the circle C_{10}.

Let a, b be the semi-transverse and semi-conjugate axes, and e the eccentricity of the hyperbola.

Then
$$AH = ae-a,$$
$$SH = 2ae.$$

Also it is known that R_1 lies between R_0-SH and R_0+SH,

i.e.,
$$R_0-|x_1-x_0| < R_1 < R_0+|x_1-x_0|,$$

and
$$|x_1-x_0| = 2ae.$$

Let the points on the hyperbola be given by
$$x = a\sec\theta, \quad y = b\tan\theta.$$

Then, if P be the point corresponding to $\theta = \gamma$, we have

$$SP = R_0 = e\,(a \sec \gamma) + a,$$

$$HP = R_1 = e\,(a \sec \gamma) - a.$$

If A be inside C_{10}, then $HA < HD$; therefore

$$HA < R_0 - |\, x_1 - x_0 \,|,$$

i.e.,

$$ae - a < R_0 - 2ae,$$

$$3ae - a < R_0;$$

but

$$2a = SP - HP = R_0 - R_1;$$

therefore

$$a = \tfrac{1}{2}\,(R_0 - R_1),$$

and

$$e < \frac{R_0 + a}{3a};$$

therefore

$$e < \frac{3R_0 - R_1}{3\,(R_0 - R_1)},$$

or, using the values of R_0, R_1 above,

$$3e < e \sec \gamma + 2. \qquad \text{(II.)}$$

Hence, if γ be large enough, A will lie inside C_{10}.

6. If, however, the relation (II.) is not satisfied, let x_2 be taken inside C_{10}, at a small distance ϵ from its circumference, and lying between x_1 and x_0.

Then

$$R_0 - |\, x_1 - x_0 \,| = |\, x_1 - x_0 \,| - |\, x_2 - x_0 \,| + \epsilon.$$

If A be not inside the circle C_{20} throughout which the relation (I.) holds, take x_3 inside C_{20}, at the same distance ϵ from its circumference, and lying between x_2 and x_0; then

$$R_0 - |\, x_2 - x_0 \,| = |\, x_2 - x_0 \,| - |\, x_3 - x_0 \,| + \epsilon,$$

and so on, until the relation

$$R_0 - |\, x_{n-1} - x_0 \,| = |\, x_{n-1} - x_0 \,| - |\, x_n - x_0 \,| + \epsilon$$

is reached.

Then

$$|\, x_2 - x_0 \,| = 2\,|\, x_1 - x_0 \,| - (R_0 - \epsilon),$$

$$|\, x_3 - x_0 \,| = 2^2\,|\, x_1 - x_0 \,| - (2^2 - 1)(R_0 - \epsilon),$$

$$|\, x_4 - x_0 \,| = 2^3\,|\, x_1 - x_0 \,| - (2^3 - 1)(R_0 - \epsilon),$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$|\, x_n - x_0 \,| = 2^{n-1}\,|\, x_1 - x_0 \,| - (2^{n-1} - 1)(R_0 - \epsilon)$$

$$= (R_0 - \epsilon) - 2^{n-1}\left[R_0 - \epsilon - |\, x_1 - x_0 \,|\right].$$

Hence it is possible after a *finite* number of circles have been constructed in this manner to find one C_{n0}, such that the relation (I.) holds throughout it, and such that

$$|x_n - x_0| < \tfrac{1}{2}(SA + SD),$$

i.e.,

$$|x_n - x_0| < SF,$$

where F is the middle point of AD.

Then x_n is between S and F.

Then the circle C_{n0} contains the point A in its interior.

Hence the point x_2 can be taken at A, and then the circles C_{20}, C_{21} corresponding to this position of x_2 coincide, and the relation (I.) holds throughout them.

It is to be noticed that this has been accomplished by taking a *finite* number of successive positions of x_2.

7. It will now be proved that it is unnecessary to consider positions of x_2 which do not lie on the hyperbolic arc PQ.

If x_2 be anywhere between the arc PQ of the hyperbola and the arc PDQ of the circle C_0, say at M, let SM meet the hyperbolic arc at O, and the arc PDQ of the circle C_0 at the point T; then the circle whose centre is M and which touches C_0 internally at T lies wholly inside the circle whose centre is O and which touches C_0 internally at T. Hence x_2 may be taken at O on the hyperbolic arc instead of at M, and all extensions of area obtained by supposing x_2 at M will be included in those obtained by supposing x_2 at O.

A similar conclusion is obtained when x_2 lies between the hyperbolic arc PQ and the arc PEQ of the circle C_1.

8. It is necessary to obtain a sequence of points on the hyperbolic arc such that each point lies inside the circle whose centre is at the preceding point of the sequence, and which touches both C_0 and C_1 internally.

If the points on the hyperbolic arc are given by the equations

$$x = a \sec\theta, \quad y = b \tan\theta,$$

it will be proved that such a sequence of points is determined by the values $0, \theta_1, \theta_2, \ldots, \theta_n, \theta_{n+1}, \ldots$ of the parameter θ which are such that

$$\frac{\sin(\gamma - \theta_{n+1})}{\sin(\gamma - \theta_n)} = \frac{\sin(\gamma - \theta_1)}{\sin\gamma}, \qquad \text{(III.)}$$

where γ is the value of the parameter which corresponds to the

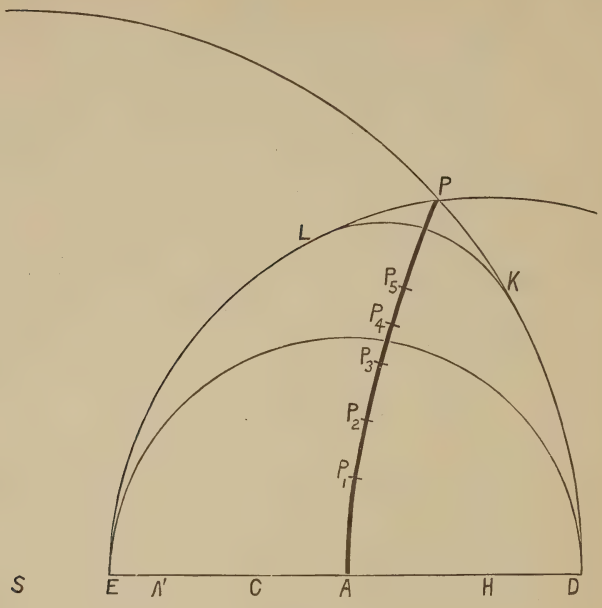

FIG. 2.

intersection P (see Fig. 2*) of the circles C_0 and C_1; and θ_1 is an acute angle which satisfies the condition

$$\sin \tfrac{1}{2}\theta_1 < \tfrac{1}{2} \sin \gamma \sin \tfrac{1}{2}\gamma. \qquad \text{(IV.)}$$

Putting $n = 0$ in (III.), it follows that $\theta_0 = 0$; so that the vertex A of the hyperbolic arc is the first point of the sequence.

It follows from (III.) that

$$\sin (\gamma - \theta_n) = \sin (\gamma - \theta_1) \left(\frac{\sin (\gamma - \theta_1)}{\sin \gamma} \right)^{n-1};$$

therefore $$\lim_{n=\infty} \sin (\gamma - \theta_n) = 0;$$

therefore $$\lim_{n=\infty} \theta_n = \gamma.$$

Hence the sequence of points on the hyperbolic arc has an intersection of the circles C_0 and C_1 for a limit-point. This is the point P in the figure.

* The description and dimensions of this figure are given in Art. 10 below.

The distance between consecutive points $P_n P_{n+1}$ of the sequence is

$$\sqrt{(a\sec\theta_{n+1}-a\sec\theta_n)^2+(b\tan\theta_{n+1}-b\tan\theta_n)^2}$$

$$=\frac{2\sin\tfrac{1}{2}(\theta_{n+1}-\theta_n)\sqrt{a^2\sin^2\tfrac{1}{2}(\theta_{n+1}+\theta_n)+b^2\cos^2\tfrac{1}{2}(\theta_{n+1}-\theta_n)}}{\cos\theta_n\cos\theta_{n+1}}.$$

The radius of the circle whose centre is P_n and which touches both C_0 and C_1 internally (viz., the circle C_{n0}) is

$$a(e\sec\gamma+1)-a(e\sec\theta_n+1)=(\sec\gamma-\sec\theta_n)\sqrt{a^2+b^2}.$$

It is obvious that

$$\sqrt{a^2\sin^2\tfrac{1}{2}(\theta_n+\theta_{n+1})+b^2\cos^2\tfrac{1}{2}(\theta_{n+1}-\theta_n)}<\sqrt{a^2+b^2},$$

and it will be proved that

$$\frac{2\sin\tfrac{1}{2}(\theta_{n+1}-\theta_n)}{\cos\theta_n\cos\theta_{n+1}}<\sec\gamma-\sec\theta_n. \qquad \text{(V.)}$$

When this has been done it will follow that $P_n P_{n+1}$ is less than the radius of the circle C_{n0}.

The inequality (V.) follows from a transformation of equation (III.) by means of the condition (IV.).

Equation (III.) can be written thus,

$$\frac{2\sin\tfrac{1}{2}(\theta_{n+1}-\theta_n)}{\cos\theta_n\cos\theta_{n+1}}=(\sec\gamma-\sec\theta_n)\cos\tfrac{1}{2}(\gamma-\theta_n)\left(\frac{\cos(\gamma-\tfrac{1}{2}\theta_1)}{\cos[\gamma-\tfrac{1}{2}(\theta_n+\theta_{n+1})]}\right)$$

$$\times\left(\frac{\cos\gamma}{\cos\theta_{n+1}}\right)\left[\frac{\sin\tfrac{1}{2}\theta_1}{\tfrac{1}{2}\sin\gamma\sin\tfrac{1}{2}(\gamma+\theta_n)}\right]. \qquad \text{(VI.)}$$

This is shown by some simple trigonometrical reductions which need not be given.

Now $\cos\tfrac{1}{2}(\gamma-\theta_n)<1$;

$$\frac{\cos\gamma}{\cos\theta_{n+1}}<1, \quad \text{since} \quad \gamma>\theta_{n+1};$$

$$\frac{\cos(\gamma-\tfrac{1}{2}\theta_1)}{\cos[\gamma-\tfrac{1}{2}(\theta_n+\theta_{n+1})]}<1, \quad \text{since} \quad \theta_1<\theta_n+\theta_{n+1};$$

$$\frac{\sin\tfrac{1}{2}\theta_1}{\tfrac{1}{2}\sin\gamma\sin\tfrac{1}{2}(\gamma+\theta_n)}<1, \quad \text{since} \quad \sin\tfrac{1}{2}\theta_1<\tfrac{1}{2}\sin\gamma\sin\tfrac{1}{2}\gamma$$

$$<\tfrac{1}{2}\sin\gamma\sin\tfrac{1}{2}(\gamma+\theta_n).$$

Hence (V.) follows from (VI.).

Hence each point P_{n+1} of the sequence lies inside the circle whose centre is at the preceding point P_n of the sequence and which touches both C_0 and C_1 internally.

It may happen that this circle will contain not only P_{n+1}, but also a finite number of the following points of the sequence.

Now, let x_2 start from the vertex A (see Fig. 2) of the hyperbolic arc, and move along that arc up to the last point (say, P_3) of the sequence inside the circle ED whose centre is A and which touches C_0 and C_1 both internally. For each position of x_2 on the hyperbolic arc, let the circle be described whose centre is at x_2 and which touches both C_0 and C_1 internally. Let the circle corresponding to the position of x_2 at P_3 touch C_0 at K and C_1 at L. Then the part of the envelope of the circles described which is above SH is bounded by the arc EL of C_1, the arc LK, the arc KD of C_0, and the straight line DE. The relation (I.) is valid throughout this area.

To extend the area further, the last point of the sequence inside the circle KL is taken. With it as centre, a circle (not shown in the figure) is described to touch the circles C_0 and C_1 both internally at K', L', suppose. Then, reasoning as before, the relation (I.) is valid in the area bounded by the arcs KL, LL', $L'K'$, $K'K$; and so on. Here it is to be noted that the inclusion of the whole of the space between C_0 and C_1 in the region for which (I.) is valid requires the consideration of an *infinite* number of positions of x_2.

9. It is interesting to examine the limiting values of the ratios of the radius of the circle C_{n0} to P_nP and to P_nP_{n+1} respectively as P_n approaches P.

$$\text{Now } \frac{\text{radius of } C_{n0}}{P_nP} = \frac{ae\,(\sec\gamma - \sec\theta_n)}{\sqrt{(a\sec\gamma - a\sec\theta_n)^2 + (b\tan\gamma - b\tan\theta_n)^2}}$$

$$= \frac{e}{\sqrt{1 + \frac{b^2}{a^2}\left(\frac{\sin(\gamma-\theta_n)}{\cos\theta_n - \cos\gamma}\right)^2}}$$

$$= \frac{e}{\sqrt{1 + \frac{b^2}{a^2}\left(\frac{\cos\frac{1}{2}(\gamma-\theta_n)}{\sin\frac{1}{2}(\gamma+\theta_n)}\right)^2}}.$$

When n becomes infinitely great, θ_n approaches the limit γ and the above ratio approaches the finite proper fraction

$$\sqrt{\frac{e^2 - e^2\cos^2\gamma}{e^2 - \cos^2\gamma}}.$$

The limiting value of

$$\frac{P_n P_{n+1}}{\text{radius of } C_{n0}} = \mathrm{L}_{n=\infty} \frac{2 \sin \frac{1}{2}(\theta_{n+1}-\theta_n)}{\cos\theta_n \cos\theta_{n+1}(\sec\gamma-\sec\theta_n)} \times \sqrt{\frac{a^2 \sin^2 \frac{1}{2}(\theta_{n+1}+\theta_n)+b^2\cos^2\frac{1}{2}(\theta_{n+1}-\theta_n)}{a^2+b^2}}$$

$$= \cos(\gamma-\tfrac{1}{2}\theta_1)\left(\frac{\sin\frac{1}{2}\theta_1}{\frac{1}{2}\sin^2\gamma}\right)\sqrt{\frac{a^2\sin^2\gamma+b^2}{a^2+b^2}}, \text{ by (VI.),}$$

which is a finite positive proper fraction, since

$$\sin\tfrac{1}{2}\theta_1 < \tfrac{1}{2}\sin\gamma \sin\tfrac{1}{2}\gamma < \tfrac{1}{2}\sin^2\gamma.$$

10. *Description of Fig.* 2.

The figure is drawn to one half of the dimensions given below.

S is the point x_0.

The radius of convergence of the series $P(x-x_0)$ is $SD = 6'$.

H is the point x_1, where $SH = 5'$.

The radius of convergence of $P(x-x_0 \mid x_1)$ is $4'$.

P is a point of intersection of the circles C_0, C_1.

AP is an arc of the hyperbola whose foci are S and H, which passes through P; A being the vertex of this hyperbolic arc.

C is the centre of the hyperbola.

$CA = 1'$.

If a be the semi-transverse axis, b the semi-conjugate axis, and e the eccentricity of the hyperbola, then

$$2ae = 5',$$

$$a = 1';$$

therefore

$$e = 2{\cdot}5,$$

$$b = a\sqrt{e^2-1} = a\frac{\sqrt{21}}{2} = 2{\cdot}29'.$$

The value of the parameter belonging to P is $60°$; so that $\gamma = 60°$.

The condition $\sin\frac{1}{2}\theta_1 < \frac{1}{2}\sin\gamma\sin\frac{1}{2}\gamma$ is satisfied by taking $\theta_1 = 24°$.

Then

$$\theta_2 = 36°\,29',$$

$$\theta_3 = 44°\,18',$$

$$\theta_4 = 49°\,25',$$

$$\theta_5 = 52°\,50'.$$

These five values of the parameter correspond to the points

P_1, whose coordinates are 1·1, 1·02;
P_2 „ „ 1·24, 1·69;
P_3 „ „ 1·4, 2·24;
P_4 „ „ 1·54, 2·67;
P_5 „ „ 1·65, 3·02;

the coordinates of P are 2, 3·97.

Über den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck. *Von* David Hilbert in Göttingen.
Received August 22nd, 1902.

Unter der *axiomatischen* Erforschung einer mathematischen Wahrheit verstehe ich eine Untersuchung, welche nicht dahin zielt, im Zusammenhange mit jener Wahrheit neue oder allgemeinere Sätze zu entdecken, sondern die vielmehr die Stellung jenes Satzes innerhalb des Systems der bekannten Wahrheiten und ihren logischen Zusammenhang in der Weise klarzulegen sucht, dass sich sicher angeben lässt, welche Voraussetzungen zur Begründung jener Wahrheit notwendig und hinreichend sind.

So habe ich beispielsweise in meiner Festschrift *Grundlagen der Geometrie** die ebenen Schnittpunktsätze, nämlich den speciellen Pascalschen Satz für das Geradenpaar und den Desarguesschen Satz von den perspectiv liegenden Dreiecken einer axiomatischen Untersuchung unterworfen, und in gleicher Weise haben auf meine Anregung hin *M. Dehn*† den Satz von der Winkelsumme im Dreieck und *G. Hamel*‡ den Satz von der Geraden als der kürzesten Verbindung zwischen zwei Punkten behandelt.

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* Leipzig, 1899; vgl. auch die mit Zusätzen versehenen Übersetzungen ins Französische (*Annales de l'Ecole Normale*, 1900) und ins Englische (Chicago, 1902).

† *Mathematische Annalen*, Bd. LIII., 1900.

‡ Inaugural-Dissertation, Göttingen, 1901, und eine demnächst in den *Mathematischen Annalen* erscheinende Abhandlung.

Die vorliegende Note betrifft die Stellung des Satzes von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck in der ebenen Euklidischen Geometrie. Zur Erleichterung des Verständnisses stelle ich—im wesentlichen, wie in meinen *Grundlagen der Geometrie*—die Axiome der ebenen Euklidischen Geometrie zusammen wie folgt:

I. *Axiome der Verknüpfung.*

1. Zwei von einander verschiedene Punkte A und B bestimmen stets eine Gerade.

2. Irgend zwei von einander verschiedene Punkte einer Geraden bestimmen diese Gerade.

3. Auf jeder Geraden giebt es wenigstens zwei Punkte. Es giebt wenigstens drei Punkte, welche nicht in einer Geraden liegen.

II. *Axiome der Anordnung.*

1. Wenn A, B, C Punkte einer Geraden sind, und B zwischen A und C liegt, so liegt auch B zwischen C und A.

2. Wenn A und B zwei Punkte einer Geraden sind, so giebt es wenigstens einen Punkt C, der zwischen A und B liegt, und wenigstens einen Punkt D, so dass B zwischen A und D liegt.

3. Unter irgend drei Punkten einer Geraden giebt es stets einen und nur einen Punkt, der zwischen den beiden anderen liegt.*

Definition.—Die zwischen zwei Punkten A und B gelegenen Punkte heissen auch die Punkte der Strecke AB.

4. Es seien A, B, C drei nicht in gerader Linie gelegene Punkte und a eine Gerade, die keinen der Punkte A, B, C trifft; wenn dann diese Gerade durch einen Punkt der Strecke AB geht, so geht sie gewiss auch durch einen Punkt der Strecke BC oder der Strecke AC.

III. *Axiome der Congruenz.*

1. Wenn A, B zwei Punkte der Geraden a sind, und A' ein Punkt einer Geraden a' ist, so kann man auf einer gegebenen

* Dass das in meiner Festschrift *Grundlagen der Geometrie* mit II. 4 bezeichnete Axiom eine Folge der übrigen Axiome ist, hat neuerdings E. H. Moore, *Transactions of the American Mathematical Society*, 1902, gezeigt.

Hälfte der Geraden a' von A' aus stets einen und nur einen Punkt B' finden, so dass die Strecke AB (oder BA) der Strecke $A'B'$ (oder $B'A'$) congruent oder gleich ist; in Zeichen

$$AB \equiv A'B'.$$

Jede Strecke ist sich selbst congruent,

$$AB \equiv AB \equiv BA \equiv BA.$$

2. Wenn die Strecke AB sowohl der Strecke $A'B$ als auch der Strecke $A''B''$ congruent ist, so ist auch $A'B'$ der Strecke $A''B''$ congruent.

3. Es seien AB und BC zwei Strecken ohne gemeinsame Punkte auf a, und ferner $A'B'$ und $B'C'$ zwei Strecken ohne gemeinsame Punkte auf einer Geraden a'; wenn dann

$$AB \equiv A'B' \quad \text{und} \quad BC \equiv B'C',$$

so ist auch $$AC \equiv A'C'.$$

Definition.—Zwei von einem Punkte ausgehende Halbstrahlen nennen wir einen Winkel und bezeichnen ihn entweder mit $\angle (h, k)$ oder $\angle (k, h)$.

4. Es sei ein Winkel (h, k), eine Gerade a', und eine bestimmte Seite von a' gegeben. Es bedeute h' einen Halbstrahl der Geraden a', der vom Punkte O' ausgeht: dann giebt es einen, und nur einen Halbstrahl k', so dass

$$\angle (h, k) \equiv \angle (h', k')$$

ist, und zugleich alle inneren Punkte des Winkels (h', k') auf der gegebenen Seite von a' liegen. Jeder Winkel ist sich selbst congruent,

d.h. $$\angle (h, k) \equiv \angle (h, k) \equiv \angle (k, h).$$

5. Wenn ein Winkel (h, k) sowohl dem Winkel (h', k') als auch dem Winkel (h'', k'') congruent ist, so ist auch der Winkel (h', k') dem Winkel (h'', k'') congruent.

6*. Wenn für zwei Dreiecke ABC und $A'B'C'$ die Congruenzen

$$AB \equiv A'B', \quad AC \equiv A'C' \quad \text{und} \quad \angle BAC \equiv \angle B'A'C'$$

gelten, so gilt auch stets

$$\angle ABC \equiv \angle A'B'C' \quad \text{und} \quad \angle ACB \equiv \angle A'C'B'.$$

Es ist nun für die nachfolgende Untersuchung wesentlich, dem letzten Congruenzaxiom, nämlich dem Axiom III. 6* über die Dreieckscongruenz eine engere Fassung zu erteilen, indem wir die Aussage desselben nur für Dreiecke mit gleichem Umlaufsinn als gültig hinstellen. Um diesen einschränkenden Zusatz scharf zu formuliren, nehmen wir irgend eine durch zwei Punkte A, B bestimmte Gerade in der Ebene beliebig an, und bezeichnen eine der beiden Halbebenen, in die diese Gerade die Ebene teilt, als rechts von der Geraden AB in der Richtung von A nach B, und dieselbe Halbebene zugleich auch als links von BA in Richtung von B nach A; die andere Halbebene bezeichnen wir als links von der Geraden AB, und zugleich als rechts von der Geraden BA gelegen. Ist nun C irgend ein Punkt auf der rechten Halbebene von AB, so bezeichnen wir diejenige Halbebene von AC, auf welcher der Punkt B liegt als die linke Halbebene von AC. Auf diese Weise können wir durch analoge Festsetzungen schliesslich in eindeutig bestimmter Weise für jede Gerade angeben, welche Halbebene rechts oder links von dieser Geraden in gegebener Richtung gelegen ist. Zugleich wird von den Schenkeln irgend eines Winkels in eindeutig bestimmter Weise stets der eine als der rechte Schenkel und der andere als der linke Schenkel zu bezeichnen sein, nämlich so, dass der rechte Schenkel auf der rechten Halbebene von derjenigen Geraden liegt, die durch den anderen Schenkel nach Lage und Richtung bestimmt ist, während der linke Schenkel links von derjenigen Geraden liegt, die durch den ersteren Schenkel nach Lage und Richtung bestimmt ist.

Stellen wir nun die Aussage des Axioms III. 6* nur dann als gültig hin, wenn in den beiden Dreiecken die Seiten AB und $A'B'$, bez. AC und $A'C'$ zugleich die rechten bez. die linken Schenkel der Winkel BAC und $B'A'C'$ sind, so erhalten wir jenes Axiom in der folgenden engeren Fassung :—

6. Wenn für zwei Dreiecke ABC und $A'B'C'$ die Congruenzen

$$AB \equiv A'B', \quad AC \equiv A'C' \quad \text{und} \quad \angle BAC \equiv \angle B'A'C'$$

gelten, so gilt auch stets

$$\angle ABC \equiv \angle A'B'C' \quad \text{und} \quad \angle ACB \equiv A'C'B',$$

vorausgesetzt, dass AB und $A'B'$ die rechten Schenkel, AC und $A'C'$ die linken Schenkel der Winkel BAC resp. $B'A'C'$ sind.

Das letzte Axiom in seiner ursprünglichen weiteren Fassung III. 6*

hat sofort den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck zur Folge. Auch ist leicht zu erkennen, dass umgekehrt mit Hilfe dieses Satzes vom gleichschenkligen Dreieck aus jenem Congruenzaxiom in der engeren Fassung III. 6 und den vorangehenden Axiomen das Axiom III. 6* in der weiteren Fassung notwendig folgt.

Das engere Axiom III. 6 zusammen mit allen früheren Axiomen gestattet den Nachweis des Satzes von der Gleichheit der Scheitelwinkel und der Möglichkeit der Halbirung einer jeden Strecke, sowie den Beweis des Euklidischen Satzes, dass der Aussenwinkel eines Dreiecks stets grösser ist als jeder der beiden inneren Winkel.

IV. *Axiom der Parallelen* (Euklidisches Axiom).

Durch einen Punkt A ausserhalb einer Geraden a lässt sich nur *eine* Gerade ziehen, welche a nicht schneidet.

Der Satz von der Gleichheit der Wechselwinkel, und mithin auch der Satz, dass die Winkelsumme im Dreieck zwei Rechte beträgt, folgen aus dem Parallelenaxiom mit Hilfe der vorangehenden Axiome, auch wenn wir das letzte Congruenzaxiom nur in der engeren Fassung III. 6 zu Grunde legen.

V. *Axiome der Stetigkeit.*

1. *Axiom des Messens* (Archimedisches Axiom).

Es sei A_1 ein beliebiger Punkt auf einer Geraden zwischen den beliebig gegebenen Punkten A und B; man construire dann die Punkte $A_2, A_3, A_4, \ldots$, so dass A_1 zwischen A und A_2, ferner A_2 zwischen A_1 und A_3, ferner A_3 zwischen A_2 und A_4 u.s.w. liegt, und überdies die Strecken

$$AA_1,\ A_1A_2,\ A_2A_3,\ A_3A_4,\ \ldots$$

einander gleich sind: dann giebt es in der Reihe der Punkte $A_2, A_3, A_4, \ldots$ stets einen solchen Punkt A_n, dass B zwischen A und A_n liegt.

2. *Axiom der Nachbarschaft.*

Ist irgend eine Strecke AB vorgelegt, so giebt es stets ein Dreieck, in dessen Innerem keine zu AB congruente Strecke sich finden lässt.

Dieses Axiom der Nachbarschaft ist eine notwendige Folge aus dem Satze von der Gleichheit der Basiswinkel im gleichschenkligen

Dreiecke, wie man aus dem Umstande erkennt, dass wegen dieses Satzes vom gleichschenkligen Dreieck die Summe zweier Seiten in jedem Dreiecke grösser als die dritte Seite ausfällt.

Ich stelle nun folgende Behauptung auf:

Mit Benutzung der sämtlichen Axiome I.–V. ist es möglich, den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck zu beweisen, auch wenn wir das Axiom von der Dreieckscongruenz nur in der engeren Fassung III. 6 zu Grunde legen.

Um dies einzusehen, wählen wir zwei sich schneidende Gerade als Axen eines schiefwinkligen Coordinatensystems, und ordnen den Punkten dieser Geraden auf Grund der linearen Congruenzaxiome III. 1–3 und des Archimedischen Axioms V. 1 reelle Zahlen zu; die sämtlichen Punkte der Ebene sind dann durch Zahlenpaare darstellbar.

Das engere Congruenzaxiom III. 6 zusammen mit V. 1 gestattet leicht die Begründung der Proportionenlehre, und aus dieser folgt, dass die Geraden durch lineare Gleichungen darstellbar sind.

Fassen wir nun den Punkt mit den Coordinaten $x = 1$ und $y = s$ in's Auge, und betrachten diejenige Congruenz der Ebene mit sich im engeren Sinne, bei welcher der Coordinatenanfang sich selbst und die x-Achse der Verbindungsgerade des Coordinatenanfanges mit jenem Punkte 1, s entspricht, so erkennen wir leicht, dass diese Drehung durch ein Formelsystem von der Gestalt

$$x' = \alpha_s x + \beta_s y,$$

$$y' = \gamma_s x + \delta_s y$$

vermittelt wird, worin α_s, β_s, γ_s, δ_s Zahlen bedeuten, die durch s eindeutig bestimmt sind. Indem wir nun einer jeden Geraden durch den Coordinatenanfang ihren Treffpunkt $(1, t)$ mit der Geraden $x = 1$ zuordnen, erhalten wir zu jeder solchen Drehung der Ebene um den Coordinatenanfang eine Transformation der Punkte der Geraden $x = 1$ von der Gestalt

$$t' = \frac{\gamma_s + \delta_s t}{\alpha_s + \beta_s t}.$$

Die Coefficienten α_s, β_s, γ_s, δ_s definiren Functionen von s, die sicher für alle rationalen Werte von s definirt sind; man sieht dann leicht ein, dass diese Functionen sich nach dem Princip der Stetigkeit so ergänzen lassen müssen, dass man aus ihnen stetige und für alle reellen Werte von s, sowie für $s = \infty$ eindeutig definirte Functionen erhält. Wenn das geschehen ist, so liefert die Formel

$$t' = \frac{\gamma_s + \delta_s t}{\alpha_s + \beta_s t}$$

eine Gruppe von linearen Transformationen, die gewiss folgende Eigenschaften besitzt:—

1. Es giebt ausser der Identität keine Transformation der Gruppe, welche einen Wert der Variabeln t festlässt.

2. Es giebt stets eine Transformation der Gruppe, welche einen gegebenen Wert von t in irgend einen anderen bestimmten Wert dieser Variabeln überführt.

In meiner Abhandlung über die Grundlagen der Geometrie* (§ 18) habe ich bewiesen, dass eine Gruppe mit diesen Eigenschaften notwendig holoedrisch isomorph ist mit der Gruppe der gewöhnlichen Drehungen eines Kreises in sich, und folglich auch mit derjenigen Transformationsgruppe, die durch die Formel

$$T' = \frac{ST-1}{T+S}$$

dargestellt wird, wo T, T' die Variabeln und S den Parameter bedeutet. Hieraus können wir nun entnehmen, dass t eine linear gebrochene Funktion von T ist; es sei etwa

$$t = \frac{AT+B}{CT+D},$$

wobei A, B, C, D Constante bedeuten. Führen wir daher mittelst der Formeln

$$x = AX+BY,$$

$$y = CX+DY$$

statt der Coordinaten x, y die Coordinaten X, Y ein, so entspricht einer bestimmten Drehung um den Coordinatenanfang die Formel

$$\frac{Y'}{X'} = \frac{X+SY}{SX-Y},$$

und hieraus entnehmen wir die Drehungsformeln

$$X' = C_S(SX-Y),$$

$$Y' = C_S(X+SY),$$

wobei C_S eine von S abhängige Grösse bezeichnet.

Unter Zuhülfenahme des Axioms der Nachbarschaft V. 2 folgt, dass die Determinante dieser eine Drehung vermittelnden Trans-

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* *Mathematische Annalen*, 1903.

formationsformeln notwendig $= 1$ sein muss; mithin ist

$$C_s = \frac{1}{\sqrt{1+S^2}},$$

und die Gruppe der Drehungen stellt sich daher in den Coordinaten X, Y, wie folgt, dar:

$$X' = \frac{S}{\sqrt{1+S^2}} X - \frac{1}{\sqrt{1+S^2}} Y,$$

$$Y' = \frac{1}{\sqrt{1+S^2}} X + \frac{S}{\sqrt{1+S^2}} Y.$$

Aus diesen Formeln entnehmen wir die Gültigkeit aller Thatsachen der gewöhnlichen Euklidischen Geometrie und insbesondere den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck.

Wir haben im Vorstehenden kurz den Beweis dafür angedeutet, dass das Axiom von der Dreieckscongruenz im weiteren Sinne III. 6* eine Folge dieses Axioms im engeren Sinne III. 6 ist, wenn man die Stetigkeitsaxiome V. 1 und V. 2, zu Hülfe nimmt.

Nunmehr entsteht die Frage, *ob auch ohne die Axiome der Stetigkeit V. 1 und V. 2 die weitere Fassung des Axioms von der Dreieckscongruenz sich als eine notwendige Folge der engeren Fassung desselben ergiebt. Die nachfolgende Untersuchung wird zeigen, dass dies nicht der Fall ist, selbst dann nicht, wenn man noch die Proportionenlehre als gültig voraussetzt. Die Geometrie, welche ich zu diesem Zwecke im Folgenden construire, beantwortet nicht nur die eben aufgeworfene Frage, sondern verbreitet überhaupt, wie ich glaube, über den logischen Zusammenhang des Satzes vom gleichschenkligen Dreieck mit den anderen in Betracht kommenden elementaren Sätzen der ebenen Geometrie neues Licht.*

Es sei t ein Parameter und α irgend ein Ausdruck mit einer endlichen oder unendlichen Gliederzahl von der Gestalt

$$\alpha = a_0 t^n + a_1 t^{n+1} + a_2 t^{n+2} + \ldots;$$

darin mögen $a_0 (\neq 0)$, a_1, a_2, ... beliebige reelle Zahlen bedeuten und n sei eine beliebige ganze rationale Zahl $\left(\begin{matrix}\geq \\ = \\ <\end{matrix} 0\right)$. Die Gesammtheit aller Ausdrücke von dieser Gestalt α sehen wir als ein complexes Zahlensystem T an, indem wir folgende Festsetzungen treffen. Man addire, subtrahire, multiplicire, dividire irgend welche Zahlen des Systems T, als warën sie gewöhnliche Potenzreihen, die nach steigenden Potenzen der Variabeln t fortschreiten. Die ent-

stehenden Summen, Differenzen, Produkte und Quotienten sind dann wiederum Ausdrücke von der Gestalt α, und mithin Zahlen des complexen Zahlensystems T. Eine Zahl α in T heisse $<$ oder >0, je nachdem in dem betreffenden Ausdrucke α der erste Coefficient $a_0 <$ oder $>$ als 0 ausfällt. Sind irgend zwei Zahlen α, β des complexen Systems T vorgelegt, so heisse $\alpha < \beta$ bez. $\alpha > \beta$, je nachdem $\alpha - \beta < 0$ oder > 0 ausfällt. Es leuchtet ein, dass bei diesen Festsetzungen alle formalen Regeln und Gesetze, wie bei den gewöhnlichen reellen Zahlen, giltig sind; dagegen gilt für unser System T das Archimedische Axiom nicht, da ja wie gross auch die positive reelle Zahl A gewählt sei, stets $At < 1$ bleibt; unser complexes Zahlensystem T ist ein Nicht-Archimedisches System.

Sind α, β irgend zwei Zahlen des Systems T, und bedeutet i die imaginäre Einheit $\sqrt{-1}$, so heisse

$$\gamma = \alpha + i\beta$$

eine imaginäre Zahl zum complexen System T.

Ist ferner τ ein Ausdruck von der Gestalt

$$\tau = a_0 t^n + a_1 t^{n+1} + a_2 t^{n+2} + \ldots,$$

wo $a_0 (\neq 0)$, a_1, a_2, ... reelle Zahlen bedeuten und der Exponent n der niedrigsten Potenz von t positiv ausfällt, so heisse τ *eine unendlich kleine Zahl des complexen Systems T.*

Offenbar lässt sich stets die unendliche Reihe

$$1 + \frac{(1+i)\tau}{1!} + \frac{(1+i)^2\tau^2}{2!} + \frac{(1+i)^3\tau^3}{3!} + \ldots$$

nach steigenden Potenzen von t ordnen, und stellt dann eine imaginäre Zahl zum complexen System T dar; wir bezeichnen diese imaginäre Zahl mit $e^{(1+i)\tau}$. Ist dann noch ϑ irgend eine gewöhnliche reelle Zahl, so setzen wir

$$e^{i\vartheta} \cdot e^{(1+i)\tau} = e^{i\vartheta + (1+i)\tau}.$$

Nunmehr construiren wir mit Hülfe des complexen Zahlensystems T eine ebene Geometrie, wie folgt:

Wir denken uns ein Paar von Zahlen (x, y) des Systems T als einen Punkt und die Verhältnisse von irgend drei Zahlen $(u : v : w)$ aus T, falls u, v nicht beide 0 sind, als eine Gerade; ferner möge das Bestehen der Gleichung

$$ux + vy + w = 0$$

ausdrücken, dass der Punkt (x, y) auf der Geraden $(u : v : w)$ liegt.

Berücksichtigen wir die obige Festsetzung, derzufolge die Zahlen in T sich ihrer Grösse nach anordnen lassen, so können wir leicht solche Festsetzungen für unsere Punkte und Geraden treffen, dass auch die Axiome II. der Anordnung sämmtlich erfüllt sind. In der That, stellen (x_1, y_1), (x_2, y_2), (x_3, y_3), ... irgend welche Punkte auf einer Geraden dar, so möge dies ihre Reihenfolge auf der Geraden sein, wenn die Zahlen $x_1, x_2, x_3, \ldots$ oder $y_1, y_2, y_3, \ldots$ in dieser Reihenfolge entweder beständig abnehmen oder wachsen; um ferner die Forderung des letzten Axioms der Anordnung II. 4 zu erfüllen, haben wir nur nötig festzusetzen, dass alle Punkte (x, y), für die $ux+vy+w$ kleiner oder grösser als 0 ausfällt, auf der einen bez. auf der anderen Seite der Geraden $(u : v : w)$ gelegen sein sollen.

Wenn zwei Punkte A, B beide zugleich entweder auf der x-Axe oder auf der y-Axe liegen, so nennen wir die absolute Differenz der Abzissen bez. der Ordinaten dieser beiden Punkte ihre Entfernung oder auch die Länge der durch sie bestimmten Strecke. Nunmehr definiren wir das Abtragen von Strecken und Winkeln durch die Parallelverschiebungen und Drehungen der Ebene in sich, wie folgt. Die Parallelverschiebung wird durch ein Transformation von der Gestalt

$$x' = x+\alpha,$$

$$y' = y+\beta$$

oder

$$x'+iy' = x+iy+\alpha+i\beta$$

vermittelt, worin α, β Zahlen in T, nämlich die Projektionen der Parallelverschiebung in Richtung der Axen sind. Die Drehung der Ebene um den Coordinatenanfangspunkt (0, 0) werde durch die Formel

$$x'+iy' = e^{i\vartheta+(1+i)\tau}(x+iy)$$

vermittelt, worin ϑ irgend eine reelle Zahl bedeutet, und τ irgend eine unendlich kleine Zahl im complexen System T bedeutet; $\vartheta+\tau$ heisse der Drehungswinkel dieser Drehung.

Zunächst ist klar, dass bei allen Verschiebungen und Drehungen der Ebene jede Gerade wiederum in eine Gerade übergeht, und zugleich die Anordnung der Punkte auf einer Geraden sowie ihre Lage zu den Geraden ebenfalls unverändert bleibt.

Es seien nun (x, y) und (x', y') irgend zwei vom Coordinatenanfangspunkt verschiedene Punkte. Die Punkte der Verbindungsgeraden von (x', y') mit dem Coordinatenanfangspunkt werden durch die Coordinaten $\mu x'$, $\mu y'$ dargestellt, worin μ irgend eine Zahl des complexen Systems T bedeutet. Wir wollen nun zeigen, dass es

stets eine Drehung um den Coordinatenanfangspunkt giebt, durch welche der Punkt (x, y) in einen Punkt jener Verbindungsgeraden übergeht. Zu dem Zwecke untersuchen wir die Gleichung

$$e^{i\vartheta+(1+i)\tau}(x+iy) = \mu(x'+iy').$$

Durch Division mit der conjugirt imaginären Gleichung folgt

$$e^{2i(\vartheta+\tau)}\frac{x+iy}{x-iy} = \frac{x'+iy'}{x'-iy'},$$

oder

$$e^{2i(\vartheta+\tau)} = \xi+i\eta,$$

worin

$$\xi+i\eta = \frac{x-iy}{x+iy}\,\frac{x'+iy'}{x'-iy'}$$

gesetzt ist. Wegen

$$\xi^2+\eta^2 = 1$$

folgt, dass ξ, η solche Zahlen in T sind, bei deren Darstellung durch t die niedrigsten Exponenten der Potenzen von t gewiss ≥ 0 sind, und mindestens einer derselben $= 0$ ausfällt. Wir können mithin setzen

$$\xi = a+\xi',$$

$$\eta = b+\eta',$$

worin a, b gewöhnliche reelle Zahlen mit der Quadratsumme 1 und ξ', η' unendlich kleine Zahlen in T bedeuten.

Die reelle Zahl ϑ ist durch die Gleichung

$$e^{2i\vartheta} = a+ib$$

bis auf ganze Vielfache von π, und ferner τ als unendlich kleine Zahl in T durch die Gleichung

$$e^{2i\tau} = 1+\frac{\xi'+i\eta'}{a+ib}$$

völlig eindeutig bestimmt. Nach Einsetzung von ϑ und τ in die ursprüngliche Gleichung folgt μ eindeutig als eine Zahl in T. Wir sehen daraus, dass es stets eine Drehung von der gewünschten Beschaffenheit giebt, und dass diese Drehung bis auf Drehungen um den Winkel π eindeutig bestimmt ist.

Nunmehr nennen wir irgend welche Strecken und Winkel einander congruent oder gleich, sobald es gelingt, dieselben durch Verschiebungen und Drehungen der Ebene miteinander zur Deckung zu bringen. Unsere obige Entwickelung zeigt dann, dass die sämmtlichen Axiome III. erfüllt sind, wobei das letzte Congruenzaxiom im engeren Sinne III. 6 zu verstehen ist. Auch gilt wie man sieht das Parallelenaxiom IV., und aus der Formel für die Drehung kann

ferner leicht ersehen werden, dass auch das Axiom V. 2 der Nachbarschaft in unserer Geometrie gültig ist.

Wir fassen dies wie folgt zusammen:

In unserer Geometrie gelten die sämmtlichen oben aufgestellten Axiome der gewöhnlichen ebenen Geometrie mit Ausnahme des Archimedischen Axioms V. 1, so jedoch, dass das Axiom über die Dreieckscongruenz dabei in der engeren Fassung III. 6 zu verstehen ist.

Wir leiten noch einige weitere Sätze ab, die in unserer Geometrie gültig sind:

In unserer Geometrie giebt es einen rechten Winkel, und jeder beliebige Winkel ist halbirbar.

In der That, $\frac{\pi}{2}$ ist gewiss ein rechter Winkel, weil derselbe bei einer Drehung um $\frac{\pi}{2}$ in seinen Nebenwinkel übergeht. Ebenso leuchtet ein, dass, wenn $\vartheta+\tau$ einen beliebigen Winkel bedeutet, $\frac{\vartheta}{2}+\frac{\tau}{2}$ die Hälfte desselben ist.

Wir führen nun den Begriff der Spiegelung an einer Geraden a wie folgt ein. Fällen wir von irgend einem Punkte A auf irgend eine Gerade a das Loth und verlängern dieses um sich selbst über den Fusspunkt B hinaus bis A', so heisse A' der Spiegelpunkt von A. Wir wählen zunächst zur Geraden a die x-Axe und für A einen Punkt im positiven Quadranten mit den Coordinaten α, β. Der Winkel AOB zwischen dem Halbstrahl OA und der positiven x-Axe sei $\vartheta+\tau$, und zwar möge der Punkt $x=\gamma$ auf der x-Axe bei der Drehung um den Winkel $\vartheta+\tau$ in den Punkt A übergehen, so dass

$$e^{i\vartheta+(1+i)\tau}\gamma=\alpha+i\beta$$

wird. Der Spiegelpunkt A' des Punktes A in Bezug auf die x-Axe hat die Coordinaten α, $-\beta$. Führen wir mithin die Drehung um den Winkel $\vartheta+\tau$ aus, so entsteht aus dem Punkte

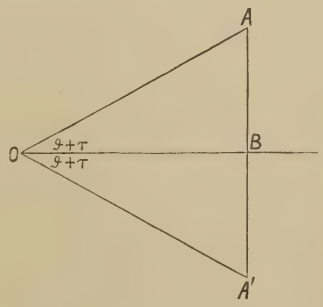

Fig. 1.

A' ein Punkt der durch die imaginäre Zahl

$$e^{i\vartheta+(1+i)\tau}(\alpha-i\beta) = \frac{\alpha+i\beta}{\gamma}(\alpha-i\beta) = \frac{\alpha^2+\beta^2}{\gamma}$$

dargestellt wird, d. h. der entstehende Punkt liegt auf der x-Axe; folglich ist der Winkel $A'OB$ ebenfalls gleich $\vartheta+\tau$, und stimmt mithin mit dem Winkel AOB überein. Wir sprechen dies Resultat wie folgt aus:

Wenn in zwei symmetrisch liegenden rechtwinkligen Dreiecken die beiden Katheten übereinstimmen, so sind auch die entsprechenden Winkel an der Hypotenuse einander gleich.

Wir folgern hieraus zugleich den allgemeineren Satz:

Im Spiegelbilde einer Figur stimmen die Winkel stets mit den entsprechenden Winkeln der ursprünglichen Figur überein.

Es sei a irgend eine Gerade durch den Anfangspunkt O, und $\vartheta+\tau$ sei der Winkel zwischen a und der x-Axe: dann erhält man zu dem beliebigen Punkt (x, y) den Spiegelpunkt in Bezug auf die Gerade a, indem man die Ebene zunächst um den Winkel $-\vartheta-\tau$ dreht, sodann an der x-Axe spiegelt, und schliesslich die Ebene um den Winkel $\vartheta+\tau$ dreht. Die Ausführung dieser Operationen liefert der Reihe nach aus der imaginären Zahl $x+iy$ die Zahlen

$$e^{-i\vartheta-(1+i)\tau}(x+iy), \quad e^{i\vartheta-(1-i)\tau}(x-iy),$$

$$e^{i\vartheta+(1+i)\tau}e^{i\vartheta-(1-i)\tau}(x-iy) = e^{2i(\vartheta+\tau)}(x-iy);$$

mithin wird die Spiegelung an der Geraden a durch die Formel

$$x'+iy' = e^{2i(\vartheta+\tau)}(x-iy)$$

vermittelt.

Nun mögen a_1, a_2 ebenfalls zwei Gerade durch den Anfangspunkt O und $\vartheta_1+\tau_1$ bez. $\vartheta_2+\tau_2$ die Winkel bedeuten, die diese Gerade mit der x-Axe einschliessen. Führen wir der Reihe nach die Spiegelungen an den Geraden a, a_1, a_2 aus, so wird die entsprechende Transformation des Punktes (x, y) durch die Formel

$$x'+iy' = e^{2i(+\vartheta-\vartheta_1+\vartheta_2+\tau-\tau_1+\tau_2)}(x-iy)$$

vermittelt, und hieraus entnehmen wir das folgende Resultat:

Wenn man an irgend drei durch einen Punkt gehenden Geraden eine Spiegelung ausführt, so ist die dann entstehende Transformation der Ebene wiederum eine Spiegelung.

Aus diesem Satze beweisen wir leicht die weiteren Sätze:

Die Mittelsenkrechten auf den drei Seiten eines Dreieckes treffen sich in einem Punkt.

Wenn in einem Vierecke $ABCD$ ohne einspringende Ecken die Winkel ACB und ADB über einstimmen, so sind auch die Winkel BAC und BDC einander gleich.

Die genannten Sätze genügen, um die Proportionenlehre ohne Hülfe des Satzes vom gleichschenkligen Dreiecke zu begründen: man hat nur nötig, den in meiner Festschrift *Grundlagen der Geometrie* (§§ 14–16) dargelegten Beweis in geeigneter Weise abzuändern, was leicht geschehen kann. Wir erkennen hieraus, dass es zur Begründung der Proportionenlehre in der ebenen Euklidischen Geometrie nicht des vollständigen Axioms von der Dreieckscongruenz im weiteren Sinne III. 6* bedarf, sondern dass dazu diejenigen Bestandteile dieses Axioms hinreichen, die in den eben genannten Sätzen enthalten sind.

Der Fundamentalsatz der Proportionenlehre lautet:

Schneiden zwei Parallele auf den Schenkeln eines beliebigen Winkels die Strecken a, b bez. a', b', ab, so gilt die Proportion

$$a : b = a' : b'.$$

Umgekehrt, wenn vier Strecken a, b, a', b' diese Proportion erfüllen, und a, a' und b, b' je auf einem Schenkel eines beliebigen Winkels abgetragen werden, so sind die Verbindungsgeraden der Endpunkte von a, b bez. von a', b' einander parallel.

Dieser Satz konnte auch direkt aus dem Umstande abgeleitet werden, dass in unserer Geometrie die Geraden durch lineare Gleichungen definirt sind, und zugleich erkennen wir hieraus diese Thatsachen:

In unserer Geometrie gelten die Schnittpunktsätze von Pascal und Desargues, und daher überhaupt alle Sätze der projektiven Geometrie.

Wir kommen nun zu der wesentlichsten Frage, *ob in unserer Geometrie der Satz von der Gleichheit der Basiswinkel des gleichschenkligen Dreieckes gilt.*

Zum Zwecke dieser Untersuchung sei a eine Gerade durch den Coordinatenanfang O, die mit der x-Axe den Winkel $\vartheta + \tau$ einschliesst; ferner sei A ein Punkt auf a so, dass die Strecke $OA = 1$ ausfällt, und A' sei der Spiegelpunkt von A in Bezug auf die

x-Axe (vgl. Figur 2). Um die Coordinaten (x, y) des Punktes A zu berechnen, bedenken wir, dass die Drehung um den Winkel $-\vartheta-\tau$ den Punkt A in den Punkt mit den Coordinaten 1, 0 überführen muss; wir haben mithin

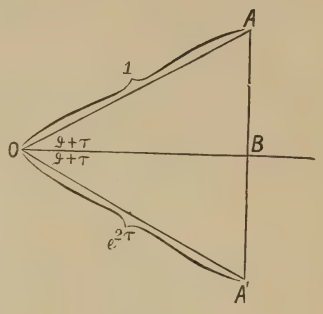

Fig. 2.

$$e^{-i\vartheta-(1+i)\tau}(x+iy) = 1,$$

und folglich

$$x+iy = e^{i\vartheta+(1+i)\tau},$$

d.h. $x = OB = e^{\tau}\cos(\vartheta+\tau)$,

$$y = AB = A'B = e^{\tau}\sin(\vartheta+\tau).$$

Um die Länge der Strecke OA' zu berechnen, bedenken wir, dass A' die Coordinaten x, $-y$ besitzt; die Drehung um den Winkel $\vartheta+\tau$ liefert mithin

$$e^{i\vartheta+(1+i)\tau}(x-iy) = e^{i\vartheta+(1+i)\tau-i\vartheta+(1-i)\tau} = e^{2\tau},$$

d.h. die Strecke OA' fällt gleich $e^{2\tau}$ aus, und sie unterscheidet sich mithin von der Strecke 1 um eine unendlich kleine von 0 verschiedene complexe Zahl.

Wir ersehen hieraus, dass im allgemeinen in zwei symmetrisch liegenden rechtwinkligen Dreiecken mit übereinstimmenden Katheten die Hypotenusen von einander verschieden ausfallen, und mithin bei der Spiegelung an einer Geraden die Strecken im Spiegelbilde denjenigen in der ursprünglichen Figur *nicht* notwendig gleich sind.

In unserer Geometrie gilt nicht der Satz vom gleichschenkligen Dreieck, und also auch nicht das Axiom von der Congruenz der Dreiecke im weiteren Sinne.

Wir erörtern noch in unserer Geometrie die Euklidische Lehre vom Flächeninhalt der Polygone und den Satz von der Geraden als der kürzesten Verbindung von zwei Punkten.

Was die erstere Frage betrifft, so mögen zwei Polygone *flächengleich* heissen, wenn sie in eine endliche Anzahl von Dreiecken zerlegt werden können, die paarweise einander congruent sind, und zwei Polygone mögen *inhaltsgleich* heissen, wenn es möglich ist, zu denselben flächengleiche Polygone zuzufügen, so dass die beiden zusammengesetzten Polygone einander flächengleich sind.

Aus diesen Definitionen wird leicht die Inhaltsgleichheit zweier Dreiecke mit gleicher Grundlinie und gleicher Höhe bewiesen.

Auch gilt in unserer Geometrie der Pythagoräische Lehrsatz, demzufolge die beiden Quadrate über den Katheten irgend eines rechtwinkligen Dreieckes zusammen inhaltsgleich dem Quadrate über der Hypotenuse sind. Denn wir erkennen, dass in dem Euklidischen Beweise des Pythagoräischen Lehrsatzes durchweg nur die Congruenz von gleichliegenden Dreiecken, und mithin nur das Axiom über die Dreieckscongruenz im engeren Sinne benutzt wird.

Wenden wir den Pythagoräischen Lehrsatz auf die beiden vorhin construirten symmetrisch liegenden rechtwinkligen Dreiecke OAB und $OA'B$ (Figur 2) an, so folgt, dass die Quadrate über den beiden Hypotenusen $OA = 1$ und $OA' = e^{2\tau}$ einander inhaltsgleich sind, d.h. es giebt in unserer Geometrie Quadrate, deren Seiten sich um unendlich kleine von 0 verschiedene Zahlen unterscheiden und die dennoch einander inhaltsgleich sind.

Infolge dieses Umstandes hat in unserer Geometrie der fundamentale Satz Euklids, wonach zwei inhaltsgleiche Dreiecke mit gleicher Grundlinie stets von gleicher Höhe sind, ebenfalls keine Gültigkeit.

Unsere Geometrie führt uns mithin zu folgender Erkenntnis:

Es ist unmöglich auf das Axiom der Dreieckscongruenz im engeren Sinne die Euklidische Lehre vom Flächeninhalt zu begründen, selbst wenn man die Gültigkeit der Proportionenlehre hinzunimmt.

In meiner Festschrift *Grundlagen der Geometrie* habe ich den Satz Euklids von der Gleichheit der Höhen in inhaltsgleichen Dreiecken auf gleicher Grundlinie dadurch bewiesen, dass ich den Begriff des Flächenmasses einführte. Dieses Flächenmass habe ich als das halbe Produkt aus Grundlinie und Höhe definirt. Der Nachweis, dass diese Grösse davon unabhängig ist, welche Seite des Dreiecks man als Grundlinie wählt, erfordert in der That die Benutzung symmetrisch gelegener ähnlicher Dreiecke, und mithin das Axiom über die Dreieckscongruenz im weiteren Sinne.

Da in unserer Geometrie die bekannte Beziehung zwischen der Hypotenuse und den Katheten eines rechtwinkligen Dreiecks, welche in der gewöhnlichen Geometrie aus den Pythagoräischen Lehrsatz geschlossen wird, nicht gilt, so möchte ich unsere Geometrie eine *nicht-Pythagoräische Geometrie* nennen.

Eine weitere Folgerung, die Euklid aus dem Satze von der Gleich-

heit der Basiswinkel im gleichschenkligen Dreiecke zieht, ist der Satz, dass in jedem Dreiecke die Summe zweier Seiten grösser als die dritte ausfällt. Um die Gültigkeit dieses Satzes in unserer Geometrie zu prüfen, nehmen wir in dem rechtwinkligen Dreiecke $OA'B$ (Figur 2), $\vartheta = 0$, so dass die Hypotenuse OA' mit der Kathete OB den unendlich kleinen von 0 verschiedenen Winkel τ einschliesst. Wir erhalten dann die Summe der beiden Katheten durch die complexe Zahl

$$e^{\tau}(\cos\tau+\sin\tau) = \left(1+\tau+\frac{\tau^2}{2}+\ldots\right)\left(1-\frac{\tau^2}{2}+\ldots+\tau-\frac{\tau^3}{6}+\ldots\right)$$
$$= 1+2\tau+\tau^2+\ldots$$

ausgedrückt, und diese Zahl ist offenbar um eine unendlich kleine von 0 verschiedene Zahl kleiner als die Hypotenuse

$$OA' = e^{2\tau} = 1+2\tau+2\tau^2+\ldots .$$

Der Satz, wonach die Summe zweier Seiten in jedem Dreieck grösser als die dritte ist, gilt also nicht in unserer Geometrie.

Wir erkennen hieraus die wesentliche Abhängigkeit dieses Satzes von dem Axiom über die Dreieckscongruenz im weiteren Sinne.

Wir fassen die hauptsächlichsten aus unserer nicht-Pythagoräischen Geometrie entspringenden Resultate zusammen wie folgt:

Verstehen wir das Axiom über die Dreieckscongruenz im engeren Sinne und nehmen wir von den Stetigkeitsaxiomen nur das Axiom der Nachbarschaft als gültig an, dann ist der Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck nicht beweisbar, selbst dann nicht, wenn wir die Lehre von den Proportionen als gültig voraussetzen. Ebenso wenig folgt die Euklidische Lehre von den Flächeninhalten; auch der Satz, wonach die Summe zweier Seiten im Dreieck grösser als die dritte ausfällt, ist keine notwendige Folge der gemachten Annahmen.

Wir wollen noch eine andere nicht-Pythagoräische Geometrie construiren, die sich von der eben behandelten Geometrie dadurch unterscheidet, dass in ihr das Archimedische Axiom, dagegen nicht das Axiom von der Nachbarschaft gültig ist.

Man betrachte den Bereich Ω aller derjenigen algebraischen Zahlen, welche hervorgehen, indem man von der Zahl 1 ausgeht und eine endliche Anzahl von Malen die vier Rechnungsoperationen: Addition, Subtraction, Multiplikation, Division und die fünfte Operation $\sqrt{1+\omega^2}$ anwendet, wobei ω jedesmal eine Zahl bedeuten kann, die vermöge jener fünf Operationen bereits entstanden ist.

Wir denken uns ein Paar von Zahlen x, y aus Ω als einen Punkt und die Verhältnisse von irgend drei Zahlen $(u : v : w)$ aus Ω, falls

u, v nicht beide 0 sind, als eine Gerade; ferner möge das Bestehen der Gleichung

$$ux+vy+w=0$$

ausdrücken, dass der Punkt (x, y) auf der Geraden $(u : v : w)$ liegt.

Die Anordnung der Punkte und Geraden, sowie die Festsetzung über die Parallelverschiebung treffen wir genau wie vorhin, dagegen weichen wir von der früheren Festsetzung hinsichtlich der Drehung um einen Punkt in folgender Weise ab:

Wir fassen die Punkte der Geraden $x=1$ ins Auge; da dieselben ein abzählbares System bilden, so können wir sie in diese Reihe bringen: $P_1, P_2, P_3, \ldots$. Es sei dann allgemein ϑ_k der Winkel unter welchem die Verbindungslinie des Punktes P_k mit dem Coordinatenanfang die x-Axe schneidet. Wir wählen nun aus der Reihe $\vartheta_1, \vartheta_2, \vartheta_3, \ldots$ eine solche Reihe $\vartheta_{k_1}, \vartheta_{k_2}, \vartheta_{k_3}, \ldots$ aus, dass für keinen Wert von n eine Gleichung von der Gestalt

$$\vartheta_{k_n} = r\pi + r_1\vartheta_{k_1} + r_2\vartheta_{k_2} + \ldots + r_{n-1}\vartheta_{k_{n-1}}$$

besteht, wo $r, r_1, r_2, \ldots, r_{n-1}$ rationale Zahlen sind, während andrerseits für jedes ϑ_k sich ein Wert von n bestimmen lässt, so dass

$$\vartheta_k = r\pi + r_1\vartheta_{k_1} + r_2\vartheta_{k_2} + \ldots + r_n\vartheta_{k_n}$$

besteht, wobei $r, r_1, r_2, \ldots, r_n$ rationale Zahlen sind.

Die Drehung unserer Ebene um den Winkel ϑ_k werde dann durch die Formel

$$x'+iy' = e^{i\vartheta_k+r_1}(x+iy)$$

vermittelt.

Es stellt sich heraus, dass in dieser Geometrie sämtliche Axiome I. bis IV. erfüllt sind, wenn das Axiom über die Dreieckscongruenz im engeren Sinne III. 6 verstanden wird. Ausserdem gilt das Archimedische Axiom V. 1, dagegen nicht das Axiom der Nachbarschaft V. 2. Auch die oben genannten Sätze und die aus diesen zu ziehenden Folgerungen, insbesondere die Proportionenlehre ist gültig. Dagegen gilt nicht der Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck, und folglich auch nicht die Euklidische Lehre vom Flächeninhalt. Der Satz, dass die Summe zweier Seiten grösser als die dritte ist, gilt in unserer Geometrie ebenfalls nicht, da ja aus diesem Satze das Axiom der Nachbarschaft V. 2 notwendig folgen würde.

Zur Gültigkeit des Satzes von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck sind mithin beide Stetigkeitsaxiome V. 1, V. 2 erforderlich; in der That haben wir in dem früheren Beweise am Anfange dieser Untersuchung jedes dieser beiden Axiome benutzt.

Mit Hülfe der beiden Stetigkeitsaxiome V. 1 und V. 2 ist es noch nicht möglich, die ebene Geometrie als identisch mit der gewöhnlichen analytischen "Cartesischen" Geometrie nachzuweisen, in der *jedem* Zahlenpaar ein Punkt der Ebene entspricht. Zu diesem Nachweise ist aber weder das Axiom vom Dedekindschen Schnitte, noch das Axiom von der Existenz der Grenzpunkte notwendig; vielmehr genügt dasjenige Axiom allgemein philosophischen Charakters, welches ich bereits in der französischen Uebersetzung meiner *Grundlagen der Geometrie** aufgestellt habe, und welches wie folgt lautet:

V. 3. *Axiom der Vollständigkeit.*

Zu dem System der Punkte und Geraden ist es nicht möglich ein anderes System von Dingen hinzuzufügen, so dass in dem durch Zusammensetzung entstehenden System sämtliche aufgeführten Axiome I.–IV., V. 1, 2 erfüllt sind; oder kurz:

Die Elemente der Geometrie bilden ein System von Dingen, welches bei Aufrechterhaltung sämtlicher Axiome keiner Erweiterung mehr fähig ist.

On the Groups defined for an Arbitrary Field by the Multiplication Tables of certain Finite Groups. By L. E. Dickson. Received and read April 10th, 1902. Received, in modified form,† October 20th, 1902.

In his paper‡ "On the Continuous Group that is defined by any given Group of Finite Order," Burnside obtained by a new method the most fundamental ones of the theorems of Frobenius§ on group-matrices and group-determinants, and gave a new aspect to the theory. A more general theory for an arbitrary field (domain of rationality) was later developed by the writer.‖ An exceptional case

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* *Annales de l'Ecole Normale*, 1900. Vgl. auch meinen Vortrag über den Zahlbegriff: *Berichte der deutschen Mathematiker-Vereinigung*, 1900.

† The original manuscript was lost in transit. From rough notes on a part of it, the present paper has been prepared.

‡ *Proc. Lond. Math. Soc.*, Vol. xxix., pp. 207–224, 546–565.

§ *Berliner Sitzungsberichte*, 1896, pp. 985–1021, 1343–1382; 1897, pp. 994–1015; 1898, pp. 501–515; 1899, pp. 330–339, 482–500; 1900, pp. 516–534; 1901, pp. 303–315. An elementary exposition of Frobenius's theory is given by the writer in the *Annals of Mathematics*, October, 1902.

‖ *Trans. Amer. Math. Soc.*, Vol. iii. (1902), pp. 285–301. Additional developments are given in the *University of Chicago Decennial Publications*, Vol. ix., pp. 35–51 (separate preprints, pp. 1–17, Oct. 1, 1902).

not treated in the paper just cited is that of a field having a modulus which divides the order of the given finite group. As this exceptional case appears to offer special difficulties, it seems desirable to have a detailed treatment of a number of examples both for the general case and for the various exceptional cases. Such are the contents of §§ 1–7; the methods employed are very elementary and entirely independent of the general theory. The next step in the extension of the general theory is very naturally a detailed study of the explicit developments in the earlier investigations which cease to hold true for the above exceptional cases. The contents of §§ 8–12 are of this nature.

Canonical Forms of the Group-Matrix for the Symmetric Group g_6 on Three Letters.

1. The body of a left-hand multiplication table for g_6 is*

$$(1)\quad \begin{Bmatrix} I & \alpha & \beta & \gamma & \delta & \epsilon \\ \beta & I & \alpha & \delta & \epsilon & \gamma \\ \alpha & \beta & I & \epsilon & \gamma & \delta \\ \gamma & \delta & \epsilon & I & \alpha & \beta \\ \delta & \epsilon & \gamma & \beta & I & \alpha \\ \epsilon & \gamma & \delta & \alpha & \beta & I \end{Bmatrix}.$$

Let T_1 be a linear transformation on $\xi_1, \ldots, \xi_6$, the matrix of whose coefficients is of the form (1) with $I, \alpha, \ldots, \epsilon$ in a given field F.

Employing the standard notation, let $B_{i,j,\lambda}$ denote the linear transformation which alters only the variable ξ_i, replacing it by $\xi_i + \lambda\xi_j$.

The inverse of $B_{i,j,\lambda}$ is evidently $B_{i,j,-\lambda}$. Transforming T_1, of matrix (1), by

$$(t_1)\quad B_{2,1,-1}B_{3,1,-1}B_{4,1,-1}B_{5,1,-1}B_{6,1,-1},$$

we obtain a transformation T_2 with the matrix of coefficients

$$(2)\quad \begin{Bmatrix} \rho & \alpha & \beta & \gamma & \delta & \epsilon \\ 0 & I-\alpha & \alpha-\beta & \delta-\gamma & \epsilon-\delta & \gamma-\epsilon \\ 0 & \beta-\alpha & I-\beta & \epsilon-\gamma & \gamma-\delta & \delta-\epsilon \\ 0 & \delta-\alpha & \epsilon-\beta & I-\gamma & \alpha-\delta & \beta-\epsilon \\ 0 & \epsilon-\alpha & \gamma-\beta & \beta-\gamma & I-\delta & \alpha-\epsilon \\ 0 & \gamma-\alpha & \delta-\beta & \alpha-\gamma & \beta-\delta & I-\epsilon \end{Bmatrix},$$

* Weber, *Algebra*, second edition, Vol. II., p. 124.

where ρ and ρ_1, used below, have the values

$$\rho = I+\alpha+\beta+\gamma+\delta+\epsilon, \quad \rho_1 = I+\alpha+\beta-\gamma-\delta-\epsilon.$$

Transforming T_2, of matrix (2), by the product

$$(t_2) \quad B_{2,3,1}B_{2,4,-1}B_{2,5,-1}B_{2,6,-1},$$

we obtain a transformation T_3 with the matrix

$$(3) \quad \begin{Bmatrix} \rho & \alpha & \beta-\alpha & \gamma+\alpha & \delta+\alpha & \epsilon+\alpha \\ 0 & \rho_1 & 0 & 0 & 0 & 0 \\ 0 & \beta-\alpha & I-2\beta+\alpha & \beta+\epsilon-\alpha-\gamma & \beta+\gamma-\alpha-\delta & \beta+\delta-\alpha-\epsilon \\ 0 & \delta-\alpha & \alpha-\beta-\delta+\epsilon & I-\alpha-\gamma+\delta & 0 & \beta+\delta-\alpha-\epsilon \\ 0 & \epsilon-\alpha & \alpha-\beta+\gamma-\epsilon & \beta+\epsilon-\alpha-\gamma & I-\alpha-\delta+\epsilon & 0 \\ 0 & \gamma-\alpha & \alpha-\beta-\gamma+\delta & 0 & \beta+\gamma-\alpha-\delta & I-\alpha+\gamma-\epsilon \end{Bmatrix}.$$

Transforming T_3, of matrix (3), by the product

$$(t_3) \quad B_{3,4,-1}B_{6,3,-1},$$

we obtain a transformation T_4 with the matrix

$$(4) \quad \begin{Bmatrix} \rho & \alpha & \beta+\epsilon & \beta+\gamma & \delta+\alpha & \epsilon+\alpha \\ 0 & \rho_1 & 0 & 0 & 0 & 0 \\ 0 & \beta-\delta & I-\beta+\delta-\epsilon & 0 & \beta+\gamma-\alpha-\delta & 0 \\ 0 & \delta-\alpha & 0 & I+\epsilon-\beta-\gamma & 0 & \beta+\delta-\alpha-\epsilon \\ 0 & \epsilon-\alpha & \alpha+\gamma-\beta-\epsilon & 0 & I+\epsilon-\alpha-\delta & 0 \\ 0 & \gamma+\delta-\alpha-\beta & 0 & \alpha+\delta-\beta-\gamma & 0 & I+\gamma-\alpha-\epsilon \end{Bmatrix}.$$

If F does not have modulus 2 or 3, we transform T_4 by

$$(t_4) \quad \xi_1' = \xi_1+\tfrac{1}{6}\xi_2+\tfrac{1}{3}(\xi_3+\xi_4+\xi_5+\xi_6), \quad \xi_i' = \xi_i \ (i>1),$$

and obtain a transformation T_5 whose matrix differs from (4) only in the first row, that being now $\rho\ 0\ 0\ 0\ 0\ 0$. Transforming T_5 by $t_5 \equiv B_{4,6,-1}$, we obtain a transformation T_6 whose matrix differs from that of T_5 only in the fourth and sixth rows, those being now

$$0 \quad \beta-\gamma \quad 0 \quad I-\alpha-\delta-\epsilon \quad 0 \quad \beta+\epsilon-\alpha-\gamma,$$
$$0 \quad \gamma+\delta-\alpha-\beta \quad 0 \quad \alpha+\delta-\beta-\gamma \quad 0 \quad I+\delta-\beta-\epsilon.$$

Transforming T_6 by

$$(t_6) \quad \xi_3' = \xi_3-\tfrac{1}{3}\xi_2, \quad \xi_4' = \xi_4-\tfrac{1}{3}\xi_2, \quad \xi_5' = \xi_5+\tfrac{1}{3}\xi_2, \quad \xi_6' = \xi_6+\tfrac{2}{3}\xi_2,$$

we obtain a transformation T_7 whose matrix differs from that of T_6 only in the second column, that now having all zeros except ρ_1 in the second row. Transforming T_7 by

$$(t_7) \quad \xi_4' = -\xi_5, \quad \xi_5' = \xi_6, \quad \xi_6' = \xi_4,$$

we obtain a transformation with the matrix M:

$$M \equiv \begin{Vmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{Vmatrix} \quad \begin{matrix} (a = I-\beta+\delta-\epsilon, \\ b = \alpha-\beta+\delta-\gamma, \\ c = -\alpha+\beta-\gamma+\epsilon, \\ d = I-\alpha-\delta+\epsilon). \end{matrix}$$

The product $t = t_1 t_2 t_3 t_4 t_5 t_6 t_7$ is found to be

$$\xi_1' = \tfrac{1}{6}(\xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6), \qquad \xi_2' = \xi_1+\xi_2+\xi_3-\xi_4-\xi_5-\xi_6,$$
$$\xi_3' = -\tfrac{1}{3}(\xi_1+\xi_2-2\xi_3+2\xi_4-\xi_5-\xi_6), \quad \xi_4' = \tfrac{1}{3}(2\xi_1-\xi_2-\xi_3+\xi_4-2\xi_5+\xi_6),$$
$$\xi_5' = -\tfrac{1}{3}(\xi_1-2\xi_2+\xi_3-\xi_4+2\xi_5-\xi_6), \quad \xi_6' = -\tfrac{1}{3}(\xi_1+\xi_2-2\xi_3-\xi_4-\xi_5+2\xi_6).$$

This transformation* t therefore transforms T_1, of matrix (1), into a transformation of the canonical form M, the cases of moduli 2 and 3 being excluded. The six functions ρ, ρ_1, a, b, c, d are readily seen to be linearly independent. The group of transformations of matrix (1) is therefore simply isomorphic with the product of two general unary linear groups and the general binary linear group, all with coefficients in the given field F, and affecting different variables.

2. Let next F have modulus 2. Transforming T_2, of matrix (2), by

$$\xi_1' = \xi_1+\xi_4+\xi_5+\xi_6, \quad \xi_2' = \xi_2+\xi_3+\xi_4+\xi_5+\xi_6, \quad \xi_i' = \xi_i \ (i>2),$$

we obtain a transformation of matrix

$$\begin{Vmatrix} \rho & \gamma+\delta+\epsilon & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & \alpha+\beta & I+\alpha & \alpha+\beta+\gamma+\epsilon & \alpha+\beta+\gamma+\delta & \alpha+\beta+\delta+\epsilon \\ 0 & \alpha+\delta & \alpha+\beta+\delta+\epsilon & I+\alpha+\gamma+\delta & 0 & \alpha+\beta+\delta+\epsilon \\ 0 & \alpha+\epsilon & \alpha+\beta+\gamma+\epsilon & \alpha+\beta+\gamma+\epsilon & I+\alpha+\delta+\epsilon & 0 \\ 0 & \alpha+\gamma & \alpha+\beta+\gamma+\delta & 0 & \alpha+\beta+\gamma+\delta & I+\alpha+\gamma+\epsilon \end{Vmatrix}.$$

Transforming it by the product $B_{3,4,1} B_{6,3,1} B_{4,6,1}$, we obtain a trans-

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* The product of t on the right by

$$\xi_1' = 6\xi_1, \quad \xi_2' = \xi_2, \quad \xi_3' = -3\xi_3, \quad \xi_4' = -3\xi_4, \quad \xi_5' = -3\xi_5, \quad \xi_6' = -3\xi_6,$$

which transforms M into itself, gives a transformation with integral coefficients of determinant -2.3^5 which effects the reduction of T_1 to M. It is given, without proof, in the *Trans. Amer. Math. Soc.*, Vol. III., 1902, p. 296 (viz., matrix T).

formation with the matrix

$$(5)\qquad \begin{Vmatrix} \rho & \gamma+\delta+\epsilon & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & \beta+\delta & \alpha & 0 & b & 0 \\ 0 & \beta+\gamma & 0 & d & 0 & c \\ 0 & \alpha+\epsilon & c & 0 & d & 0 \\ 0 & \alpha+\beta+\gamma+\delta & 0 & b & 0 & \alpha \end{Vmatrix}.$$

Transforming it by t_6, which for modulus 2 becomes

$$\xi_3' = \xi_3+\xi_2, \quad \xi_4' = \xi_4+\xi_2, \quad \xi_5' = \xi_5+\xi_2,$$

we obtain a transformation whose matrix differs from (5) only in the second column, that being now $\gamma+\delta+\epsilon$, ρ, 0, 0, 0, 0. Finally, transforming by $(\xi_1\xi_2)(\xi_4\xi_5\xi_6)$, we obtain a transformation with the canonical matrix

$$M_2 \equiv \begin{Vmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ \gamma+\delta+\epsilon & \rho & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & b & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & b \\ 0 & 0 & 0 & 0 & c & d \end{Vmatrix}.$$

The six functions ρ, $\gamma+\delta+\epsilon$, α, b, c, d of I, α, β, γ, δ, ϵ are seen to be linearly independent modulo 2. The group is therefore simply isomorphic with the product of the binary group of transformations $\begin{pmatrix} \rho & 0 \\ \lambda & \rho \end{pmatrix}$ and the general binary linear group, the two affecting different variables.

3. Let, finally, F have modulus 3. For any modulus, the transformed of T_1, with matrix (1), by

$$\xi_1' = \xi_1+\xi_2+\xi_3+\xi_4+\xi_5+\xi_6, \quad \xi_2' = \xi_4+\xi_5+\xi_6, \quad \xi_3' = \xi_3, \quad \xi_4' = \xi_2,$$

$$\xi_5' = \xi_5, \quad \xi_6' = \xi_6,$$

has the matrix of coefficients

$$\begin{Vmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ \gamma+\delta+\epsilon & \rho_1 & 0 & 0 & 0 & 0 \\ \alpha & \epsilon-\alpha & I-\alpha & \beta-\alpha & \gamma-\epsilon & \delta-\epsilon \\ \beta & \delta-\beta & \alpha-\beta & I-\beta & \epsilon-\delta & \gamma-\delta \\ \delta & \beta-\delta & \gamma-\delta & \epsilon-\delta & I-\beta & \alpha-\beta \\ \epsilon & \alpha-\epsilon & \delta-\epsilon & \gamma-\epsilon & \beta-\alpha & I-\alpha \end{Vmatrix}.$$

Transforming it in succession by the three transformations (mod 3)

$$\xi_3' = \xi_3 - \xi_4 + \xi_5 - \xi_6, \quad \xi_4' = \xi_4 + \xi_5 + \xi_6, \quad \xi_5' = \xi_5 - \xi_6,$$

further simplifications arise, the final matrix being

$$\begin{Vmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ \gamma+\delta+\epsilon & \rho_1 & 0 & 0 & 0 & 0 \\ \tau & \tau & \rho & 0 & 0 & 0 \\ \beta+\delta+\epsilon & \alpha-\epsilon & \alpha-\beta+\gamma-\epsilon & \rho_1 & 0 & 0 \\ \delta-\epsilon & -\tau & \gamma+\delta+\epsilon & 0 & \rho_1 & 0 \\ \epsilon & \alpha-\epsilon & \delta-\epsilon & \gamma+\delta+\epsilon & \beta-\alpha-\gamma+\delta & \rho \end{Vmatrix},$$

where $\tau \equiv \alpha - \beta + \delta - \epsilon$. Subsequent simplifications arise by transformation by $B_{2,1,1}$, $B_{5,3,1}$, $B_{6,4,-1}$, $B_{4,5,1}$, $(\xi_2\xi_5\xi_3\xi_6\xi_4)$, $B_{3,2,1}$, $B_{6,5,1}$, in succession, the final matrix having the canonical form*

$$M_3 \equiv \begin{Vmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ \sigma & \rho_1 & 0 & 0 & 0 & 0 \\ \kappa & -\tau & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_1 & 0 & 0 \\ 0 & 0 & 0 & \tau & \rho & 0 \\ 0 & 0 & 0 & \nu & \sigma & \rho_1 \end{Vmatrix},$$

where $\sigma \equiv \alpha - \beta - \delta + \epsilon$, $\kappa \equiv \alpha + \beta + \delta + \epsilon$, $\nu = -\alpha - \beta + \delta + \epsilon$. The six functions ρ, ρ_1, σ, τ, κ, ν are readily seen to be linearly independent modulo 3. For the Galois field of order 3^n, the group is evidently of order $3^{4n}(3^n-1)^2$.

The transformations $[\rho, \rho_1, \sigma, \tau, \kappa, \nu]$, of matrix M_3, form a solvable group. Indeed, the transformations $[1, 1, \sigma, \tau, \kappa, \nu]$ form an invariant sub-group, the quotient-group being a commutative group [of order $(3^n-1)^2$ for the $GF[p^n]$]. The sub-group itself has an invariant sub-group formed by the commutative transformations $[1, 1, 0, \tau, \kappa, \nu]$. In fact,

$$[1, 1, 0, \tau, \kappa, \nu]\,[1, 1, 0, \tau', \kappa', \nu'] = [1, 1, 0, \tau+\tau', \kappa+\kappa', \nu+\nu'],$$

while $\begin{pmatrix} 1 & 0 & 0 \\ \sigma & 1 & 0 \\ \kappa & -\tau & 1 \end{pmatrix}$, whose inverse is $\begin{pmatrix} 1 & 0 & 0 \\ -\sigma & 1 & 0 \\ -\kappa-\tau\sigma & \tau & 1 \end{pmatrix}$, transforms a ternary transformation, leaving fixed the first two variables, into a transformation leaving them fixed; likewise $\begin{pmatrix} 1 & 0 & 0 \\ \tau & 1 & 0 \\ \nu & \sigma & 1 \end{pmatrix}$, whose inverse is $\begin{pmatrix} 1 & 0 & 0 \\ -\tau & 1 & 0 \\ -\nu+\sigma\tau & -\sigma & 1 \end{pmatrix}$, transforms $(\xi_1' = \xi_4, \ \xi_5' = \xi_5 + \lambda\xi_4, \ \xi_6' = \xi_6 + \mu\xi_4)$ into a similar transformation.

* As a check, note that the transformations of matrix M_3 form a group.

Canonical Matrices for the Cyclic G_3.

4. The body of a left-hand multiplication table for G_3 is

$$(6) \quad \begin{pmatrix} I & \alpha & \beta \\ \beta & I & \alpha \\ \alpha & \beta & I \end{pmatrix}.$$

Let ω be an imaginary cube root of unity and set

$$\delta_1 = I+\alpha+\beta, \quad \delta_2 = I+\omega\alpha+\omega^2\beta, \quad \delta_3 = I+\omega^2\alpha+\omega\beta,$$

$$\eta_1 = \xi_1+\xi_2+\xi_3, \quad \eta_2 = \xi_1+\omega^2\xi_2+\omega\xi_3, \quad \eta_3 = \xi_1+\omega\xi_2+\omega^2\xi_3.$$

If F does not have modulus 3, δ_1, δ_2, δ_3 are linearly independent with respect to F, and likewise η_1, η_2, η_3. Expressed in terms of the latter, the transformation on ξ_1, ξ_2, ξ_3, with matrix (6), takes the form

$$(7) \quad \eta_1' = \delta_1\eta_1, \quad \eta_2' = \delta_2\eta_2, \quad \eta_3' = \delta_3\eta_3.$$

Hence, if ω belongs to F, the group of transformations of matrix (6) is simply isomorphic with the direct product of three groups each a general unary linear group. If ω is not in F, it serves to extend* F to a field F_2. Then the conjugate of ω with respect to F is $\bar{\omega} = \omega^2$; so that $\eta_3 = \bar{\eta}_2$, $\delta_3 = \bar{\delta}_2$. The group is therefore simply isomorphic with the direct product of the general unary group in F and the general unary group in F_2. Thus, if F is the $GF[p^n]$, so that F_2 is the $GF[p^{2n}]$, the orders of the two factor groups are p^n-1 and $p^{2n}-1$ respectively.

If $p = 3$, a transformation of matrix (6) is transformed by

$$\xi_1' = -\xi_1-\xi_2-\xi_3, \quad \xi_2' = \xi_2-\xi_3, \quad \xi_3' = -\xi_3,$$

of period 2, into a transformation of canonical matrix

$$(8) \quad \begin{pmatrix} \delta & 0 & 0 \\ \alpha-\beta & \delta & 0 \\ \alpha & \alpha-\beta & \delta \end{pmatrix}.$$

The transformations of matrix (8) form a *commutative* group. Since α, $\alpha-\beta$, $\delta \equiv I+\alpha+\beta$ are linearly independent modulo 3, the order of the group for the $GF[3^n]$ is $3^{2n}(3^n-1)$.

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* To avoid introducing ω, note that a transformation of matrix (6) is transformed into one of matrix $\begin{pmatrix} \delta & 0 & 0 \\ 0 & I-\alpha & \alpha-\beta \\ 0 & \beta-\alpha & I-\beta \end{pmatrix}$ by $\begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

Canonical Matrices for the Cyclic G_6.

5. The body of a left-hand multiplication table for G_6 is

$$(9)\quad \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \alpha_5 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_4 & \alpha_5 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 & \alpha_5 & \alpha_0 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_0 & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_0 \end{pmatrix}.$$

Let ρ be a primitive sixth root of unity and set

$$\left.\begin{aligned} \delta_i &\equiv \alpha_0+\rho^i\alpha_1+\rho^{2i}\alpha_2+\rho^{3i}\alpha_3+\rho^{4i}\alpha_4+\rho^{5i}\alpha_5 \\ \eta_i &\equiv \xi_0+\rho^{-i}\xi_1+\rho^{-2i}\xi_2+\rho^{-3i}\xi_3+\rho^{-4i}\xi_4+\rho^{-5i}\xi_5 \end{aligned}\right\} \quad (i=0, 1, \ldots, 5).$$

The determinant of the coefficients of $\alpha_0, \ldots, \alpha_5$ in $\delta_0, \ldots, \delta_5$ equals the product of the differences $\rho^r-\rho^s$, and is zero if, and only if, $x^6=1$ has a double root in the field F, *i.e.*, if F has as modulus either 2 or 3.* Excluding these cases, $\delta_0, \ldots, \delta_5$ are linearly independent functions of $\alpha_0, \ldots, \alpha_5$; likewise, $\eta_0, \ldots, \eta_5$ are linearly independent functions of $\xi_0, \ldots, \xi_5$. A transformation on $\xi_0, \ldots, \xi_5$ with coefficients of matrix (9) gives rise to the canonical transformation on $\eta_0, \ldots, \eta_5$:

$$(10)\quad \eta_i' = \delta_i\eta_i \quad (i=0, 1, \ldots, 5).$$

The structure of the group is now quite evident.† When F is the $GF[p^n]$, the order of the group is $(p^n-1)^6$ or $(p^n-1)^2(p^{2n}-1)^2$ according as ρ does or does not belong to the $GF[p^n]$, viz., according as $p^n=6k+1$ or $p^n=6k+5$.

6. Next, let F have modulus 2. Introduce the variables‡

$$\begin{aligned} Y_0 &= \xi_0+\xi_1+\xi_2+\xi_3+\xi_4+\xi_5, & Y_1 &= \xi_0+\xi_2+\xi_4, \\ Y_2 &= \xi_0+\omega\xi_1+\omega^2\xi_2+\xi_3+\omega\xi_4+\omega^2\xi_5, & Y_3 &= \xi_0+\omega^2\xi_2+\omega\xi_4, \\ Y_4 &= \xi_0+\omega^2\xi_1+\omega\xi_2+\xi_3+\omega^2\xi_4+\omega\xi_5, & Y_5 &= \xi_0+\omega\xi_2+\omega^2\xi_4, \end{aligned}$$

ω being a root of $\omega^2+\omega+1=0$, and therefore a cube root of unity. Then the transformation on $\xi_0, \ldots, \xi_5$ with matrix (9) becomes

$$(11)\quad \begin{cases} Y_0' = \delta_0 Y_0, & Y_2' = \delta_2 Y_2, & Y_4' = \delta_4 Y_4, \\ Y_1' = \lambda Y_0+\delta_0 Y_1, & Y_3' = \mu Y_2+\delta_2 Y_3, & Y_5' = \nu Y_4+\delta_4 Y_5, \end{cases}$$

* Compare Dickson, "Linear Groups," Corollary of § 74, p. 54.
† *University of Chicago Decennial Publications*, Vol. IX., p. 38 (preprints, p. 4).
‡ The determinant of their coefficients is congruent to unity modulo 2.

where $\lambda \equiv \alpha_1+\alpha_3+\alpha_5$, $\mu \equiv \omega^2\alpha_1+\alpha_3+\omega\alpha_5$, $\nu \equiv \omega\alpha_1+\alpha_3+\omega^2\alpha_5$.

Here λ, μ, ν are linearly independent functions of α_1, α_2, α_5, since the determinant of their coefficients is $\equiv 1 \pmod 2$. Then δ_0, δ_2, δ_4, λ, μ, ν are seen to be linearly independent modulo 2. If ω belongs to F, the group of transformations (11) is the direct product of three binary linear groups in F, each a commutative group of transformations $C \equiv \begin{pmatrix} \delta & 0 \\ \sigma & \delta \end{pmatrix}$; for the $GF[2^n]$ its order is $2^{3n}(2^n-1)^3$. If ω extends F to a larger field F_2, the group is the direct product* of a binary group in F and a binary group in F_2, each formed of transformations of type C; for the $GF[2^n]$ its order is

$$2^n(2^n-1)\,2^{2n}(2^{2n}-1).$$

7. Finally, let F have modulus 3. Introduce the variables

$$Z_0 = \xi_0+\xi_1+\xi_2+\xi_3+\xi_4+\xi_5, \quad Z_1 = \xi_1-\xi_2+\xi_4-\xi_5, \quad Z_2 = \xi_0+\xi_3,$$
$$Z_3 = \xi_0-\xi_1+\xi_2-\xi_3+\xi_4-\xi_5, \quad Z_4 = -\xi_1-\xi_2+\xi_4+\xi_5, \quad Z_5 = \xi_0-\xi_3.$$

The transformation on $\xi_0, \ldots, \xi_5$ with matrix (9) becomes

$$(12) \quad \begin{cases} Z_0' = \delta_0 Z_0, & Z_3' = \delta_3 Z_3, \\ Z_1' = qZ_0+\delta_0 Z_1, & Z_4' = sZ_3+\delta_3 Z_4, \\ Z_2' = rZ_0+qZ_1+\delta_0 Z_2, & Z_5' = tZ_3+sZ_4+\delta_3 Z_5, \end{cases}$$

where $q \equiv -\alpha_1+\alpha_2-\alpha_4+\alpha_5$, $r \equiv -\alpha_1-\alpha_2-\alpha_4-\alpha_5$,
$s \equiv \alpha_1+\alpha_2-\alpha_4-\alpha_5$, $t \equiv \alpha_1-\alpha_2-\alpha_4+\alpha_5$.

To show that q, r, s, t are linearly independent functions modulo 2, we note that

$$q+t \equiv \alpha_4-\alpha_5, \quad q-t \equiv \alpha_1-\alpha_2, \quad r+s \equiv \alpha_4+\alpha_5, \quad r-s \equiv \alpha_1+\alpha_2 \pmod 3;$$

so that α_1, α_2, α_4, α_5 may be expressed in terms of q, r, s, t. It follows that

$$\delta_0 \equiv \alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5, \quad \delta_3 \equiv \alpha_0-\alpha_1+\alpha_2-\alpha_3+\alpha_4-\alpha_5,$$

and q, r, s, t are six linearly independent functions of $\alpha_0, \ldots, \alpha_5$, modulo 3. Hence the group of transformations (12) is the direct product of two groups, affecting different variables, each a *commutative* group of the type

$$\begin{pmatrix} \delta & 0 & 0 \\ \lambda & \delta & 0 \\ \kappa & \lambda & \delta \end{pmatrix} \quad [\delta, \lambda, \kappa \text{ arbitrary in } F;\ \delta \neq 0].$$

* Given δ_0 and λ arbitrarily in F, δ_2 and μ arbitrarily in F_2, the values of $\alpha_0, \ldots, \alpha_5$ follow uniquely.

Remarks on the General Theory.

8. In the writer's treatment* of a group matrix of order n for an arbitrary field F, the case in which F has a modulus which divides n was excluded. The question of the first invariant factor of the characteristic determinant† of the special group-determinant D depends upon the nature of the greatest common divisor of all its first minors. The adjoint minor of an element ϵ_k in the determinant D equals $\frac{1}{nn_k}\frac{\partial D}{\partial \epsilon_k}$, where n_k is the number of operators in the k-th set of conjugates in the given group of order n. The necessity of a variation in the treatment will be evident from the following examples.

9. For the symmetric group g_6 on three letters, the special group-determinant D is the determinant of matrix (1) when $\alpha = \beta$, $\gamma = \delta = \epsilon$. From the form of matrix M of § 1, we conclude that

$$D \equiv (I+2\alpha+3\gamma)(I+2\alpha-3\gamma)(I-\alpha)^4.$$

Let I_1, α_1, γ_1 denote the adjoint minors of I, α, γ, respectively (by a theorem the same wherever I, α, γ occur in D). Then‡

$$I_1 = \tfrac{1}{6}\frac{\partial D}{\partial I} = (I^2+3I\alpha+2\alpha^2-6\gamma^2)(I-\alpha)^3,$$

$$\alpha_1 = \tfrac{1}{12}\frac{\partial D}{\partial \alpha} = (3\gamma^2-I\alpha-2\alpha^2)(I-\alpha)^3,$$

$$\gamma_1 = \tfrac{1}{18}\frac{\partial D}{\partial \gamma} = -\gamma\,(I-\alpha)^4.$$

For I, α, γ arbitrary in F, the greatest common divisor of I_1, α_1, γ_1 is therefore $(I-\alpha)^3$, if F does not have modulus 3; but is $(I-\alpha)^4$, if F has modulus 3.

10. For the quaternion group (Weber, *Algebra*, 2nd ed., pp. 216–218),

$$D = \sigma_1\sigma_2\sigma_3\sigma_4\,(x_1-x_2)^4,$$

$$\sigma_1 = x_1+x_2+2x_3+2x_5+2x_7, \quad \sigma_2 = x_1+x_2+2x_3-2x_5-2x_7,$$

$$\sigma_3 = x_1+x_2-2x_3+2x_5-2x_7, \quad \sigma_4 = x_1+x_2-2x_3-2x_5+2x_7.$$

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* *Trans. Amer. Math. Soc.*, Vol. III., 1902, pp. 285–301. See p. 293.

† It is derived from D by replacing the identity element I by $I-\rho$.

‡ The values of I_1 and γ_1 were also computed direct as determinants of order 5.

Denote the adjoint minor of x_i by X_i. Then

$$X_1 = (x_1-x_2)^3\{x_1(x_1+x_2)^3-2(x_1+x_2)(3x_1+x_2)(x_3^2+x_5^2+x_7^2)$$
$$-32x_3^2x_5^2+24x_2x_3x_5x_7+40x_1x_3x_5x_7+8(x_3^2+x_5^2-x_7^2)^2\},$$
$$X_3 = (x_1-x_2)^4\{4x_3^3-4x_3x_5^2-4x_3x_7^2+4x_5x_7(x_1+x_2)-x_3(x_1+x_2)^2\}.$$

Since $X_5 \equiv \frac{1}{16}\frac{\partial D}{\partial x_5}$ is derived from $X_3 \equiv \frac{1}{16}\frac{\partial D}{\partial x_3}$ by interchanging x_3 with x_5, D being thereby unaltered, the expression for X_5 may be written down by inspection. Similarly for X_7 and X_2. It follows readily that the greatest common divisor of X_1, X_2, X_3, X_5, X_7 is $(x_1-x_2)^3$, if F does not have modulus 2; but is $(x_1-x_2)^6$ if F has modulus 2. The special character of modulus 2 is also shown by the fact that it is the only case in which the factors σ_1, σ_2, σ_3, σ_4, x_1-x_2 are all equal.

11. For the alternating group G_{12} on four letters, we have*

$$D = (I+3\epsilon+4\alpha+4\beta)(I+3\epsilon+4\rho\alpha+4\rho^2\beta)(I+3\epsilon+4\rho^2\alpha+4\rho\beta)(I-\epsilon)^9,$$

where $\rho = \frac{1}{2}(-1+\sqrt{-3})$ is a cube root of unity. Hence

$$D = \{(I+3\epsilon)^3+64\alpha^3+64\beta^3-48\alpha\beta(I+3\epsilon)\}(I-\epsilon)^9.$$

For the adjoint minors I_1, α_1, β_1, ϵ_1, we get

$$I_1 = \tfrac{1}{12}\frac{\partial D}{\partial I} = (I-\epsilon)^8\{(I+3\epsilon)^2(I+2\epsilon)+48\alpha^3+48\beta^3-40\alpha\beta I-104\alpha\beta\epsilon\},$$

$$\alpha_1 = \tfrac{1}{48}\frac{\partial D}{\partial \alpha} = (I-\epsilon)^9(4\alpha^2-I\beta-3\beta\epsilon),$$

$$\beta_1 = \tfrac{1}{48}\frac{\partial D}{\partial \beta} = (I-\epsilon)^9(4\beta^2-I\alpha-3\alpha\epsilon),$$

$$\epsilon_1 = \tfrac{1}{36}\frac{\partial D}{\partial \epsilon} = (I-\epsilon)^8\{-\epsilon(I+3\epsilon)^2-16\alpha^3-16\beta^3+8\alpha\beta I+40\alpha\beta\epsilon\}.$$

The greatest common divisor of I_1, α_1, β_1, ϵ_1 is therefore $(I-\epsilon)^8$ if F does not have modulus 2; but is $(I-\epsilon)^{10}$ if F has modulus 2, since then

$$I_1 = I(I-\epsilon)^{10}, \quad \alpha_1 = \beta(I-\epsilon)^{10}, \quad \beta_1 = \alpha(I-\epsilon)^{10}, \quad \epsilon_1 = \epsilon(I-\epsilon)^{10}.$$

The exceptional character of the case of modulus 2 is also shown by

* Using table of characters, Weber, *Algebra*, 2nd ed., II., p. 206.

two further facts. The linear factors of D are then all equal. Again, the general group-determinant then decomposes entirely into three linear factors each to the fourth power. For any field, the theory of Frobenius gives* for the general group-determinant of G_{12},

$$\Delta = ff_1f_2\Phi^3,$$

$$f \equiv I+\epsilon+\epsilon_1+\epsilon_2+\alpha+\alpha_1+\alpha_2+\alpha_3+\beta+\beta_1+\beta_2+\beta_3,$$

$$f_1 \equiv I+\epsilon+\epsilon_1+\epsilon_2+\rho\,(\alpha+\alpha_1+\alpha_2+\alpha_3)+\rho^2\,(\beta+\beta_1+\beta_2+\beta_3),$$

$$f_2 \equiv I+\epsilon+\epsilon_1+\epsilon_2+\rho^2\,(\alpha+\alpha_1+\alpha_2+\alpha_3)+\rho\,(\beta+\beta_1+\beta_2+\beta_3),$$

$$\begin{aligned}\Phi \equiv\ & I^3+\epsilon^3+\epsilon_1^3+\epsilon_2^3+\alpha^3+\alpha_1^3+\alpha_2^3+\alpha_3^3+\beta^3+\beta_1^3+\beta_2^3+\beta_3^3\\
&-I^2\epsilon-I^2\epsilon_1-I^2\epsilon_2-I\epsilon^2-I\epsilon_1^2-I\epsilon_2^2-\epsilon^2\epsilon_1-\epsilon^2\epsilon_2-\epsilon_1^2\epsilon-\epsilon_1^2\epsilon_2\\
&-\epsilon_2^2\epsilon-\epsilon_2^2\epsilon_1-\alpha^2\alpha_1-\alpha^2\alpha_2-\alpha^2\alpha_3-\alpha_1^2\alpha-\alpha_1^2\alpha_2-\alpha_1^2\alpha_3-\alpha_2^2\alpha-\alpha_2^2\alpha_1\\
&-\alpha_2^2\alpha_3-\alpha_3^2\alpha-\alpha_3^2\alpha_1-\alpha_3^2\alpha_2-\beta^2\beta_1-\beta^2\beta_2-\beta^2\beta_3-\beta_1^2\beta-\beta_1^2\beta_2-\beta_1^2\beta_3\\
&-\beta_2^2\beta-\beta_2^2\beta_1-\beta_2^2\beta_3-\beta_3^2\beta-\beta_3^2\beta_1-\beta_3^2\beta_2\\
&+2\epsilon\epsilon_1\epsilon_2+2I\epsilon\epsilon_1+2I\epsilon\epsilon_2+2I\epsilon_1\epsilon_2+2\alpha\alpha_1\alpha_2+2\alpha\alpha_1\alpha_3+2\alpha\alpha_2\alpha_3\\
&+2\alpha_1\alpha_2\alpha_3+2\beta\beta_1\beta_2+2\beta\beta_2\beta_3+2\beta\beta_1\beta_3+2\beta_1\beta_2\beta_3\\
&+I(\alpha\beta_1+\alpha\beta_2+\alpha\beta_3+\alpha_1\beta+\alpha_1\beta_2+\alpha_1\beta_3+\alpha_2\beta+\alpha_2\beta_1+\alpha_2\beta_3\\
&\qquad+\alpha_3\beta+\alpha_3\beta_1+\alpha_3\beta_2)\\
&+(\epsilon+\epsilon_1+\epsilon_2-3I)(\alpha\beta+\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3)\\
&-3\epsilon\,(\alpha\beta_1+\alpha_1\beta+\alpha_2\beta_3+\alpha_3\beta_2)\\
&-3\epsilon_1\,(\alpha\beta_2+\alpha_1\beta_3+\alpha_2\beta+\alpha_3\beta_1)\\
&-3\epsilon_2\,(\alpha\beta_3+\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta)\\
&+\epsilon\,(\alpha\beta_2+\alpha\beta_3+\alpha_1\beta_2+\alpha_1\beta_3+\alpha_2\beta+\alpha_2\beta_1+\alpha_3\beta+\alpha_3\beta_1)\\
&+\epsilon_1(\alpha\beta_1+\alpha\beta_3+\alpha_1\beta+\alpha_1\beta_2+\alpha_2\beta_1+\alpha_2\beta_3+\alpha_3\beta+\alpha_3\beta_2)\\
&+\epsilon_2\,(\alpha\beta_1+\alpha\beta_2+\alpha_1\beta+\alpha_1\beta_3+\alpha_2\beta+\alpha_2\beta_3+\alpha_3\beta_1+\alpha_3\beta_2).\end{aligned}$$

For the case of modulus 2, we find that

$$\Phi \equiv ff_1f_2 \pmod 2, \qquad \Delta \equiv f^4f_1^4f_2^4 \pmod 4.$$

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* The computation of Φ was checked as follows. For $\epsilon=\epsilon_1=\epsilon_2$, $\alpha=\alpha_1=\alpha_2=\alpha_3$, $\beta=\beta_1=\beta_2=\beta_3$, $\Phi=(I-\epsilon)^3$, thereby agreeing with the above result for the special group-determinant D. Again, the coefficients of I^3, $I^2\epsilon_i$, $I^2\alpha_i$, $I^2\beta_i$ in Φ agree with those given by using the characters for the factor Φ.

12. For the symmetric group G_{24} on four letters,*

$$D = \alpha\epsilon\beta^9\gamma^9\delta^4,$$

$$\alpha = a+3b+6c+8d+6e, \quad \epsilon = a+3b-6c+8d-6e,$$

$$\beta = a-b+2c-2e, \quad \gamma = a-b-2c+2e, \quad \delta = a+3b-4d.$$

By direct computation (see § 8), we find for the adjoint minors

$$d_1 = \beta^9\gamma^9\delta^3\{3(c+e)^2-d(a+3b+8d)\},$$

$$c_1 = \beta^8\gamma^8\delta^4\{4eb^2+4eab-c(a^2+2ab+5b^2)+(c^2-e^2)(20c+16e)$$
$$-(c-e)(8ad+24bd+32d^2)\},$$

$$b_1 = \beta^8\gamma^8\delta^3\{(a-b)(a+3b+8d)(8d^2-ab-3b^2-4bd)+24(c^2-e^2)^2$$
$$+(a-b)(c+e)^2(3a+33b-36d)$$
$$-(c-e)^2(a+3b+8d)(a+3b+4d)\},$$

$$a_1 = \beta^8\gamma^8\delta^3\{(a-b)(a+3b+8d)(a^2+5ab+4ad+6b^2+8bd-24d^2)$$
$$-(c-e)^2(a+3b+8d)(a+3b+4d)+24(c^2-e^2)^2$$
$$+(a-b)(c+e)^2(-33a-75b+108d)\}.$$

Since D is unaltered by the interchange of c with e, we derive $e_1 \equiv \frac{1}{144}\frac{\partial D}{\partial e}$ from $c_1 \equiv \frac{1}{144}\frac{\partial D}{\partial c}$ by interchanging c with e.

If F does not have modulus 2 or 3, the greatest common divisor g of a_1, b_1, c_1, d_1, e_1 is $\beta^8\gamma^8\delta^3$, in accord with the general theory. If F has modulus 2, so that $\alpha \equiv \beta \equiv \gamma \equiv \delta \equiv \epsilon \equiv a+b \pmod 2$, then

$$d_1 = \alpha^{21}\{(c+e)^2+d(a+b)\}, \qquad c_1 = \alpha^{20}(ca^2+cb^2),$$
$$b_1 = \alpha^{19}\{b(a+b)^3\}, \qquad a_1 = \alpha^{19}\{a(a+b)^3\}.$$

Hence the greatest common divisor g is $(a+b)^{21}$. For modulus 3,

$$d_1 = \beta^9\gamma^9\delta^3\{-d(a-d)\},$$

$$c_1 = \beta^8\gamma^8\delta^4\{eb(a+b)+c(b^2+ab-a^2)+(e-c)(c^2-e^2-ad-d^2)\},$$

$$b_1 = \beta^8\gamma^8\delta^3(a-d)\{(a-b)(-ab-bd-d^2)-(c-e)^2(a+d)\},$$

$$a_1 = \beta^8\gamma^8\delta^3(a-d)(a+d)\{(a-b)^2-(c-e)^2\},$$

and $\alpha = \epsilon = \delta = a-d$. Since $a-d$ is not a factor of the expression in brackets in c_1, the greatest common divisor g is $\beta^8\gamma^8\delta^4$. Hence the cases of moduli 2 and 3 are both special.

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* Frobenius, *Berliner Sitzungsberichte*, 1896, p. 1012.

The Theory of Cauchy's Principal Values. (Third Paper: *Differentiation and Integration of Principal Values.*) *By* G. H. Hardy. Received February 15th, 1902. Read March 13th, 1902. Revised November, 1902.

1. In my second paper* I stated the general conditions under which

$$P\int_a^A f(x, \alpha)\, dx \tag{1}$$

is a continuous function of α. In this paper I shall deal with two of the most important special cases of this general problem, which lead to theorems corresponding to the ordinary rules for differentiation and integration under the integral sign. That (1) is convergent for all values of α in question will be presupposed in all that follows.

Differentiation under the Sign of the Principal Value.

2. The first question which will engage us is that of finding sufficient conditions for the truth of the equation

$$\frac{d}{d\alpha}P\int_a^A f(x, \alpha)\, dx = P\int_a^A \frac{\partial f(x, \alpha)}{\partial \alpha}\, dx, \tag{1}$$

which is a generalization of Leibniz's theorem. This equation asserts that, if

$$\Delta_h f(x, \alpha) = \frac{1}{h}\{f(x, \alpha+h) - f(x, \alpha)\},$$

$$\lim_{h=0} P\int_a^A \Delta_h f(x, \alpha)\, dx = P\int_a^A \lim_{h=0} \Delta_h f(x, \alpha)\, dx,$$

that is to say that

$$P\int_a^A \Delta_h f(x, \alpha)\, dx \tag{2}$$

is a continuous function of h for $h = 0$. Hence sufficient conditions for (1) are (II., § 25) (i.) that $f(x, \alpha)$ is continuous except on certain curves, and (ii.) that (2) is uniformly convergent in an interval $(-H, H)$.

This general criterion is, however, not very easy to apply.

* In this paper the first and second papers (*Proceedings*, Vol. xxxiv., pp, 16, 55) will be referred to as I., II.

3. I shall assume at present that A is finite, and that we can find a positive value of H such that $f(x, \alpha)$ and its derivates $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial \alpha}$ are continuous functions of both variables in the rectangle

$$(a, A, \alpha_0 - H, \alpha_0 + H),$$

except on a finite number of curves $x = X_i(\alpha)$. We may without loss of generality suppose that there is only one such curve. It is also convenient to take $a + A > 2X$ for $\alpha = \alpha_0$; we can then choose H so small that this condition is satisfied throughout $(\alpha_0 - H, \alpha_0 + H)$.

Finally, we suppose *either* that

$$\lim_{\epsilon = 0} \{f(X - \epsilon, \alpha) - f(X + \epsilon, \alpha)\} = 0$$

uniformly for all values of α in question; *or* that $\frac{dX}{d\alpha}$ is identically zero, *i.e.*, X independent of α. As X is independent of ϵ, both alternatives are included in the single condition that

$$\lim_{\epsilon = 0} \frac{dX}{d\alpha} \{f(X - \epsilon, \alpha) - f(X + \epsilon, \alpha)\} = 0$$

uniformly for all values of α in question. Then

$$P \int_a^A f\, dx = P \int_a^{2X - a} f\, dx + P \int_{2X - a}^A f\, dx,$$

and

$$P \int_a^{2X - a} f\, dx = \lim_{\epsilon = 0} \int_\epsilon^{X - a} \{f(X + y, \alpha) + (X - y, \alpha)\}\, dy$$

$$= \int_0^{X - a} \phi(y, \alpha)\, dy,$$

if

$$\phi(y, \alpha) = f(X + y, \alpha) + f(X - y, \alpha).$$

Now ϕ is continuous and has continuous first derivates except for $y = 0$. And

$$\int_\epsilon^{X - a} \frac{\partial \phi}{\partial \alpha} dy = \int_\epsilon^{X - a} \left\{ \left[\frac{\partial f(t, \alpha)}{\partial \alpha} + \frac{dX}{d\alpha} \frac{\partial f(t, \alpha)}{\partial t} \right]_{t = X + y} + \left[\frac{\partial f(t, \alpha)}{\partial \alpha} + \frac{dX}{d\alpha} \frac{\partial f(t, \alpha)}{\partial t} \right]_{t = X - y} \right\} dy$$

$$= \left(\int_a^{X - \epsilon} + \int_{X + \epsilon}^{2X - a} \right) \left\{ \frac{\partial f(x, \alpha)}{\partial \alpha} + \frac{dX}{d\alpha} \frac{\partial f(x, \alpha)}{\partial x} \right\} dx$$

$$= \left(\int_a^{X - \epsilon} + \int_{X + \epsilon}^{2X - a} \right) \frac{\partial f}{\partial \alpha} dx + \frac{dX}{d\alpha} \{f(2X - a) - f(X + \epsilon) + f(X - \epsilon) - f(a)\}.$$

The right-hand side tends uniformly to the limit

$$P\int_a^{2X-a}\frac{\partial f}{\partial \alpha}dx+\frac{dX}{d\alpha}\{f(2X-a)-f(a)\}.$$

when ϵ tends to zero. Hence

$$\int_0^{X-a}\frac{\partial \phi}{\partial \alpha}dy$$

is uniformly convergent; and therefore

$$\frac{d}{d\alpha}\int_0^{X-a}\phi\,dy=\int_0^{X-a}\frac{\partial \phi}{\partial \alpha}dy+\frac{dX}{d\alpha}\phi(X-a),$$

by the ordinary theorem of differentiation under the integral sign. Therefore

$$\begin{aligned}\frac{d}{d\alpha}P\int_a^A f\,dx&=\frac{d}{d\alpha}\int_0^{X-a}\phi\,dy+\frac{d}{d\alpha}\int_{2X-a}^A f\,dx\\&=P\int_a^{2X-a}\frac{\partial f}{\partial \alpha}dx+2\frac{dX}{d\alpha}f(2X-a)\\&\quad+\int_{2X-a}^A\frac{\partial f}{\partial \alpha}dx-2\frac{dX}{d\alpha}f(2X-a)\\&=P\int_a^A\frac{\partial f}{\partial \alpha}\partial x.\end{aligned}$$

THEOREM 1.—*If* $f(x,\alpha)$, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial \alpha}$ *are continuous in*

$$(a,\ A,\ \alpha_0-H,\ \alpha_0+H),$$

except along a finite number of curves $x=X_i(\alpha)$, *which do not meet* $x=a$, *or* $x=A$, *or one another, and have at every point a definite direction never parallel to* x; *if, moreover,*

$$\lim_{\epsilon=0}\frac{dX}{d\alpha}\{f(X_i-\epsilon,\alpha)-f(X_i+\epsilon,\alpha)\}=0$$

uniformly for all values of α *in* $(\alpha_0-H,\ \alpha_0+H)$, *and*

$$P\int_a^A\frac{\partial f}{\partial \alpha}dx$$

is uniformly convergent in $(\alpha_0-H,\ \alpha_0+H)$; *then*

$$\frac{d}{d\alpha}P\int_a^A f\,dx=P\int_a^A\frac{\partial f}{\partial \alpha}dx.$$

We have supposed A finite. But no new difficulty arises from supposing the upper limit infinite, if there are only a finite number of curves across which the integrals are not unconditionally convergent. For then we can choose A so that

$$X_i(\alpha) < A, \quad \alpha_0 - H \leqq \alpha \leqq \alpha_0 + H;$$

and

$$\frac{d}{d\alpha} P\int_{a]}^{x} f\,dx = \frac{d}{d\alpha} P\int_{a}^{A} + \frac{d}{d\alpha}\int_{A}^{x}.$$

The first term is

$$P\int_{a}^{A} \frac{\partial f}{\partial \alpha}\,dx,$$

by Theorem 1; and the second is

$$\int_{A}^{x} \frac{\partial f}{\partial \alpha}\,dx,$$

if this integral is uniformly convergent; so that

$$\frac{d}{d\alpha} P\int_{a}^{x} f\,dx = P\int_{a}^{} \frac{\partial f}{\partial \alpha}\,dx.$$

Case 1 (X_i *independent of* α).

4. The simplest way to satisfy the conditions of the theorem is to suppose that* $f(x, \alpha) = \Omega_\nu(x - X)\,\Theta(x, \alpha)$,

Θ being a function which has continuous derivates $\dfrac{\partial\Theta}{\partial\alpha}$, $\dfrac{\partial\Theta}{\partial x}$, and X independent of α.

5. (i.) If

$$I(\alpha) = P\int_0^p \log\left(1 + \frac{\alpha}{x}\right)\frac{dx}{x-t} \quad (0 < t < p,\ 0 < \alpha),$$

$$\frac{dI}{d\alpha} = P\int_0^p \frac{dx}{(x+\alpha)(x-t)} = \frac{1}{\alpha+t}\log\left(\frac{p-t}{t}\,\frac{\alpha}{p+\alpha}\right).$$

Integrating from $\alpha = 0$ to $\alpha = q$ and putting $q = p$, we find

$$P\int_0^p \log\left(1 + \frac{p}{x}\right)\frac{x\,dx}{x^2-t^2} = \tfrac{1}{2}\log\left(\frac{p}{t}+1\right)\log\left(\frac{p}{t}-1\right).$$

(ii.) It is easy to prove by differentiation that

$$P\int_0^\infty \log\left(1 + \frac{\alpha^2}{x^2}\right)\frac{dx}{x^2-a^2} = -\frac{\pi}{a}\tan^{-1}\frac{\alpha}{a}.$$

(iii.) If $0 < \alpha < 1$,

$$\int_0^x \frac{x^{\alpha-1}\log x}{1-x}\,dx = \frac{d}{d\alpha} P\int_0^\infty \frac{x^{\alpha-1}\,dx}{1-x} = -\left(\frac{\pi}{\sin\alpha\pi}\right)^2.$$

In the latter case the symbol of the principal value is unnecessary in the derived integral. In general, however, when X is independent of α, both integrals are only principal values

(iv.)

$$P\int_0^\infty \frac{x^{\alpha-1} - x^{\beta-1}}{1-x}\,\frac{dx}{\log x} = \log\left(\frac{\sin\alpha\pi}{\sin\beta\pi}\right).$$

* The functions Ω_ν, ψ_ν are defined in I. (§§ 8, 9).

Case 2 (X_i *dependent on* α).

6. The more interesting case of the theorem is, however, that in which X_i depends upon α.

THEOREM 2.—*If* $f(x, \alpha)$, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial \alpha}$ *are continuous except upon the curves* $x = X_i(\alpha)$, *and* $f(x, \alpha)$ *can, near any of these curves, be expressed in the form*

$$\psi_\nu \{x - X_i(\alpha)\}\, \Theta(x, \alpha),$$

where ψ_ν *is a product of logarithmic factors only, and* Θ, $\dfrac{\partial \Theta}{\partial x}$, $\dfrac{\partial \Theta}{\partial \alpha}$ *are continuous without exception, then*

$$\frac{d}{d\alpha}\int_a^A f(x, \alpha)\, dx = P\int_a^A \frac{\partial f}{\partial \alpha}\, dx.$$

For
$$\psi_\nu(x - X) = \prod_1^\nu |\, l^i (x - X)\,|^{s_i};$$

in this product some or all of the signs of the absolute value may possibly be omitted. And

$$\lim_{\epsilon = 0} \{f(X - \epsilon) - f(X + \epsilon)\} = \lim_{\epsilon = 0} \psi_\nu(\epsilon) \{\Theta(X - \epsilon) - \Theta(X + \epsilon)\} = 0$$

uniformly for all values of α.

Moreover

$$\frac{\partial f}{\partial \alpha} = \psi_\nu(x - X)\frac{\partial \Theta}{\partial \alpha} + \frac{\Theta}{x - X}\sum_{i=1}^{\nu} s_i \prod_{\kappa=1}^{i} |\, l^\kappa (x - X)\,|^{s_\kappa - 1} \prod_{i+1}^{\nu} |\, l^\kappa (x - X)\,|^{s_\kappa}.$$

The first term is unconditionally integrable. The rest consists of a finite number of terms of the form

$$\Omega_\nu(x - X)\, \Theta(x, \alpha).$$

Hence (II., § 15)
$$P\int_a^A \frac{\partial f}{\partial \alpha}\, dx$$

is uniformly convergent, and so the conditions of Theorem 1 are satisfied. In this case it is always an ordinary integral whose derivate is expressed as a principal value.

7. (i.) It is easily verified that

$$\int_a^A l(x - \alpha)\, dx = \tfrac{1}{2}\int_a^A \log(x - \alpha)^2\, dx = (A - \alpha)\, l(A - \alpha) - (a - \alpha)\, l(a - \alpha) - (A - a) \qquad (0 < a < A);^*$$

and
$$\frac{d}{d\alpha}\int_a^A l(x - \alpha)\, dx = l(a - \alpha) - l(A - \alpha) = -P\int_a^A \frac{dx}{x - \alpha}.$$

* By lx I denote $\log |x|$ (I., § 8).

(ii.) It is easy to prove that, if n is integral and $0 < \alpha < \pi$,

$$\int_0^\pi l(\cos x - \cos\alpha)\cos nx\,dx = -\frac{\pi}{n}\cos n\alpha,$$

or $= 0$, according as $n >$ or $= -\pi \log 2$. Hence

$$P\int_0^\pi \frac{\cos nx\,dx}{\cos x - \cos\alpha} = \frac{\pi \sin n\alpha}{\sin\alpha} \quad \text{or} \quad 0.$$

Similarly, from

$$\int_0^\pi l(\cos x - \cos\alpha)\, l(\cos x - \cos\beta)\,dx = \pi(\log 2)^2 + \tfrac{1}{2}\pi(\alpha^2+\beta^2) - \pi^2\alpha + \tfrac{1}{3}\pi^3 \qquad (0 < \beta < \alpha < \pi),$$

we deduce

$$P\int_0^\pi \frac{l(\cos x - \cos\alpha)}{\cos x - \cos\beta}\,dx = \frac{\pi(\alpha-\pi)}{\sin\alpha} \text{ or } \frac{\pi\alpha}{\sin\alpha},$$

according as $\beta <$ or $> \alpha$. The integral is discontinuous for $\beta = \alpha$ (II., § 36), and its value for $\beta = \alpha$ is the mean of its values for $\beta = \alpha \pm 0$.*

(iii.) We find [*cf.* § 5 (i.)] that

$$\int_q^p \log\left(\frac{p-x}{x-q}\right)^2 \frac{dx}{x} = \left\{\log\left(\frac{p}{q}\right)\right\}^2 \quad (p > q).$$

(iv.)
$$P\int_0^\infty \frac{\log\{1+\alpha(x-\alpha)\}^2}{(x-a)^2}\,dx = -\frac{1}{2a}\{a\alpha\log(a\alpha)^2 + (1-a\alpha)\log(1-a\alpha)^2\},$$

unless $a\alpha = 0$ or 1. This example is instructive, as (according to the values of a and α) it may be an example of the use of Leibniz's theorem, or of either or both of Theorems 1, 2.

(v.) The following formulæ afford further illustrations; in all α, β, γ are positive:

$$\int_0^\infty \log\left(1-\frac{\alpha^2}{x^2}\right)^2 \log\left(1-\frac{\beta^2}{x^2}\right)^2 dx = 2\pi^2\alpha\ (\alpha<\beta),\ 2\pi^2\beta\ (\alpha>\beta);$$

$$P\int_0^\infty \log\left(1-\frac{\beta^2}{x^2}\right)^2 \log\left(1-\frac{\gamma^2}{x^2}\right)\frac{dx}{\alpha^2-x^2}$$

$$= \frac{\pi^2}{\alpha}\left\{l\left(\frac{\gamma^2}{\alpha^2}-1\right) + 2l\left(1+\frac{\beta}{\alpha}\right)\right\},\ \frac{\pi^2}{\alpha}\left\{l\left(\frac{\beta^2}{\alpha^2}-1\right) + 2l\left(1+\frac{\gamma}{\alpha}\right)\right\},$$

$$\frac{2\pi^2}{\alpha}l\left(1+\frac{\beta}{\alpha}\right),\ \frac{\pi^2}{\alpha}l\frac{\alpha+\beta}{\alpha-\beta},\ \frac{2\pi^2}{\alpha}l\left(1+\frac{\gamma}{\alpha}\right),\ \frac{\pi^2}{\alpha}l\frac{\alpha+\gamma}{\alpha-\gamma},$$

according as $\alpha<\beta<\gamma$, $\alpha<\gamma<\beta$, $\beta<\alpha<\gamma$, $\beta<\gamma<\alpha$, $\gamma<\alpha<\beta$, or $\gamma<\beta<\alpha$;

$$\int_0^\infty \log\left(1-\frac{\alpha^2}{x^2}\right)^2 \log\left(1-\frac{\beta^2}{x^2}\right)^2 \log\left(1-\frac{\gamma^2}{x^2}\right)^2 dx$$

$$= 4\pi^2\left\{\alpha\log\left(\frac{\gamma^2}{\alpha^2}-1\right) + \gamma\log\frac{\gamma+\alpha}{\gamma-\alpha} + 2\alpha\log\left(1+\frac{\beta}{\alpha}\right) + 2\beta\log\left(1+\frac{\alpha}{\beta}\right)\right\}$$

$$(\alpha \leqq \beta \leqq \gamma).$$

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* In II., § 36, the discontinuity is assigned only half its true value.

8. THEOREM 3.—*If* $f(x, \alpha) = \Omega_\nu(x-\alpha)\,\Theta(x, \alpha)$,

Θ *being a function whose derivates of the first two orders are continuous in* $(a, A, \alpha_0-H, \alpha_0+H)$, *where* $a<\alpha_0-H<\alpha_0+H<A$, *then*

$$I(\alpha) = P\int_a^A f(x, \alpha)\,dx$$

will have a continuous derivate equal to

$$P\int_a^A \Omega_\nu(x-\alpha)\left\{\frac{\partial\Theta}{\partial x}+\frac{\partial\Theta}{\partial\alpha}\right\}dx+f(a, \alpha)-f(A, \alpha).$$

It is assumed that f and its derivates are continuous except for $x=\alpha$. A product of logarithmic factors such as occurs in $\Omega_\nu(u)$ may become infinite for values of u other than 0, such as 1, e, ... We suppose that such a contingency is avoided by a suitable choice of the range (a, A) and the exponents s_i.

Suppose $2\alpha-a<A$; then, by the transformation used in § 3,

$$I(\alpha) = \int_0^{\alpha-a}\phi(y, \alpha)\,dy+\int_{2\alpha-a}^A f\,dx,$$

where $$\phi(y, \alpha) = f(\alpha+y, \alpha)+f(\alpha-y, \alpha).$$

Now, for values of y other than zero,

$$\frac{\partial\phi}{\partial\alpha} = \left[\frac{\partial f(t, \alpha)}{\partial\alpha}+\frac{\partial f(t, \alpha)}{\partial t}\right]_{t=\alpha+y}+\left[\frac{\partial f}{\partial\alpha}+\frac{\partial f}{\partial t}\right]_{t=\alpha-y}$$

$$= \Omega_\nu(y)\left\{\left[\frac{\partial\Theta}{\partial\alpha}+\frac{\partial\Theta}{\partial t}\right]_{t=\alpha+y}-\left[\frac{\partial\Theta}{\partial\alpha}+\frac{\partial\Theta}{\partial t}\right]_{t=\alpha-y}\right\}$$

$$= 2y\Omega_\nu(y)\left[\frac{\partial^2\Theta}{\partial\alpha\,\partial t}+\frac{\partial^2\Theta}{\partial t^2}\right]_{t=\alpha+\theta y} \qquad (-1\leqq\theta\leqq 1).$$

This is only logarithmically infinite for $y=0$, and

$$\int_0^{\alpha-a}\frac{\partial\phi}{\partial\alpha}\,dy$$

is uniformly convergent. Hence

$$\frac{d}{d\alpha}\int_0^{\alpha-a}\phi\,dy = \int_0^{\alpha-a}\frac{\partial\phi}{\partial\alpha}\,dy+\{f(2\alpha-a, \alpha)+f(a, \alpha)\}.$$

Also $$\frac{d}{d\alpha}\int_{2\alpha-a}^A f\,dx = \frac{d}{d\alpha}\int_{\alpha-a}^{A-\alpha} f(\alpha+y, \alpha)\,dy$$

$$= \int_{\alpha-a}^{A-\alpha}\left[\frac{\partial f}{\partial\alpha}+\frac{\partial f}{\partial t}\right]_{t=\alpha+y}dy-\{f(A, \alpha)+f(2\alpha-a, \alpha)\}.$$

Therefore

$$\frac{dI}{d\alpha} = \int_0^{A-\alpha} \left[\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial t}\right]_{t=\alpha+y} dy + \int_0^{\alpha-a} \left[\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial t}\right]_{t=\alpha-y} dy + f(a,\alpha) - f(A,\alpha).$$

Finally, $$\int_\epsilon^{A-\alpha} \left[\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial t}\right]_{t=\alpha+y} dy + \int_\epsilon^{\alpha-a} \left[\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial t}\right]_{t=\alpha-y} dy$$

$$= \left(\int_{\alpha+\epsilon}^{A} + \int_a^{\alpha-\epsilon}\right) \left\{\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial x}\right\} dx;$$

and, as the left hand tends uniformly to its limit for $\epsilon = 0$, the right hand tends uniformly to its limit

$$P\int_a^A \left(\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial x}\right) dx.$$

This principal value is therefore uniformly convergent and continuous, and the theorem follows. If we had taken f in the more general form

$$\Omega_\nu \{x - X(\alpha)\}\, \Theta(x, \alpha),$$

the derivate would have been

$$P\int_a^A \Omega_\nu (x-X) \left\{\frac{\partial \Theta}{\partial \alpha} + \frac{dX}{d\alpha}\frac{\partial \Theta}{\partial x}\right\} dx + \frac{dX}{d\alpha}\{f(a,\alpha) - f(A,\alpha)\}.$$

9. (i.) If $$I(\alpha) = P\int_a^A \frac{dx}{x-\alpha} = \log\frac{A-\alpha}{\alpha-a} \quad (a < \alpha < A),$$

$$\frac{dI}{d\alpha} = \frac{1}{a-\alpha} - \frac{1}{A-\alpha},$$

which is evidently correct.

(ii.) From $$P\int_0^\pi \frac{dx}{\cos x - \cos\alpha} = 0 \quad (0 < \alpha < \pi)$$

we deduce by successive differentiation

$$P\int_0^\pi \frac{\sin x - \sin\alpha}{(\cos x - \cos\alpha)^2} dx = -\frac{2}{\sin^2\alpha}, \quad P\int_0^\pi \frac{(\sin x - \sin\alpha)^2}{(\cos x - \cos\alpha)^3} dx = \frac{4\cos\alpha}{\sin^3\alpha}.$$

(iii.) From $$P\int_0^\infty \frac{dx}{x^2 - \alpha^2} = 0 \quad (\alpha > 0)$$

we deduce $$P\int_0^\infty \frac{dx}{(x-\alpha)(x+\alpha)^{n+1}} = -\frac{1}{2^n \alpha^{n+1}}\left(1 + \frac{2}{2} + \frac{2^2}{3} + \ldots + \frac{2^{n-1}}{n}\right).$$

(iv.) If $$I = P\int_{-\infty}^{\infty} \frac{\sin\mu x}{x-\alpha} dx \quad (\mu, \alpha > 0),$$

we find $$\frac{d^2I}{d\alpha^2} + p^2 I = 0, \quad I = \pi \cos p\alpha,$$

and so $$P\int_0^\infty \frac{x \sin px}{x^2-\alpha^2}\,dx = \tfrac{1}{2}\pi \cos \alpha p, \qquad P\int_0^\infty \frac{\cos px}{x^2-\alpha^2}\,dx = -\frac{\pi}{2\alpha}\sin \alpha p.$$

Infinite Limits; the general case.

10. None of the preceding theorems cover the case in which the number of singular curves is infinite.*

THEOREM 4.—*If* $f(x, \alpha)$, $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial \alpha}$ *are continuous throughout any finite part of*

$$(a, \infty, \alpha_0 - H, \alpha_0 + H),$$

except upon a finite number of curves $x = X_i(\alpha)$, *which satisfy the same conditions as in* 1, *and*

$$\lim_{\epsilon = 0} \frac{dX_i}{d\alpha}\{f(X_i - \epsilon, \alpha) - f(X_i + \epsilon, \alpha)\} = 0$$

uniformly for all values of α *in* $(\alpha_0 - H, \alpha_0 + H)$, *and*

$$P\int_a^\infty \frac{\partial f}{\partial \alpha}\,dx$$

is uniformly convergent, then will

$$\frac{d}{d\alpha} P\int_a^\infty f\,dx = P\int_a^\infty \frac{\partial f}{\partial \alpha}\,dx.$$

For let $\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots$ be any series of descending positive quantities whose limit is 0. We can choose a value of a_1 such that

$$P\int_a^{a_1} \frac{\partial f}{\partial \alpha}\,dx$$

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* For the purposes of these theorems I shall alter slightly the meaning of the expressions *uniformly convergent*, *regularly convergent*, defined in my last paper for principal values whose upper limit is ∞.

In the definition of uniform convergence (II., § 19) condition (ii.) is to read "however small be the positive quantity σ, and however great be H, we can find a value of $A > H$, such that ..."

In the definition of regular convergence (II., § 21) condition (iii.) is to read "however small be the positive quantity σ, and however great be H, we can find (1) a value of $A > H$, (2) ....

The definitions, in their original form, suffice for the deductions drawn from them in my former paper. But we have not been and shall not be concerned with any principal values which satisfy the original and not the modified form of them.

is uniformly convergent, and

$$\left| P\int_{a_1}^{x} \right| < \sigma_1$$

for all values of α in $(a_0-H,\ a_0+H)$; a value of a_2, such that $P\int_{a_1}^{a_2}$ is uniformly convergent, and $\left| P\int_{a_2}^{x} \right| < \sigma_2$; and so on. We can suppose $a_1<a_2<\ldots$, $\lim a_n=\infty$. Then none of the curves $x=X_i$ can meet any of the lines $x=a_1, a_2, \ldots$; and the conditions of I. are satisfied in each of the regions

$$(a_n,\ a_{n+1},\ a_0-H,\ a_0+H),$$

and

$$\frac{d}{d\alpha}P\int_{a_n}^{a_{n+1}} f\,dx = P\int_{a_n}^{a_{n+1}} \frac{\partial f}{\partial \alpha}\,dx.$$

Moreover the series

$$\sum_0^x \frac{d}{d\alpha}P\int_{a_n}^{a_{n+1}} f\,dx$$

$(a_0=a)$ is uniformly convergent. Therefore

$$\frac{d}{d\alpha}P\int_a^x f\,dx = \frac{d}{d\alpha}\sum_0^x P\int_{a_n}^{a_{n+1}} f\,dx = \sum_0^x \frac{d}{d\alpha}P\int_{a_n}^{a_{n+1}} f\,dx$$

$$= \sum_0^\infty P\int_{a_n}^{a_{n+1}} \frac{\partial f}{\partial \alpha}\,dx = P\int_a^\infty \frac{\partial f}{\partial \alpha}\,dx.$$

11. (i.) Thus, if

$$I(\alpha) = P\int_0^\infty \sin \alpha x \tan x \frac{dx}{x^2},$$

$$\frac{dI}{d\alpha} = P\int_0^\infty \cos \alpha x \tan x \frac{dx}{x} = \tfrac{1}{2}\pi.$$

And, as $I(\alpha)$ is continuous for $\alpha = 0$,

$$I(\alpha) = \tfrac{1}{2}\pi\alpha.$$

Similarly

$$P\int_0^\infty \frac{\cos \alpha x - \cos \beta x}{x^3} \tan x\,dx = \tfrac{1}{4}\pi(\alpha^2-\beta^2).$$

Similarly we can establish the following results:—(ii.) If $a>0$,

$$P\int_0^\infty \log\left(1+\frac{a^2}{x^2}\right)\frac{dx}{\cos x} = 2\pi \tan^{-1}\tanh \tfrac{1}{2}a,$$

$$P\int_0^x \log\left(1-\frac{a^2}{x^2}\right)^2 \frac{dx}{\cos x} = 0,\ \pi^2,\ 2\pi^2,$$

according as $(2n-\frac{1}{2})\pi < a < (2n+\frac{1}{2})\pi$, $a=(n+\frac{1}{2})\pi$, or $(2n+\frac{1}{2})\pi < a < (2n+\frac{3}{2})\pi$. In the latter case we prove first that the integral is constant where continuous, and evaluate its discontinuities by II., § 36.

(iii.) If $$0 < (m+\tfrac{1}{2})\pi < \beta < \alpha < (m+\tfrac{3}{2})\pi,$$

$$P\int_0^\infty \frac{\log\left(1-\frac{\alpha^2}{x^2}\right)^2}{\beta^2-x^2}\,\frac{dx}{\cos x} = 2\pi^2\left[\frac{1}{2\beta\cos\beta}+\sum_0^m \frac{(-)^\kappa}{\beta^2-\left\{(\kappa+\frac{1}{2})\pi\right\}^2}\right].$$

(iv.) $$P\int_0^\infty \frac{1}{\cos x-\cos\alpha}\,\frac{dx}{a^2+x^2} = \frac{\pi}{2a\,(\cosh a-\cos\alpha)}.$$

12. Theorem 5.—*If the conditions of* 4 *are satisfied, except that*

$$P\int_a^\infty \frac{\partial f}{\partial \alpha}\,dx$$

is only regularly convergent,

$$\frac{d}{d\alpha}P\int_a^x f\,dx = P\int_a^x \frac{\partial f}{\partial \alpha}\,dx.$$

For let $\sigma_1, \sigma_2, \ldots$ be a series of descending positive quantities whose limit is 0. We can choose a value of a_1, a set of positive quantities $p_{1,i}$, each less than some fixed quantity p_0, and a division of (a_0-H, a_0+H) into two sets of partial intervals $\theta_{1,i}$, $\eta_{1,i}$, such that

$$P\int_a^{a_1}\frac{\partial f}{\partial \alpha}\,dx$$

is uniformly convergent in $\theta_{1,i}$ and

$$P\int_a^{a_1-p_{1,i}}\frac{\partial f}{\partial \alpha}\,dx$$

in $\eta_{1,i}$; and, moreover, $\left|P\int_a^{a_1}\right| < \sigma_1$ in $\theta_{1,i}$, and $\left|P\int_a^{a_1-p_{1,i}}\right| < \sigma$ in $\eta_{1,i}$. If then we define $a_1'(\alpha)$ as being $= a_1$ in θ_1 and $a_1-p_{1,i}$ in $\eta_{1,i}$, $P\int_a^{a_1'}$ is uniformly convergent, and

$$\left|P\int_a^{a_1'}\right| < \sigma_1$$

for all values of α in (a_0-H, a_0+H). We next choose a_2, $p_{2,i}$, $\theta_{2,i}$, $\eta_{2,i}$, similarly corresponding to σ_2, and define $a_2'(\alpha)$ in the same way; and so on. And we can suppose that the least value of $a_{n+1}'(\alpha)$ is greater than the greatest of $a_n'(\alpha)$, and that $\lim_{n=\infty} a_n'(\alpha) = \infty$ uniformly for all values of α. Then, if $a_0' = a$,

$$P\int_a^\infty f\,dx = \sum_0^\infty P\int_{a_n'}^{a_{n+1}'}, \quad P\int_a^\infty \frac{\partial f}{\partial \alpha}\,dx = \sum_0^\infty P\int_{a_n'}^{a_{n+1}'};$$

and the second series is uniformly convergent. So the theorem follows as before.*

13. If, *e.g.*,

$$I(\alpha) = \int_0^\infty \frac{\log\cos^2 \alpha x}{x^2}\,dx \quad (\alpha > 0),$$

$$\frac{dI}{d\alpha} = -2P\int_0^\infty \frac{\tan \alpha x}{x}\,dx = -\pi;$$

$$I(\alpha) = -\alpha\pi.$$

Again, it is easy to prove that

$$\int_0^\infty \cos ax \log\cos^2 \alpha x\,\frac{dx}{1+x^2} = \pi\cosh a \log(1+e^{-2\alpha}) - \pi e^{-a}\log 2,$$

$$P\int_0^\infty \cos ax \log\cos^2 \alpha x\,\frac{dx}{1-x^2} = \pi\alpha\cos a - \pi\sin a\log 2,$$

if $0 < a < 2\alpha$, and in the second formula $2\alpha < \pi$. Differentiating,

$$P\int_0^\infty \cos ax \tan \alpha x\,\frac{x\,dx}{1+x^2} = \frac{\pi\cosh a}{e^{2\alpha}+1},$$

$$P\int_0^\infty \cos ax \tan \alpha x\,\frac{x\,dx}{1-x^2} = -\tfrac{1}{2}\pi\cos a.$$

If

$$I(\alpha) = \int_0^\infty \log\left(1-\frac{a^2}{x^2}\right)^2 \log\cos^2 \alpha x\,dx,$$

$$\frac{dI}{d\alpha} = -2P\int_0^\infty \log\left(1-\frac{a^2}{x^2}\right)^2 \tan \alpha x\,dx,$$

$$\frac{d^2I}{d\alpha\,da} = -8aP\int_0^\infty \frac{x\tan \alpha x}{a^2-x^2}\,dx = 4a\pi \quad \left(0 < a,\ \ 0 < \alpha < \frac{\pi}{2a}\right).$$

Hence

$$\frac{dI}{d\alpha} = 2a^2\pi, \quad I = 2a^2\alpha\pi.$$

14. A good deal of the substance of §§ 2–13 appeared in a paper "On Differentiation and Integration under the Integral Sign" (*Quarterly Journal of Mathematics*, No. 125, 1900, p. 66).

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* There is a difficulty here which should be expressly mentioned. If

$$\int_{\alpha_n}^{\alpha'_{n+1}} u_n, \quad P\int_{\alpha_n}^{\alpha'_{n+1}} = u'_n,$$

$\frac{du_n}{d\alpha} = u'_n$ for $\alpha = \alpha_0$, and indeed throughout an interval $(\alpha_0 - H_n,\ \alpha_0 + H_n)$, which may, however, decrease indefinitely as n increases. Since α'_n is not continuous throughout $(\alpha_0 - H,\ \alpha_0 + H)$, neither u_n nor u'_n is as a rule continuous throughout that interval for all values of n; and for certain values of α, which may approach indefinitely near to α_0 as n increases, $\frac{du_n}{d\alpha}$ ceases to be determinate at all. The series Σu_n may none the less be differentiated for $\alpha = \alpha_0$. I omit the formal proof of this statement, as it is detailed and perhaps tedious.

Integration under the sign of the Principal Value.

15. If $f(x, y)$ is a continuous function of both variables in the rectangle (a, A, b, B),

$$\int_b^B dy \int_a^A f(x, y)\, dx = \int_a^A dx \int_b^B f(x, y)\, dy.$$

But this equation is true under much more general conditions, which have been studied by many writers, among whom I may particularly mention M. Ch. de la Vallée-Poussin. It is sufficient, for instance, (when the limits are finite), that $f(x, y)$ be finite throughout any part of (a, A, b, B) which does not contain any point situated on a set of curves satisfying certain conditions, and

$$\int_a^A f(x, y)\, dx, \quad \int_b^B f(x, y)\, dy$$

be uniformly convergent.

I shall not, however, enter into any discussion of these general conditions in this paper. The question which concerns us now is: *Under what circumstances is* (1) *true when some or all of the integrals contained in it are only principal values?* And I have already pointed out that it is not worth while to attempt to state theorems connected with the principal value with all the generality we can give them. What is worth doing is to distinguish and examine the various simple cases which occur when we are dealing with functions which present themselves naturally in analysis.

We shall suppose then that $f(x, y)$ behaves in a normal manner throughout (a, A, b, B), except in the immediate neighbourhood of a finite number of simple curves, along which it becomes infinite in such a way that its integral with respect to x or y is not unconditionally convergent across them.

16. The simplest case is that in which these curves are straight lines parallel to the axes.

Let us suppose, in the first place, that $f(x, y)$ is continuous except for $x = \alpha$. Then, however small be the positive quantity ϵ,

$$\left(\int_a^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^A\right) dx \int_b^B f\, dy = \int_b^B dy \left(\int_a^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^A\right) f\, dx.$$

Further, let us suppose that

$$P\int_a^A f\, dx$$

is uniformly convergent. Then, however small be σ, we can choose ϵ so that

$$\left| P\int_{\alpha-\epsilon}^{\alpha+\epsilon} f\,dx \right| < \sigma$$

throughout (b, B). Thus the limit of the right hand is

$$\int_b^B dy\, P\int_a^A f\,dx;$$

and so the left hand also tends to a limit which is, by definition,

$$P\int_a^A dx \int_b^B f\,dy.$$

THEOREM 6.—*If $f(x, y)$ is a continuous function of both variables throughout any part of (a, A, b, B) which does not include any point of the line $x = \alpha$, and*

$$P\int_a^A f\,dx$$

is uniformly convergent, then

$$P\int_a^A dx\int_b^B f\,dy = \int_b^B dy\, P\int_a^A f\,dx.$$

17. If $a = b = 0$, $A = B = c$, and

$$f(x, y) = \frac{1}{(x-\alpha)(x+y+\beta)} \quad (0 < \beta,\ 0 < \alpha < c),$$

we find on integration that

$$P\int_0^c \log\left(1+\frac{c}{x+\beta}\right)\left(\frac{1}{x-\alpha}+\frac{1}{x+\alpha+\beta}\right)dx = \log\frac{c-\alpha}{\alpha}\log\frac{c+\alpha+\beta}{\alpha+\beta}.$$

18. Let us suppose now that B is infinite, and that, for any finite value of $B' > b$,

$$P\int_a^A dx\int_b^{B'} f\,dy = \int_b^{B'} dy\, P\int_a^A f\,dx;$$

then, if $\int_b^\infty f\,dy$ is convergent, except for $x = \alpha$, and

$$\lim_{B'=\infty} P\int_a^A dx\int_{B'}^\infty f\,dy = 0,$$

this equation passes over in the limit into

$$P\int_a^A dx\int_b^\infty f\,dy = \int_b^\infty dy\, P\int_a^A f\,dx.$$

Suppose, for instance, that

$$f = \frac{\psi(x, y)}{x - \alpha},$$

ψ being a function whose derivate $\frac{\partial \psi}{\partial x}$ is continuous throughout (a, A, b, ∞), and that

$$\int_b^\infty \psi \, dy, \quad \int_b^x \frac{\partial \psi}{\partial x} \, dy$$

are uniformly convergent in (a, A); then

$$\Psi(x) = \int_{B'}^x \psi \, dy$$

is continuous, and has a continuous derivate represented by

$$\int_{B'}^x \frac{\partial \psi}{\partial x} \, dy.$$

Also

$$P \int_a^A dx \int_{B'}^\infty f \, dy$$

is determinate, and equal to

$$\left(\int_a^{\alpha - \epsilon} + \int_{\alpha + \epsilon}^A \right) \int_{B'}^x + P \int_{\alpha - \epsilon}^{\alpha + \epsilon} \int_{B'}^\infty$$

for any small positive value of ϵ. The last term

$$= 2\epsilon \Psi'(\alpha + \theta\epsilon) \quad (-1 \leq \theta \leq 1);$$

and this is numerically $< \epsilon K$, where K is a quantity independent of ϵ and of B'. We can therefore make it less than any assigned positive quantity $\frac{1}{2}\sigma$ by choice of a value of ϵ independent of B'. We can then choose B' so that

$$\left| \int_{B_1}^\infty \psi \, dy \right| < \frac{\sigma\epsilon}{2(A - a)}$$

for all values of x in $(a, \alpha - \epsilon)$ and $(\alpha + \epsilon, A)$, and all values of $B_1 \geq B'$; and therefore

$$\left| \left(\int_a^{\alpha - \epsilon} + \int_{\alpha + \epsilon}^A \right) \frac{dx}{x - \alpha} \int_{B_1}^\infty \psi \, dy \right| < \tfrac{1}{2}\sigma.$$

Hence

$$\left| P \int_a^A dx \int_{B_1}^\infty f \, dy \right| < \sigma;$$

so that

$$\lim_{B' = \infty} P \int_a^A dx \int_{B'}^\infty f \, dy = 0.$$

This is equally true if for $\frac{1}{x-a}$ we substitute the more general factor $\Omega_\nu(x-a)$.

19. The extension of §§ 16, 18 to the case in which $A = \infty$ does not present any fresh difficulty which is particularly interesting to us now. For we have only to combine the equations

$$P\int_a^A dx\int_b^B f\,dy = \int_b^B dy\,P\int_a^A f\,dx$$

and

$$\int_A^\infty dx\int_b^B f\,dy = \int_b^B dy\int_A^\infty f\,dx\,;$$

and the difficulties which may meet us when we try to prove this last are not those with which this paper is concerned.

20. We can prove, for instance, without much difficulty, that the theorem holds if $a = b = 0$, $A = B = \infty$, and

$$f(x, y) = \frac{\cos xy}{(x^2-\alpha^2)(1+y^2)} \quad (\alpha > 0).$$

We deduce

$$\int_0^\infty \frac{\sin \alpha y\,dy}{1+y^2} = -\frac{2\alpha}{\pi}\int_0^\infty \frac{dy}{1+y^2}\,P\int_0^\infty \frac{\cos xy}{x^2-\alpha^2}\,dx$$

$$= -\frac{2\alpha}{\pi}\,P\int_0^\infty \frac{dy}{x^2-\alpha^2}\int_0^\infty \frac{\cos xy}{1+y^2}\,dy$$

$$= \tfrac{1}{2}\left\{e^{-\alpha}\,li\,(e^{\alpha}) - e^{\alpha}\,li\,(e^{-\alpha})\right\},$$

a result proved otherwise by Schlömilch and Kronecker.

21. It is to be observed that we need not insist on the condition laid down in § 18, that

$$\int_b^\infty \frac{\partial \psi}{\partial x}\,dy$$

should be uniformly convergent, if we can assure ourselves that

$$\Psi(x) = \int_{B'}^\infty \psi\,dy$$

has a continuous derivate always numerically less than some quantity independent of x and B'.

This case occurs, *e.g.*, if

$$f(x, y) = \frac{\cos xy}{(x^2-\alpha^2)\,y^\mu} \quad (0 < \mu < 1),$$

when we deduce

$$P\int_0^\infty \frac{t^{\nu-1}dt}{1-t} = \pi\cot\nu\pi \quad (0 < \nu < 1).$$

22. Let us suppose next that the limits are finite, and f continuous except along $x = \alpha$ and $y = \beta$, where $u < \alpha < A$, $b < \beta < B$; and that

$$P\int_a^A f\,dx, \quad P\int_b^B f\,dy$$

are uniformly convergent, except for $y = \beta$ and $x = \alpha$ respectively. Then

$$P\int_a^A dx\left(\int_b^{\beta-\theta} + \int_{\beta+\theta}^B\right) f\,dy = \left(\int_b^{\beta-\theta} + \int_{\beta+\theta}^B\right) dy\, P\int_a^A f\,dx,$$

however small be θ. This equation will pass over in the limit into

$$(1) \quad P\int_a^A dx\, P\int_b^B f\,dy = P\int_b^B dy\, P\int_a^A f\,dx,$$

provided only

$$P\int_a^A dx\, P\int_{\beta-\theta}^{\beta+\theta} f\,dx$$

be determinate, and tend to zero for $\theta = 0$. And it is clear that this will be the case if the same is true of

$$P\int_{\alpha-\epsilon}^{\alpha+\epsilon} P\int_{\beta-\theta}^{\beta+\theta} = P\int_{-\epsilon}^{\epsilon} d\xi\, P\int_{-\theta}^{\theta} f(\alpha+\xi, \beta+\eta)\,d\eta.$$

Now let us suppose that

$$f(x, y) = \frac{\psi(x, y)}{(x-\alpha)(y-\beta)},$$

where ψ is a function which has continuous first derivates throughout (a, A, b, B). To save unnecessary discussion we shall assume that $\psi(x, y)$ is capable of expansion in a Taylor's series in the neighbourhood of (α, β). Then

$$\psi(\alpha+\xi, \beta+\eta) = \psi(\alpha, \beta) + \xi\psi_1(\xi, \eta) + \eta\psi_2(\xi, \eta),$$

where ψ_1, ψ_2 are functions which also behave regularly near (α, β). We consider the integrals arising from these three terms separately.

(i.) Since

$$P\int_{-\eta}^{\eta} \frac{d\eta}{\eta} = 0,$$

the first term contributes nothing.

(ii.) The second contributes

$$\int_{-\epsilon}^{\epsilon} d\xi\, P\int_{-\eta}^{\eta} \frac{\psi_1(\xi, \eta)}{\eta}\,d\eta = 2\theta\int_{-\epsilon}^{\epsilon} \frac{\partial\psi_1(\xi, \eta')}{\partial\eta'}\,d\xi \quad (-\epsilon \leq \eta' \leq \epsilon),^*$$

which is determinate and tends to zero for $\theta = 0$.

* I., § 12.

(iii.) The third contributes

$$P\int_{-\epsilon}^{\epsilon}\frac{d\xi}{\xi}\int_{-\theta}^{\theta}\psi_2(\xi,\eta)\,d\eta.$$

This is also determinate and tends to zero for $\theta=0$; for

$$\int_{-\theta}^{\theta}\psi_2\,d\eta$$

is a function of ξ which possesses a continuous derivate

$$\int_{-\theta}^{\theta}\frac{\partial\psi_2}{\partial\xi}\,d\eta,$$

and

$$P\int_{-\epsilon}^{\epsilon}\frac{d\xi}{\xi}\int_{-\theta}^{\theta}\psi_2\,d\eta=2\epsilon\left[\int_{-\theta}^{\theta}\frac{\partial\psi_2}{\partial\xi}\,d\eta\right]_{\xi=\xi'},$$

where

$$-\epsilon\leqq\xi'\leqq\epsilon.$$

Hence in this case

$$P\int_a^A dx\,P\int_b^B f\,dy=P\int_b^B dy\,P\int_a^A f\,dx.$$

And it is easy to see that the same conclusion holds in the more general case in which

$$f(x,y)=\Omega_\mu(x-\alpha)\,\Omega_\nu(y-\beta)\,\psi(x,y).$$

23. So long as there is but a finite number of singular lines parallel to either axis, the extension of (1) of § 22 to the case in which A or B or both are infinite does not present any fresh difficulty, as the rectangle (a,∞,b,∞) can be divided up into a finite number of partial rectangles each of which satisfies the conditions of one or other of the preceding sections.

24. If

$$a=b=0,\quad A=B=\infty\quad(\alpha,\beta>0),$$

and

$$f(x,y)=\frac{1}{(x+y)(x-\alpha)(y-\beta)},$$

we obtain, after some reduction,

$$P\int_0^\infty\frac{\log u\,du}{(u+\alpha)(u-\beta)}=\frac{1}{\alpha+\beta}\left\{\pi^2+(\log\alpha)^2-(\log\beta)^2\right\}.$$

25. It sometimes happens that the formula

$$\int_a^A dx\int_b^B f\,dy=\int_b^B dy\int_a^A f\,dx$$

is only true if we introduce the sign of the principal value before

one of the outer integral signs. We may have

$$(1)\quad \left(\int_a^{\alpha-\epsilon}+\int_{\alpha+\epsilon}^{A}\right)\int_b^B=\int_b^B\left(\int_a^{\alpha-\epsilon}+\int_{\alpha+\epsilon}^{A}\right),$$

and $\int_a^A$, $\int_b^B\int_a^A$ convergent, while $\int_a^A\int_b^B$ is not so. If, however,

$$\lim\int_b^B\int_{\alpha-\epsilon}^{\alpha+\epsilon}=0,$$

equation (1) will pass over in the limit into

$$P\int_a^A dx\int_b^B f\,dy=\int_b^B dy\int_a^A f\,dx.$$

This case is not of the same type as those which we have been discussing.

26. Suppose $$b=0,\quad B=\infty,$$

and $$f(x,y)=e^{-y(1+\cos x)}\sin(x-y\sin x)\,\phi(x),$$

$\phi(x)$ being a function of x which possesses a continuous derivate. Then, so long as x is not an odd multiple of π,

$$\int_0^\infty f\,dy=\frac{\sin x\,(1+\cos x)-\cos x\,\sin x}{(1+\cos x)^2+\sin^2 x}\,\phi(x)=\tfrac{1}{2}\tan\tfrac{1}{2}x\,\phi(x).$$

Hence, if A is not an odd integer,

$$P\int_a^A\tfrac{1}{2}\tan\tfrac{1}{2}x\,\phi(x)\,dx=\int_0^\infty e^{-y}\,dy\int_a^A e^{-y\cos x}\sin(x-y\sin x)\,\phi(x)\,dx\,;$$

provided the condition of the preceding section is satisfied.

Now $$e^{-y\cos x}\sin(x-y\sin x)=\sum_1^\infty(-)^{n-1}\frac{y^n\sin nx}{n!},$$

and so
$$\begin{aligned}P\int_a^A\tfrac{1}{2}\tan\tfrac{1}{2}x\,\phi(x)\,dx&=\int_0^\infty e^{-y}\,dy\int_a^A\phi(x)\sum_1^\infty(-)^{n-1}\frac{y^n\sin nx}{n!}\,dx\\&=\int_0^\infty e^{-y}\,dy\sum_0^\infty(-)^{n-1}\frac{y^n}{n!}\int_a^A\sin nx\,\phi(x)\,dx\\&=\sum_0^\infty(-)^{n-1}\frac{1}{n!}\int_0^\infty e^{-y}y^n\,dy\int_a^A\sin nx\,\phi(x)\,dx\\&=\sum_0^\infty(-)^{n-1}\int_a^A\sin nx\,\phi(x)\,dx,\end{aligned}$$

provided this series be convergent. As this result has already been obtained by another method in II. (§ 28), I shall not delay over the proof that the inversion of the order of integration is as a matter of fact legitimate.

27. So far I have supposed that the singular lines are all straight lines parallel to an axis. I shall now consider the case in which they are continuous curves never parallel to either axis. The

simplest such curve is the line

$$x = y,$$

and I shall begin by supposing that this is the only singular curve. It will not be difficult to generalize the results.

28. Suppose, then, that $f(x, y)$ is a function whose derivates $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are continuous throughout any part of the square

$$(a, A, a, A)$$

which does not include any point of the line $x = y$; and that, near $x = y$, $f(x, y)$ may be expressed in the form

$$\Omega_\nu (x-y)\, \Theta (x, y),$$

Θ being a function whose derivates $\frac{\partial \Theta}{\partial x}$, $\frac{\partial \Theta}{\partial y}$ are continuous without exception.

Then $P\int_a^A f\, dy$ is uniformly convergent in $x = (a, A)$, except for $x = a, A$; and $P\int_a^A f\, dx$ in $y = (a, A)$, except for $y = a, A$. And the function

$$\phi (x, y) = P\int_a^y f\, dy$$

is a continuous function of both variables except along the line $x = y$, and as y approaches x becomes infinite (if at all) in such a way that

$$\lim_{\eta = 0} \eta^\mu \phi (x, x-\eta) = 0$$

for any positive value of μ. And

$$\int_a^A \phi (x, y)\, dx$$

is uniformly convergent in (a, A).

Similarly, if $$\psi (x, y) = P\int_a^x f\, dx,$$

$\int_a^A \psi (x, y)\, dy$ is uniformly convergent in (a, A). We exclude $x = y$ from the field of integration by the two lines $x-y = \pm\epsilon$. Applying the ordinary theorem, that the order of integration is indifferent, to that part of the remainder of the field which lies between

$$y = a+\epsilon, \quad y = A-\epsilon,$$

we find (Fig. 1)

$$(1)\quad \int_{a+\epsilon}^{A-\epsilon} dy \left(\int_{a}^{y-\epsilon} + \int_{y+\epsilon}^{A} \right) f\,dx = \int_{a}^{A-2\epsilon} dx \int_{x+\epsilon}^{A-\epsilon} f\,dy + \int_{a+2\epsilon}^{A} dx \int_{a+\epsilon}^{x-\epsilon} f\,dy.$$

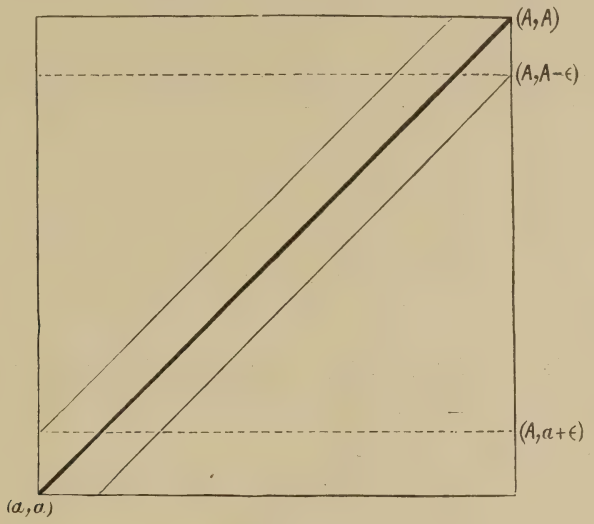

FIG. 1.

Now, if $$a+\epsilon \leqq y \leqq A-\epsilon,$$

$$P\int_{y-\epsilon}^{y+\epsilon} f\,dx = \left[\frac{\partial\Theta}{\partial x}\right]_{x=y+\theta\epsilon} \int_{y-\epsilon}^{y+\epsilon} (x-y)\,\Omega_{\nu}(x-y)\,dx,$$

where $$-1 \leqq \theta \leqq 1.$$

We can therefore choose ϵ_0 so that $P\int_{y-\epsilon}^{y+\epsilon}$ is numerically less than any assigned positive quantity $\dfrac{\sigma}{2(A-a)}$ for all values of $\epsilon \leqq \epsilon_0$. Then the left-hand side of (1) differs from

$$\int_{a+\epsilon}^{A-\epsilon} dy\, P\int_{a}^{A} f\,dx$$

by less than $\frac{1}{2}\sigma$. But we can also suppose ϵ_0 so small that

$$\int_{a}^{a+\epsilon} P\int_{a}^{A}, \quad \int_{A-\epsilon}^{A} P\int_{a}^{A}$$

are also numerically less than $\frac{1}{4}\sigma$. Then the left-hand side of (1) differs from

$$\int_{a}^{A} dy\, P\int_{a}^{A} f\,dx$$

by less than σ. And the right-hand side of (1) is

$$\int_{a+2\epsilon}^{A-2\epsilon} dx \left(\int_{x+\epsilon}^{x-\epsilon} + \int_{x+\epsilon}^{A-\epsilon} \right) f\, dy + \Delta,$$

where $$\Delta = \int_a^{a+2\epsilon} dx \int_{x+\epsilon}^{A-\epsilon} f\, dy + \int_{A-2\epsilon}^{A} dx \int_{a+\epsilon}^{x-\epsilon} f\, dy.$$

And an argument similar to that used above shows that ϵ_0 can be chosen so small that this differs from

$$\int_a^A dx\, P \int_a^A f\, dy$$

by less than σ. Hence

$$\int_a^A dy\, P \int_a^A f\, dx = \int_a^A dx\, P \int_a^A f\, dy.$$

28. Now, let $y = X(u)$, $X(u)$ being a function which has continuous derivates $\frac{dX}{du}$, $\frac{d^2X}{du^2}$, the first of which does not vanish between $y = a$, A. And suppose

$$X(b) = a, \quad X(B) = A.$$

Then $$F(x, u) = f\{x, X(u)\}$$

may be expressed in the form

$$\bar{F} + \sum_{i=1}^{n} \Omega_{\nu_i} \{x - X(u)\} \Phi_i(x, u),$$

where $\bar{F}$ is a function which is continuous throughout

$$(a, A, b, B),$$

except on $x = X(u)$, and possesses an unconditionally and uniformly convergent integral across it; and Φ_ν is a function whose derivates $\frac{\partial \Phi_\nu}{\partial x}$, $\frac{\partial \Phi_\nu}{\partial u}$ are continuous without exception.

Also $$P \int_a^A f\, dx = P \int_a^A F(x, u)\, dx,$$

and, by I., § 21, $$P \int_a^A f\, dy = P \int_b^B F(x, u) \frac{dX}{du}\, du;$$

and so $$\int_b^B du\, P \int_a^A G\, dx = \int_a^A dx\, P \int_b^B G\, du,$$

where $$G = F(x, u)\frac{dX}{du}$$

is a function of the form

$$G + \sum_{i=1}^{n} \Omega_{\nu_i} \{x - X(u)\} \Psi_i.$$

We may suppose, moreover, that G is *any* expression of this kind; for any such expression, when expressed in terms of y, becomes a sum of terms to each of which the argument of the preceding pages may be applied. And the conditions satisfied by $X(u)$ amount to this, that $x = X(u)$ is a curve of continuous curvature which is never parallel to x or u, and passes through two corners of the rectangle (a, A, b, B).

29. THEOREM 7. — *If* $f(x, y)$ *is a function whose derivates* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ *are continuous throughout*

$$(a, A, b, B),$$

except on a finite number of curves of continuous curvature

$$x = X_i(y), \quad y = Y_i(x),$$

which do not intersect and are nowhere parallel to the axes, and if $f(x, y)$, *in the immediate neighbourhood of any one of these curves, can be expressed in the form*

$$\bar{f}(x, y) + \sum_{1}^{n} \Omega_{\nu_\kappa} \{x - X_i(y)\} \Theta_\kappa(x, y),$$

or in the form $$\bar{g}(x, y) + \sum_{1}^{n} \Omega_{\nu_\lambda} \{y - Y_i(x)\} \Theta_\lambda(x, y),$$

where $\bar{f}(x, y)$ *is a function which becomes at most logarithmically infinite along* $x = X_i(y)$, *and* Θ_κ *a function whose derivates* $\frac{\partial \Theta}{\partial x}$, $\frac{\partial \Theta}{\partial y}$ *are continuous without exception, then will*

$$\int_a^A dx \left(P \int_b^B f\, dy \right) = \int_b^B dy \left(P \int_a^A f\, dx \right).$$

For (see Fig. 2) we can divide the rectangle (a, A, b, B) into a finite number of rectangles, to each of which we may apply either the equation proved in the last paragraph or the ordinary theorem as to the interchange of two integrations.

30. If $$a = b = -1, \quad A = B = 1, \quad \alpha, \beta > 1,$$

and $$f(x, y) = \frac{1}{(x-y)(x+\alpha)(y+\beta)},$$

we deduce, after some reduction,*

$$\int_{-1}^{1} \log\left(\frac{1-u}{1+u}\right)\frac{du}{\alpha+u} = \tfrac{1}{2}\left(\log\frac{\alpha+1}{\alpha-1}\right)^2.$$

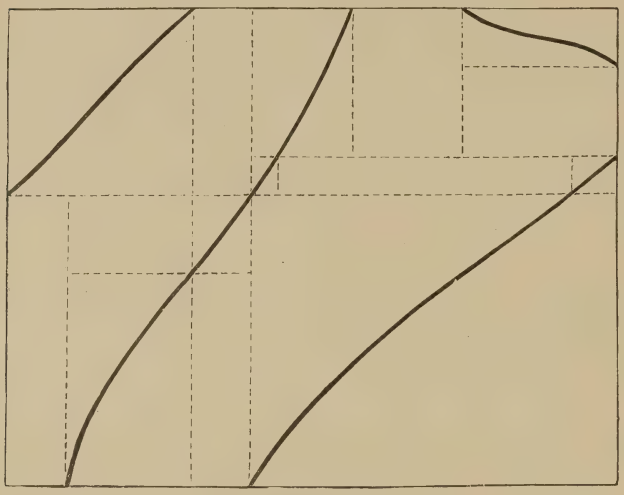

FIG. 2.

31. A considerable variety of different cases may occur when we suppose A or B infinite, but the singular curves still finite in number. If, for instance, $x = y$ is the only singular curve, no new difficulty arises when one of A, B is infinite; but when they are both infinite further discussion is necessary.

32. Let $$a = 0, \quad A = \infty, \quad b = 0, \quad B = H > 1,$$

and $$f(x, y) = \frac{\phi(y)}{x^2-y^2+1},$$

$\phi(y)$ being a function whose derivate is continuous. The singular curve is the hyperbola $x^2-y^2+1 = 0$ (Fig. 3), and satisfies the conditions of 7, except at (0, 1), where it becomes parallel to x. Hence, if $\epsilon > 0$,

$$\int_{\epsilon}^{\infty} dx\, P\int_0^H \frac{\phi(y)\,dy}{x^2-y^2+1} = \int_0^H \phi(y)\,dy\, P\int_{\epsilon}^{\infty} \frac{dx}{x^2-y^2+1}.$$

But it is not difficult to prove that

$$\lim_{\epsilon=0}\int_0^H \phi(y)\,dy\, P\int_0^{\epsilon} \frac{dx}{x^2-y^2+1} = 0.$$

Hence $$\int_0^{\infty} dx\, P\int_0^{\infty} \frac{\phi(y)\,dy}{x^2-y^2+1} = \int_0^H \phi(y)\,dy\, P\int_0^{\infty} \frac{dx}{x^2-y^2+1} = \tfrac{1}{2}\pi\int_0^1 \frac{\phi(y)\,dy}{\sqrt{(1-y^2)}},$$

from which we can derive various integrals.

* This example is worked out in detail in the paper in the *Quarterly Journal* already referred to, p. 131.

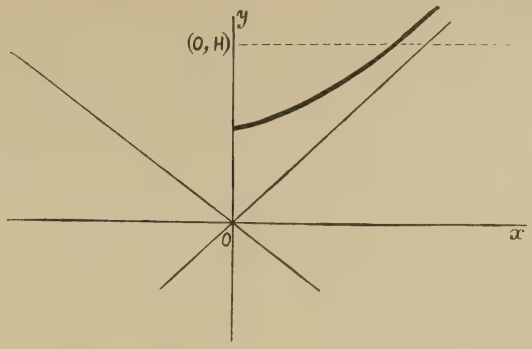

FIG. 3.

33. When A and B are both infinite there is another condition to be attended to. Let us suppose that there is one singular curve (such as $x-y=0$ or $x^2-y^2+1=0$) which extends to infinity in the positive quadrant in a definite direction not parallel to x or y. Then for any finite values of A and B

$$\int_a^A P \int_b^B = \int_b^B P \int_a^A ;$$

and we may suppose one of the upper limits, say A, replaced by ∞, as in the example of § 32. We need not stop to discuss the conditions under which this is legitimate; for, if the singular curve meets $y=B$ in (A', B), we have only to combine the two equations

$$\int_a^{A_1} P \int_b^B = \int_b^B P \int_a^{A_1}, \qquad \int_{A_1}^{\infty} \int_b^B = \int_b^B \int_{A_1}^{\infty}$$

($A_1 > A'$; see Fig. 4). Let us suppose, therefore, that

$$\int_a^{\infty} P \int_b^B = \int_b^B P \int_a^{\infty}.$$

Then, if

$$\int_a^{\infty} P \int_B^{\infty}$$

is determinate, and tends to the limit 0 for $B=\infty$, this equation ultimately passes over into

$$\int_a^{\infty} P \int_b^{\infty} = \int_b^{\infty} P \int_a^{\infty}.$$

34. Let

$$a=-\infty, \quad b=0,$$

and

$$f(x, y) = \frac{e^{pix}\phi(y)}{x-y} \quad (p>0),$$

where $\phi(y)$ is a function whose derivate is continuous. Then, if the condition of

the previous section (or that which results by interchanging x and y) is satisfied,

$$(1)\quad \int_{-\infty}^{\infty} e^{pix}\,dx\,P\int_0^{\infty}\frac{\phi(y)\,dy}{x-y} = \pi i\int_0^{\infty} e^{piy}\,\phi(y)\,dy.$$

It can be shown (I omit the proof) that this equation certainly holds if $\int_0^{\infty}\phi(y)\,dy$ is absolutely convergent. If, *e.g.*,

$$\phi(y) = \frac{1}{1+y^2},$$

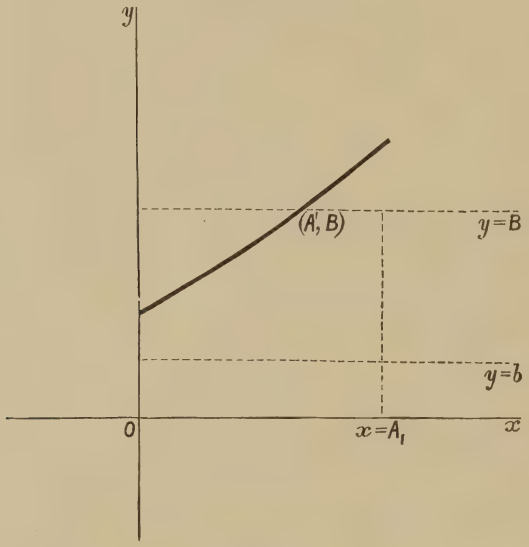

FIG. 4.

gives
$$\int_0^{\infty}\frac{\cos px\log x}{1+x^2}\,dx = \tfrac{1}{4}\left\{e^{p}\,li\,e^{-p} - e^{-p}\,li\,e^{p}\right\}.$$

But (1) also may be true if $\int_0^{\infty}\phi(y)\,dy$ is not absolutely convergent. If, *e.g.*,

$$\phi(y) = \frac{1}{y+\alpha}\quad(\alpha>0),$$

it gives $\int_0^{\infty}\log\left(\frac{x}{\alpha}\right)\frac{\cos px}{x^2-\alpha^2}\,dx = -\frac{\pi}{2\alpha}\left\{\left(\frac{\pi}{2}-Si\,p\alpha\right)\cos p\alpha - Ci\,p\alpha\sin p\alpha\right\}.$

35. Moreover (1) is only a particular case of the more general formula

$$(2)\quad \int_{-\infty}^{\infty} e^{pix}\,dx\,P\int_b^{\infty}\frac{\phi(y)}{x-\theta(y)}\,dy = \pi i\int_b^{\infty}\phi(y)\,e^{pi\theta(y)}\,dy,$$

which holds when $\theta(y)$ satisfies certain conditions.

If, *e.g.*, $\quad b=-\infty,\quad \phi(y)=\frac{1}{1+y^2},\quad \theta(y)=ay-\frac{b}{y}\quad(a,\,b>0),$

(2) becomes, after a little reduction,

$$\int_0^{\infty}\cos\left(ay-\frac{b}{y}\right)\frac{dy}{1+y^2} = \tfrac{1}{2}\pi e^{-a-b}.$$

36. There are two more questions which we must discuss in order to complete this series of investigations. In the first place, when two singular curves intersect within the field of integration, the formula

$$(1) \quad \int_a^A P \int_b^B = \int_b^B P \int_a^A$$

generally ceases to be true. If, for instance,

$$f(x, y) = \frac{\psi(x, y)}{\lambda(x, y)\,\mu(x, y)},$$

where ψ is a function with continuous derivates, and $\lambda = 0$, $\mu = 0$ are two curves which satisfy the conditions of 7, except that they intersect simply at the one point (α, β), the difference between the two sides of (1) will be

$$\frac{2\pi^2 \psi(\alpha, \beta)}{\frac{\partial(\lambda, \mu)}{\partial(\alpha, \beta)}}.$$

We shall have to discuss this case, and some other similar cases in which the corresponding "correction" or "residue" can be found.

In the second place, we must attempt to extend the theorems of the latter part of this paper to the case in which not only are the limits infinite, but the singular curves infinite in number.

Types of Perpetuants. *By* J. H. Grace.

Received and read June 12th, 1902.

1. I propose to apply the symbolical method of Aronhold directly to the discovery of the irreducible system of covariants of an indefinite number of binary forms of infinite order.

Suppose the forms are $a_x^n,\ b_x^n,\ c_x^n,\ \ldots$,

when n is indefinitely large; then the problem is to evolve a system of forms of the type

$$(ab)^\lambda (bc)^\mu (cd)^\nu \ldots a_x^p b_x^q \ldots$$

in terms of which all others can be expressed.

Since a covariant such as that written above is completely determined by its determinantal factors, we need specify these alone, and remark that the reduction of forms is simplified because the presence of the indefinitely large number of factors a_n renders possible the free application of the fundamental identities

$$(bc)\,a_x + (ca)\,b_x + (ab)\,c_x = 0\,;$$

or, as they stand for our purpose,

$$(bc) + (ca) + (ab) = 0.$$

2. Every covariant may be expressed in terms of those of the type

$$(ab)^\lambda\,(ac)^\mu\,(ad)^\nu\,\ldots,*$$

because factors of the type (bc) in which a does not occur may be eliminated by the use of the identity

$$(bc)\,a_x = (ba)\,c_x - (ca)\,b_x.$$

The original results of Cayley on the enumeration of these covariants follow at once, as is remarked in a similar way by Stroh.

3. Consider next the covariants of degree 3, namely,

$$(ab)^\lambda\,(bc)^\mu\,(ca)^\nu.$$

In virtue of the above we need only consider the type

$$(ab)^\lambda\,(ac)^\mu,$$

and of these $(ab)(ac)$ is reducible, because

$$2\,(ab)(ac) = (ab)^2 + (ac)^2 - (bc)^2.$$

Thus perpetuants of degree 3 of three different quantics are completely given by

$$(ab)^\lambda\,(ac)^\mu,$$

in which $\mu \geq 1$, $\lambda \geq 2$.

We shall next show that the perpetuants of degree 4 of four different quantics are given by

$$(ab)^\lambda\,(ac)^\mu\,(ad)^\nu,$$

when $\lambda \geq 4$, $\mu \geq 2$, $\nu \geq 1$; but to establish this we need to recall certain results from the great memoirs of Jordan (*Liouville's Journal*, 1876 and 1879).

* Provided, of course, that the symbol a occurs in the covariant; we shall henceforth consider those containing a and, in fact, a definite set of symbols.

4. If ξ, η, ζ be three quantities such that

$$\xi+\eta+\zeta=0,$$

then all homogeneous products

$$\xi^p\eta^q\zeta^r,$$

where

$$p+q+r=n,$$

can be expressed as linear combinations of

$$\xi^\rho\eta^\sigma, \quad \eta^\rho\zeta^\sigma, \quad \zeta^\rho\xi^\sigma,$$

where ρ, σ take all values subject to the condition that $\sigma \not> \frac{n}{3}$. I do not stop to reproduce the proof of this fundamental lemma of Jordan: it will suffice to refer to his papers cited above, and to remark that a different proof can be given depending on the theory of apolar forms.

5. Consider now the covariants of degree 4. As already observed, we need only take into account

$$(ab)^\lambda\,(ac)^\mu\,(ad)^\nu.$$

If now either λ or μ be less than 2—say, $\mu<2$—then, by transforming the expression

$$(ac)^\mu\,(ad)^\nu,$$

we can express it in terms of such as contain $(ac)^2$ at least and reducible forms. Hence

$$(ab)^\lambda\,(ac)^\mu\,(ad)^\nu$$

can be expressed in terms of covariants containing more factors in a, b, c and reducible forms. Again,

$$(ab)^\lambda\,(ac)^\mu$$

can be expressed linearly in terms of similar forms, each containing (ab) or (ac) to a power equal to or less than $\frac{\lambda+\mu}{3}$. Supposing this done, we see that, if

$$\lambda+\mu<6,$$

the form can be expressed in terms of others for which the sum of the exponents of (ab) and (ac) is greater and reducible forms; thus we may assume that $\lambda+\mu\geqslant 6$.

Let us now take $\lambda+\mu=N>6$ and consider the various products

$$(ab)^\lambda\,(ac)^\mu.$$

They may be divided into four classes :

(i.) $(ab)^n$, $(ab)^{n-1}(ac)$;

(ii.) $(ac)^n$, $(ac)^{n-1}(ab)$;

(iii.) $(ab)^2(ac)^{n-2}$, $(ab)^3(ac)^{n-3}$;

(iv.) all the remaining products for which $\lambda+\mu=n$.

Now, by means of the identities expressing $(bc)^n$ and $(bc)^{n-1}(ab)$ in terms of (ab) and (ac), we can express

$$(ab)^2(ac)^{n-2}, \quad (ab)^3(ac)^{n-3},$$

as linear combinations of

$$\left.\begin{array}{ll} (ab)^n, & (ab)^{n-1}(ac) \\ (ac)^n, & (ac)^{n-1}(ab) \\ (bc)^n, & (bc)^{n-1}(ab) \end{array}\right\} \qquad \text{(A)}$$

and products contained in (iv.), in all of which the exponent of (ab) is 4 at least.

But, if any of the products (A) occur, the covariant is either reducible, or the number of factors involving a, b, c can be increased, and hence we need only consider the others, *i.e.*, every covariant of degree 4 can be expressed in terms of such as contain (ab) to the power 4 at least.

This establishes the result for forms of degree 4, for it manifestly follows that all covariants can be expressed in terms of covariants

$$(ab)^\lambda (ac)^\mu (ad)^\nu,$$

in which $\lambda \geq 4$, $\mu \geq 2$, $\nu \geq 1$.

6. The extension of this to forms of degree n belonging to n different quantics is immediately evident, both as to enunciation and proof. In fact, if we have n quantics a, b, c, d, ..., the perpetuants of degree n are of the type

$$(ab)^\lambda (ac)^\mu (ad)^\nu \ldots,$$

where $\lambda \geq 2^{n-2}$, $\mu \geq 2^{n-3}$, $\nu \geq 2^{n-4}$, ...

To put the proof of this by induction into a neater form, let us assume the result for $n+1$ letters and prove it for $(n+2)$ letters.

Suppose the form is $(ab)^\lambda (ac)^\mu R$,

when R is a type belonging to n letters, and for brevity write

$$2^{n-1} = N,$$

then, just as before, $(ab)^{\lambda}(ac)^{\mu}$ can be replaced by a number of products, each containing either (ab) or (ac) to a power not greater than $\frac{\lambda+\mu}{3}$.

But $(ac)^{\mu}R$ is a covariant for $(n+1)$ letters, and can therefore be expressed in terms of reducible forms, and such as contain (ac) to the power 2^{n-1} at least; accordingly, unless $\frac{\lambda+\mu}{3} \geqslant N$, the number of factors containing a, b, c can be increased.

If, then, $$\lambda+\mu = M \geqslant 3N,$$

we have to consider the following expressions, viz.,

$$(ab)^{M},\ (ab)^{M-1}(ac),\ \ldots,\ (ab)^{M-N+1}(ac)^{N-1}, \tag{A}$$

$$(ac)^{M},\ (ac)^{M-1}(ab),\ \ldots,\ (ac)^{M-N+1}(ab)^{N-1}, \tag{B}$$

$$(ab)^{2N}(ac)^{M-2N},\ \ldots,\ (ab)^{M-N}(ac)^{N}, \tag{C}$$

$$(ab)^{N}(ac)^{M-N},\ \ldots,\ (ab)^{2N-1}(ac)^{M-2N+1}. \tag{D}$$

By means of the identical equations expressing

$$(bc)^{M},\ (bc)^{M-1}(ab),\ \ldots,\ (bc)^{M-N+1}(ac)^{N-1}$$

in terms of (ab) and (ac), we can on solution express the N products

$$(ab)^{N}(ac)^{M-N},\ \ldots,\ (ab)^{2N-1}(ac)^{M-2N+1}$$

linearly in terms of the sets A, B, C and the subsidiary set

$$(bc)^{M},\ (bc)^{M-1}(ab),\ \ldots,\ (bc)^{M-N+1}(ac)^{N-1}.*$$

Now, in A, B and the last system of products each term contains either (ab) or (ac) to a power equal to or less than $N = 2^{n-1}$, and accordingly, if one of these products occurs, the number of factors containing a, b, c can be increased. Eventually, then, we can express all the covariants in terms of reducible forms, and such as contain (ab) to the power $2N$ at least. The general result follows at once, and corroborates the result of Stroh relating to the perpetuant of lowest weight and given degree, and also the recent results of MacMahon in the *Camb. Phil. Trans.*

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* The possibility of this depends on the equations in question being linearly independent. With the notation of § 4 we can express all products of ξ, η, ζ of degree n linearly in terms of three sets, viz.,

$$\xi^{n},\ \xi^{n-1}\eta,\ \ldots,\ \xi^{n-g_1+1}\eta^{g_1-1};\ \eta^{n},\ \eta^{n-1}\zeta,\ \ldots,\ \eta^{n-g_2+1}\zeta^{g_2-1};\ \zeta^{n},\ \zeta^{n-1}\xi,\ \ldots,\ \zeta^{n-g_3+1}\xi^{g_3-1},$$

where g_1, g_2, g_3 are any three integers, such that $g_1+g_2+g_3 = n+1$. The result required in the text is a particular case of this which was first explicitly stated by Stroh, *Math. Ann.*, Vol. XXXIII. [*November 30th*, 1902].

The following presents were made to the Library during the recess :—

Brioschi, F.—"Opere matematiche," Tomo II. ; Milano, 1902.

Laplace, Marquis de.—"Philosophical Essay on Probabilities," translated by F. W. Truscott and F. L. Emory ; New York, 1902.

Laurent, H.—"Principes fondamentales de la théorie des nombres."

Macdonald, H. M.—"Electric Waves"; Cambridge, 1902.

"Imperial University Calendar," 1901–1902 ; Tokyo.

"Mém. de la Société des Sciences phys. et nat.," Sér. 6, Tome I., 1901 ; "Procès-verbaux," 1900, 1901 ; Tome V., Appendice, G. Royet : "Observations pluviométriques et thermométriques," 1900, 1901 ; Bordeaux.

"L'Enseignement Mathématique," Année IV., Nos. 4, 5 ; 1902.

Académie Royale de Belgique.—"Bulletin de la Classe des Sciences," Nos 4–7 ; Bruxelles, 1902.

"Wiadomosci Matematyczne," Tom VI., Zeszyt 4–5 ; Warsaw, 1902.

"University Bulletin," Vol. I., Nos. 1–4 ; Kansas, 1902.

"Periodico di Matematica," Vol. V., Fasc. 1, 2 ; "Supplemento," Anno 5, Fasc. 8, 9 ; Livorno, 1902.

"Year Book of the College of Mines," 1901–1902 ; Michigan.

"Mathematical Gazette," Vol. II., No. 34, 1902.

American Philosophical Society.—"Proceedings," Vol. XLI., Nos. 168, 169 ; 1902.

Kantor, S.—"Das Maximalgeschlecht der algebraischen Curven" (pamphlet).

Reprints from "Leipzig Abhandlungen," Bd. XXVII., 1902 :—

Fischer, O.—"Das statische und das kinetische Maass für die Wirkung eines Muskels."

Marchand, F.—"Ueber das Hirngewicht des Menschen."

Peter, B.—"Beobachtungen am sechszähligen Repsoldschen Heliometer der Leipziger Sternwarte."

Reprints from "Monthly Weather Review" :—

Abbé, C.—"Physical Basis of Long Range Weather Forecasts," Dec., 1901.

Bjerknes, V.—"Dynamic Principle of Circulatory Movements in the Atmosphere," Oct., 1901.

McAdie, A. G.—"Fog Studies on Mount Tamalpais," Nov., 1900.

Pockels, F.—"Theory of the Formation of Precipitation on Mountain Slopes," April, 1901.

"Memoirs of the National Academy of Sciences," Vol. VIII. ; Washington, 1902.

"Annals of Mathematics," Series 2, Vol. III., No. 4 ; 1902.

Royal Irish Academy.—"Transactions," Vol. XXXI., Part 12, 1901 ; Vol. XXXII., Parts 1, 2, 1902.

"American Journal of Mathematics," Vol. XXIV., No. 3 ; 1902.

Reprints from "Memorie delle Istituto Lombardo," Tomo XIX. ; Milano, 1902 :—

Jatta, M.—"Ricerche sulla genesi della fibrina nelle membrane difteriche."

Negri, A.—"Osservazioni sulla sostanza colorabile col rosso neutro nelle esnazie dei vertebrati."

Oehl, E.—"Sul diverso e variante grado di attività della saliva umana."

Veratti, E.—"Ricerche sulla fine struttura della fibra muscolare striata."

"Publications of the U.S. Naval Observatory," Ser. 2, Vol. II.; Washington, 1902.

Cape of Good Hope.—"Geodetic Survey of South Africa," Vol. II.

India.—"Trigonometrical Survey," Vol. XXVI., 1901; "Details of the Tidal Observations."

"Institution of Civil Engineers—List of Members."

Royal Society.—"Reports of the Malaria Committee," Series 7; London, 1902.

"Educational Times," Vol. LV., Nos. 495–498; 1902.

"Indian Engineering," Vol. XXXI., Nos. 21–26, and Index, Vol. XXXII., Nos. 1–10; 1902.

The following exchanges were received during the recess:—

"Proceedings of the Royal Society," Vol. LXX., Nos. 461–466; 1902.

"Beiblätter zu den Annalen der Physik und Chemie," Bd. XXVI., Hefte 6–10; Leipzig, 1902.

"Rendiconti del Circolo Matematico di Palermo," Tomo XVI., Fasc. 3–5; 1902.

"Bulletin de la Société Mathématique de France," Tome XXX., Fasc. 2; Paris, 1902.

"Annales de la Faculté des Sciences de Toulouse," Série 2, Tome III., Fasc. 3, 4; Paris, 1902.

"Bulletin of the American Mathematical Society," Vol. VIII., Nos. 9, 10; Vol. IX., No. 1, 1902; "Transactions," Vol. II., Nos. 2, 3, and Vol. III., No. 3, 1901–2; New York.

"Monatshefte für Mathematik und Physik," Jahrgang XIII., Hefte 3, 4; Wien, 1902.

"Reale Istituto Lombardo—Rendiconti," Ser. 2, Vol. XXXIV.; Milano, 1901.

"Bulletin des Sciences Mathématiques," Tome XXVI., Mai–Septembre; Paris, 1902.

"Rendiconto dell'Accademia delle Scienze Fisiche e Matematiche," Vol. VIII., Fasc. 4–7; Napoli, 1902.

"Journal für die reine und angewandte Mathematik," Bd. CXXIV., Heft 4; Berlin, 1902.

"Annali di Matematica," Tomo VII., Fasc. 4; Tomo VIII., Fasc. 1; Milano, 1902.

"Atti della Reale Accademia dei Lincei—Rendiconti," Ser. 5, Sem. 1, Vol. XI., Fasc. 11, 12; Sem. 2, Vol. XI., Fasc. 1–5; Rendiconto dell' Adunanza solenne del 1 Giugno 1902, Vol. II.; Roma, 1902.

"Berichte über die Verhandlungen der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig," 1901, No. 7; 1902, Nos. 1, 2.

"Revue Semestrielle des Publications Mathématiques," Tome X., Pt. 2, Oct., 1901–Av., 1902; 1902.

"Proceedings of the Physical Society," Vol. XVIII., Pt. 2; London, 1902.

"Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu Berlin," Nos. 23–40; 1902.

"Proceedings of the Cambridge Philosophical Society," Vol. XI., Pt. 6; "Transactions," Vol. XIX., Pt. 2; 1902.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Klasse, 1902, Hefte 2–4; Geschäftliche Mittheilungen, Heft 1; 1902.

"Jahrbuch über die Fortschritte der Mathematik," Bd. xxxi., Hefte 1, 2; Berlin, 1902.

"Acta Mathematica," Vol. xiv., Pt. 4, 1891; Vol. xvii., Pts. 1, 2, 1893; Vol. xxiv., 1901; Vol. xxv., Pts. 1, 2, 1901; Stockholm.

THIRTY-NINTH SESSION, 1902–1903

(since the Formation of the Society, January 16th, 1865).

November 13th, 1902.

The Ninth Annual General Meeting of The London Mathematical Society, as incorporated under the Companies Act, 1867, on October 23rd, 1894, held at 22 Albemarle Street, W.

Dr. E. W. HOBSON, F.R.S., President, in the Chair.

Twenty-five members and a visitor present.

Mrs. Alicia Stott was elected a member.

The Treasurer read his report. The reception of the Treasurer's report was moved by Mr. Kempe, seconded by Prof. Hudson, and carried *nem. con.*

The President nominated Mr. J. H. Grace to act as Auditor.

The Secretary reported that the number of members of the Society at the beginning of the previous session was 257. Of these 3 had resigned, and the names of 2 had been omitted from the list. The number of new members elected during the session was 8, bringing the number at the beginning of the present session to 260.

The President stated to the meeting the grounds on which the De Morgan Medal had been awarded by the Council to Prof. A. G. Greenhill, and presented the medal. Prof. Greenhill expressed his thanks to the Council and to the President.

The President spoke on the retirement of Mr. Tucker from the office of Honorary Secretary, and moved the following resolution:—

"That the thanks of the London Mathematical Society be offered to Mr. Robert Tucker for the eminent services which he has rendered

to the Society during the thirty-five years in which he has held the office of Honorary Secretary." This resolution was seconded by Dr. Glaisher and carried unanimously.

At the President's request Prof. A. Lodge and Mr. R. J. Dallas acted as Scrutineers of the ballot for the election of a Council and Officers for the new session. The following gentlemen were declared by them to be elected to serve on the Council:—President: Prof. Lamb; Vice-Presidents: Mr. Tucker, Dr. Hobson, Dr. Baker; Treasurer: Dr. Larmor; Hon. Secs.: Prof. Love and Prof. Burnside; other members: Mr. Campbell, Lieut.-Col. Cunningham, Dr. Glaisher, Prof. Greenhill, Mr. Macdonald, Major MacMahon, Mr. Western, Mr. Whittaker, Mr. A. Young.

Prof. Lamb took the Chair as President.

Dr. Hobson read his address "On the Infinite and the Infinitesimal in Mathematical Analysis."

The President moved that Dr. Hobson's Address be printed in the *Proceedings*. The motion was seconded by Dr. Baker and carried unanimously.

The following papers were communicated from the Chair:—

Prof. W. Burnside: On Linear Homogeneous Groups.

Prof. D. Hilbert: Ueber den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck.*

Prof. H. Lamb: Wave Propagation in Two Dimensions.

Prof. A. C. Dixon: (1) The Summation of a certain Series; (2) Expansion by means of Lamé's Functions.

Mr. W. H. Young: (1) On Sets of Intervals; (2) Note on Unclosed Sets of Points defined as the Limit of a Sequence of Closed Sets of Points.

Prof. M. J. M. Hill: The Continuation of certain Fundamental Power Series.

Prof. L. Crawford: A Geodesic on a Spheroid and an Associated Ellipse.

Prof. A. W. Conway: The Propagation of Light in a Uniaxal Crystal.

Mr. E. T. Whittaker: A new Connexion between Legendre Functions and Bessel Functions.

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* *Ante*, p. 50.

The following presents were made to the Library :—

"Educational Times," Vol. LV., No. 499; 1902.

"Indian Engineering," Vol. XXXII., Nos. 11–16; 1902.

"Mathematical Gazette," Vol. II., No. 35; 1902.

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*On the Infinite and the Infinitesimal in Mathematical Analysis.**

Presidential Address, by E. W. Hobson, Sc.D., F.R.S.,
November 13th, 1902.

Mr. President,

In the days of our forefathers, when an unsuccessful politician had reached the end of his career, it was customary to grant him one last privilege, that of delivering an address upon topics chosen by himself to the assembled multitude on Tower Hill. Although my conscience acquits me of having been guilty during my period of office of conduct traitorous to the interests of our Society, I avail myself of the corresponding privilege accorded by our custom to a retiring President.

The remark that the nineteenth century has been an age of unexampled progress in all branches of science has been so often made as to have become a commonplace. The remark is true in a pre-eminent degree of our own department of science. As is known to you all, at no earlier time has a more rapid development taken place in all parts of mathematical science, involving the creation of entirely new branches and of new and powerful general methods. However, it is not in the main of these new developments and of the extensions made in our science in the outward direction that I propose to speak this evening. In the past century, and perhaps especially during the second half of it, the attention of mathematicians has been devoted in an unusual degree to a critical examination of the foundations of the various branches of mathematical thought. In analysis, geometry, and mechanics a close scrutiny has been made of the fundamental assumptions and concepts. This scrutiny has resulted not only in a large measure of restatement of the base principles of these departments of science, but has also powerfully reacted on the methods of procedure within these departments, and has suggested new and fruitful lines of research. Although outside criticism of the founda-

* Ordered by the Council to be printed as a pamphlet and distributed to Members, to avoid the delay that would be occasioned by waiting for the appropriate issue of *Proceedings*.

tions of mathematics has at all times been abundant, the work of underpinning the edifice of our science has been for the most part carried out by the same workmen who have been engaged in the general work of the structure, and especially in building new wings. There are times when it is appropriate to draw attention in general terms to the critical side of some part of our science, and I think it will be admitted that a Presidential address is such an occasion. I have accordingly chosen as the subject of my discourse this evening "The Infinite and the Infinitesimal in Mathematical Analysis." It will be found that my intention to speak of critical rather than constructional results admits of some considerable exceptions. These exceptions will, however, illustrate the fact that pertinent criticism of fundamentals almost invariably gives rise to new construction. On such a subject as that I have chosen, I cannot hope to have anything essentially new to say; but, nevertheless, I venture to hope it may not be profitless, if I state as explicitly as I can, what seems to me to be the trend of thought in this connexion at present prevailing among mathematicians as the result of the labours of some of the most distinguished of their number during the last half century. I am strengthened in this view by my knowledge of the fact that many British mathematicians, absorbed as they rightly are in the technique of their science, and in the work of applying it to the quantitative description of natural phenomena, have not yet fully appreciated the results of recent movements in mathematical thought in this connexion. In some of the text-books in common use in this country, the symbol ∞ is still used as if it denoted a number, and one in all respects on a par with the finite numbers. The foundations of the integral calculus are treated as if Riemann had never lived and worked. The order in which double limits are taken is treated as immaterial, and in many other respects the critical results of the last century are ignored. It would, however, be unjust not to recognize the fact that a great improvement in these respects has been shown in some of the most recent of our text-books.

Essentially connected as views about the infinite and the infinitesimal are with the most fundamental notions on which analysis is based, with the concepts of number and magnitude, with the notions of continuity and discreteness, with the doctrine of limits in all the various forms in which it has appeared, with the nature of the ideal objects with which mathematical thought operates, these ideas have been a subject of unceasing controversy since the very commencement of abstract thought—a controversy which has by no means ceased at

the present time. The fact that these fundamentals lie on the border-line across which mathematics passes into the wider region of philosophy has brought it about that in all ages philosophical thinkers as well as mathematicians, before as well as after the two classes ceased to be identical, have occupied themselves with the attempt to introduce clearness into the doctrine concerning them. The kind of judgments which are made in mathematical thinking, forming as they do a class which in certain aspects are of a comparatively simple character, have at all times formed a kind of touchstone on which epistemologists have tested their general theories of knowledge.

To attempt to give, even in outline, a history of thought upon the subjects of the infinite and the infinitesimal, involving as it would the task of tracing the history of the various theories of the infinitesimal calculus, would be altogether beyond the scope of such a discourse as the present one. In order, however, to make clear what has been the precise effect of the more recent movements of thought in this order of ideas, it will be necessary for me to take a brief glance at the mode in which the subject presented itself at various times to thinkers confronted with the ordinary problems of mathematical analysis.

How, then, did the problems of analysis present themselves to the earliest mathematicians? What were the elements with which those mathematicians had to work? The two notions of number and of magnitude with which they had to operate in problems of a geometrical or kinematical character, have points of resemblance and also points of difference. Both number and magnitude appear by their very nature to be unlimited in two directions: there is no greatest number or magnitude, and (excluding zero) no smallest one. A set of numbers or of magnitudes may be contemplated, each one of which is definite and finite, and yet the set contains numbers or magnitudes which are greater than any particular number or magnitude which we may choose to assign. A similar possibility holds as regards smallness. A symbol to which are assigned successively the increasing or diminishing values of the numbers or magnitudes in the set contemplated, is said to become in the one case indefinitely great, in the other case indefinitely small. At any particular stage the symbol represents a finite number or magnitude, but the absence of a limit is designated by the phrase "becoming indefinitely great, or small." The indefinitely great thus described is the potentially infinite, *das uneigentlich Unendliche*, and expresses the mere absence of upper limit

to a variable. In this form, as expressing a mere potentiality, the infinite and the infinitesimal seem so inevitable a necessity of thought as hardly to give rise to differences of opinion, except, perhaps, upon matters of language. But when it is conceived that these mere potentialities pass into actualities, that *fixed* numbers or magnitudes exist which are infinite or infinitesimal, that the merely indefinitely great becomes an actual infinite, or the merely indefinitely small becomes an actual infinitesimal, the region of serious controversy has been reached—a controversy which is still proceeding, and about the modern aspect of which I shall have some remarks to make later on. In respect of the actually infinite, there have been exhibited at different times and by various thinkers the extremes of faith and of scepticism; there have been believers and sceptics, critics and freethinkers, idealists and empiricists. The infinite of mathematics has at times been treated with that familiarity which is bred of innocent inappreciation. Bold generalizations have been made in which rules applicable to the finite were uncritically and unconditionally extended to the indefinitely great, as if that represented an actuality necessarily subject to the same rules of operation as the finite. At other times, the desire to remain on what was felt to be the firmer ground of empirical knowledge has led almost to a denial of all validity to the conceptions of the infinite and the infinitesimal, and to all processes involving their use. It is noteworthy that both these attitudes of mind have at different times been of direct advantage to science, and that the most opposite tendencies in regard to this order of ideas have led to the advancement of knowledge.

One of the principal forms in which an indefinitely great number of operations occurs is that of infinite series, which were introduced in the seventeenth century. The mathematicians of the eighteenth century used these series freely, without troubling themselves much as to questions of convergence. Early in the nineteenth century came a rude awakening. In a letter written by Abel* in 1826 we read: "Divergent series are *in toto* an invention of the Devil, and it is a disgrace that any one should venture to found on them the smallest demonstration. One can get out of them anything one likes when one employs them, and it is they which have produced so many difficulties and so many paradoxes." And further: "I have become prodigiously attentive to all that; for, if one excepts the cases of the

* See Abel's correspondence, p. 16, in the volume *Niels Henrik Abel: Mémorial publié a l'occasion du centenaire de sa naissance.*

most extreme simplicity—for example, geometric series—there is scarcely in the whole of mathematics a single infinite series of which the sum is determined in a rigorous manner. In other words, all that is most important in mathematics is without foundation. Most of the things are exact, that is true, and it is extraordinarily surprising. I am trying to find out the reason." In our time, now that the use of divergent series has been to a large extent placed upon a sound mathematical basis, the work of Poincaré, Stieltjes, Borel, and others, has given us an answer to the question that puzzled Abel.

To the Greeks, and to later thinkers, magnitude, as given by the intuition of space, time, and especially of motion, appeared to present itself essentially as a continuum, the intuitional or sensuous continuum. On the other hand, number (and it must be recollected that the Greeks only knew rational numbers) appeared to be essentially discrete. Fractional numbers arose historically from the necessity for the representation of the sub-divisions of a unit magnitude into equal parts. The Greek discovery of the existence of magnitudes which are incommensurable with a given unit, by exhibiting the inadequacy of such discrete numbers for the complete representation of *prima facie* continuous magnitude, served to emphasize the distinction contained in the antinomy of the continuous and the discrete.

In order completely to envisage the problem of analysis as it presented itself to the minds of mathematicians from the earliest commencement of the attempts to deal with geometrical and kinematical problems numerically, we must take into account that peculiarity of the human mind in virtue of which it is in general unable to deal with an object of thought as a whole, but is obliged to consider it piecemeal, dividing it up into some kind of elements, taking account of these, and reconstructing the object mentally by a process of synthesis. This inability to grasp a scheme of relations at once as a whole, involved as it is in our essentially discursive modes of apprehension, leads to the necessity of dividing up a geometrical figure, or a portion of time, into parts regarded as elements of the whole, dealing separately with these, and of obtaining final judgments as to the integral properties of the figure or the motion, by means of a process of summation. This necessity of mathematical method led directly to a discussion of the nature of the elements of which magnitudes were to be regarded as made up: Could, for example, the straight line be legitimately regarded as made up of points? If so, of how many? Ought it not rather to be regarded as made up of infinitesimal elements, each of which possess all the properties of the finite

length? Ought such elements to be regarded as fixed or as essentially in a state of flux? Such were the questions which inevitably presented themselves as soon as men began to investigate geometrically or analytically the properties of curves and surfaces, to determine areas and volumes.

Again, in order to deal with a geometrical figure, not only had the figure to be divided up into elements, but qualitative changes in the figure had to be introduced: for example, a curve without corners had to be replaced by a rectilineal polygon with corners, if its length was to be found. Here we have the origin of the method of limits, in its geometrical and its arithmetical forms, and here we come across the central difficulty of the mode in which a limit was regarded as being actually attained. A limit, which appeared only as the unattainable end of a process of indefinite regression, and to which unending approach was made, had, by some process inaccessible to the sensuous imagination, to be regarded as actually reached; the chasm which separated the limit from the approaching magnitudes had in some mysterious way to be leapt over; the attainment of a numerical limit, and an actual qualitative change in a geometrical figure were to be regarded as somehow taking place simultaneously as the result of a process which contained no principle of termination within itself.

The germ of the methods of the infinitesimal calculus appears in a geometrical form in the method of exhaustions, employed by the Greek geometers. It was by this method that Archimedes showed that the area of the surface of a sphere is four times that of a great circle, by which he expressed the area of the surface of a right cone, and solved other problems of a similar nature. We have an example of this method in the proof in Euclid XII. 2, that the circumferences of circles are as their diameters; in this case the quantity to be determined is trapped in between two sets of polygons, the one circumscribed to, and the other inscribed in, the circle; as the number of sides of the polygons is increased the space between the two sets of polygons is exhausted; the proof that the required result is obtained is then carried out by the method of *reductio ad absurdum*. This method would, in the cases in which it can be carried out, leave nothing to be desired as regards rigour, provided the existence of the limit is *a priori* admitted.

It will be observed that the Greeks did not deem it necessary to define the length of a circle, or other curve; that every curve has a length, and every surface an area, was taken by them to be a truth

obvious from intuition. Naturally they were not led by intuition to contemplate the existence of not-rectifiable curves.

In the method of indivisibles employed by Cavallieri, Pascal, Roberval, and others, straight lines are regarded as made up of an infinite number of points, surfaces as made up of lines, and volumes of surfaces. This method was applied with considerable success before the introduction by Newton and Leibniz, of the methods of the Infinitesimal Calculus, but it appears to have been regarded, by some at least of those who employed it, in the light of a shortened mode of procedure in which the method of exhaustions is used with an abbreviated form of language, rather than as a method, the principles of which, when taken literally, were to be regarded as rigorous. Thus Pascal writes: "J'ai voulu faire cet avertissement, pour montrer que tout ce qui est démontré par les véritables règles des indivisibles, se démontrera aussi à la rigueur et à la manière des anciens; et qu'ainsi l'une de ces méthodes ne diffère de l'autre qu'en la manière de parler; ce qui ne peut blesser les personnes raisonnables quand on les a une fois averties de ce qu'on entend par là."

The infinitesimal calculus, in the form which was devised by Leibniz, has usually been regarded as the art of employing infinitesimal quantities as auxiliaries for the purpose of finding the relations between certain quantities of which the existence is assumed. In order to find the relations between certain quantities, some of which are constant, others variable, the system is imagined as having arrived at a determinate state regarded as fixed; this state is then compared with other states of the same system, which are regarded as continually approaching the first state, so as to differ arbitrarily little from it. These other states of the system are regarded only as auxiliary systems introduced to facilitate the comparison between the parts of the fixed one. The differences of corresponding quantities in all these systems can be regarded as arbitrarily small, without changing the quantities which define the fixed state, and the relations between which are to be found; these differences are the infinitesimals, and unity divided by one of these infinitesimals was regarded as giving rise to an infinite quantity. The question as to the true nature of these infinitesimals gave rise to almost endless discussions; the views which have been maintained with regard to them fall under three main heads. By some, the infinitesimals have been regarded as fixed objects, having a real existence and in a state of rest outside the ordinary realm of magnitude, two finite magnitudes which differ by an infinitesimal being regarded as equal to one another. A second

view as regards infinitesimals is that they are ordinary magnitudes essentially in a state of motion towards zero. This conception of magnitudes continually in a state of flux has been sarcastically described by P. Du Bois Reymond as follows:—"As long as the book is closed there is perfect repose, but as soon as I open it there commences a race of all the magnitudes which are provided with the letter d towards the zero limit." The third view as to the nature of the infinitesimals is that they are simply ordinary magnitudes too small to be perceived by the senses, and possessing thus only a relative smallness. This view is that of those mathematicians who regard a geometrical point as simply an object whose size is too small to be perceived by the means at our command; a line as a volume of which two of the dimensions are insensible; and so on. The empiricists of this school refuse to idealize objects of perception which form the subject of calculations, by bringing them under exact abstract definitions; the calculus thus regarded is an approximative system in which the results make no claim to absolute exactness, but only to freedom from errors which are observable. Apart altogether from the difficulties as to the true nature of the differentials, it will be observed that in the Liebnizian calculus the existence of the magnitudes between which the relations are in any special problem to be found is regarded as *a priori* known, or, in other words, no doubt is admitted as to the existence of the limit; this it has in common with the Greek method of exhaustions, of which it is essentially a translation into a more analytical and convenient form.

In the method of limits devised by Newton, and employed in a different form by later writers, infinitesimals are not employed singly, but the ratio of two quantities at the moment when they vanish is contemplated, and forms what was later known as the differential coefficient. These "ghosts of departed quantities," as Bishop Berkeley derisively designated them, whose ratio at the moment of their disappearance is the quantity dealt with, present very much the same kind of difficulty as in the Leibnizian form of the calculus. No criterion was obtained for the determinacy of such an ultimate ratio, whose existence was regarded as obvious from intuition.

In that form of the Newtonian calculus known as the method of fluxions, the appeal to intuition was made more cogent by representing the vanishing ratio in the form of a velocity; that a moving point has at every instant necessarily a definite velocity was apparently hardly doubted until comparatively recently. We now know that such a velocity has no such unconditional existence as was supposed.

Speaking of the method of vanishing ratios, Lagrange writes "Cette méthode a le grand inconvénient de considérer les quantités, dans l'état où elles cessent, pour ainsi dire, d'être quantités; car quoiqu'on conçoive toujours bien le rapport de deux quantités, tant qu'elles demeurent finies, ce rapport n'offre plus à l'esprit une idée claire et précise, aussitôt que ces termes deviennent l'un et l'autre nuls à la fois." This clear perception on the part of Lagrange of the difficulty at the root of the method of limits or of differential coefficients, was doubtless a determining factor in deciding him to embark upon his great attempt to place the calculus upon a basis independent of the idea of infinitesimals or of their ratios. The title of his great work, "Théorie des fonctions analytiques, contenant les principes du calcul différentiel, dégagés de toute considération d'infiniment petits, d'évanouissans, de limites et de fluxions, et réduits à l'analyse algébrique des quantités finies," contains the most concise statement of his aim. Although his attempt was in principle a failure, his idea of making Taylor's series the cardinal form by which functions are to be represented must be regarded as containing the germ of the theory of analytical functions which was developed with so much success at a later period.

In the various forms of the infinitesimal calculus to which I have referred, a crucial difficulty is that of the existence of the limit. That this difficulty is no merely imaginary one, but indicates a real gap in the logical basis of the systems, receives an *a posteriori* confirmation from the discoveries made in the latter half of the nineteenth century, that special restrictions in the nature of the functions employed are necessary for the validity of the ordinary processes of the calculus. The exhibition by Weierstrass and others, of continuous non-differentiable functions, the resulting investigations of the restrictive conditions over and above that of continuity which are necessary for the existence of a differential coefficient, Riemann's investigation of the conditions of integrability of a function, the various theorems discovered as to the conditions of the reversibility of the order of double limits, all indicate that the existence of a limit cannot be presumed apart from all restrictive conditions. The failure of the older analysis to exhibit the existence and nature of such restrictive conditions is a clear proof of defectiveness in the logical basis of that analysis. That the earlier mathematicians were usually able to obtain correct results by means of their methods, is due to the fact that the functions with which they operated were of a comparatively simple character; in point of fact, almost all the functions which are

required for the investigation of the problems arising from ordinary intuition satisfy the restrictive conditions first brought to light in our day.

I now come to consider the changes which have been brought about in the point of view of mathematicians with respect to the matters I have discussed, as the result of the critical efforts of recent times. In the first place, the notion of number, integral or fractional, has been placed upon a basis entirely independent of measurable magnitude, and pure analysis is regarded as a scheme which deals with number only, and has, *per se*, no concern with measurable quantity. Analysis thus placed upon an arithmetical basis is characterized by the rejection of all appeals to our special intuitions of space, time, and motion,* in support of the possibility of its operations. It is a very significant fact that the operation of counting, in connexion with which numbers, integral and fractional, have their origin, is the one, and only absolutely exact, operation of a mathematical character which we are able to undertake upon the objects which we perceive; this is due to the fact that the operation is of a highly abstract character, since in counting objects, all special qualitative or quantitative peculiarities of the objects counted are treated as irrelevant. On the other hand, all operations of the nature of measurement which we can perform in connexion with the objects of perception contain an esssential element of inexactness, corresponding to the approximative character of our sensuous intuition. The theory of exact measurement in the domain of the ideal objects of abstract geometry is not immediately derivable from intuition, but is now usually regarded as requiring for its development a previous independent investigation of the nature and relations of number. The relations of number having been developed on an independent basis, the scheme is applied by the help of the principle of congruency, or other equivalent principle, to the representation of extensive or intensive magnitude. In any such theory of measurement the non-arithmetical conception of a unit is involved. Those departments of science, including geometry, in which abstract measurement is applied are thus regarded as fields of application for analysis; but they do not directly contribute towards the development of pure analysis, although they may, no doubt, suggest to it problems for treatment in accordance with its

* It is not intended here to prejudge the questions as to the part which intuition may have in the formation of the concepts of number.

own principles. This complete separation of the notion of number, especially fractional number, from that of magnitude, involves, no doubt, a reversal of the historical and psychological orders. It is, however, no uncommon occurrence that the logical order of a subject should be very different from the historical order in which the concepts of the subject have arisen. Is it not an essential part of our scientific procedure that, in our conceptual schemes, factors are separated from one another, which intuitionally appeared in combination?

The so-called arithmetization of analysis is, and has been, accepted in somewhat various degrees by different mathematicians. The extreme arithmetizing school, of which, perhaps, Kronecker was the founder, ascribes reality, whatever that may mean, to integral numbers only, and regards fractional numbers as possessing only a derivative character, and as being introduced only for convenience of notation. The ideal of this school is that every theorem of analysis should be interpretable as giving a relation between integral numbers only. The validity and feasibility of this ideal I cannot here discuss. Some mathematicians, on the other hand, like P. Du Bois Reymond, while using to a large extent the ideas and methods of arithmetical analysis, appear still to regard the notion of continuous magnitude as a necessary part of the foundations of the subject.

The true ground of the difficulties of the older analysis as regards the existence of limits, and in relation to the application to measurable quantity, lies in its inadequate conception of the domain of number, in accordance with which the only numbers really defined were rational numbers. This inadequacy has now been removed by means of a purely arithmetical definition of irrational numbers, by means of which the continuum of real numbers has been set up as the domain of the independent variable in ordinary analysis. This definition has been given in the main in three forms—one by Heine and Cantor, the second by Dedekind, and the third by Weierstrass. Of these the first two are the simplest for working purposes, and are essentially equivalent to one another; the difference between them is that, whilst Dedekind defines an irrational number by means of a section of all the rational numbers, in the Heine-Cantor form of definition a selected convergent aggregate of such numbers is employed. The essential change introduced by this definition of irrational numbers is that, for the scheme of rational numbers, a new scheme of numbers is substituted, in

which each number, rational or irrational, is defined and can be exhibited in an indefinitely great number of ways, by means of a convergent aggregate of rational numbers. In this continuum of real numbers the notion of number is, as it were, raised to a different plane. By this conception of the domain of number the root difficulty of the older analysis as to the existence of a limit is turned, each number of the continuum being really defined in such a way that it itself exhibits the limit of certain classes of convergent sequences. It would, of course, be futile to define a number by means of a convergent aggregate, were it not shown—as has been, in fact, done—that the ordinary operations of arithmetic can be defined for such numbers in such a way as to be in agreement with the ordinary scheme of operations for the rational numbers taken on the lower plane. It should be observed that the criterion for the convergence of an aggregate is of such a character that no use is made in it of infinitesimals, definite finite numbers alone being used in the tests. The old attempts to prove the existence of limits of convergent aggregates were, in default of a previous arithmetical definition of irrational number, doomed to inevitable failure. It could not, for example, in general be shown that an unending decimal formed according to prescribed rules possessed a limit, since it was clearly impossible to infer from the existence and properties of a set of rational numbers, the existence of a number which itself is in general not rational, and was therefore undefined within the domain of operation. A considerable part of the newer analysis consists of putting the criterion for the convergence of an aggregate into various forms suitable for application in various classes of cases. The convergence of an aggregate having in any given case been established by the application of one or other of such derivative rules, the aggregate itself defines the limit. In all such proofs the only statements made are as to relations of finite numbers, no such entities as infinitesimals being recognized or employed. Such is the essence of the ϵ proofs with which we are familiar. In such applications of analysis—as, for example, the rectification of a curve—the length of the curve is defined by the aggregate formed by the lengths of a proper sequence of inscribed polygons. The length is not regarded as something whose existence is *a priori* known. In case the aggregate is not convergent, the curve is regarded as not rectifiable. If it can be shown that the lengths of these inscribed polygons form a convergent aggregate which is independent of the particular choice of the polygons of the sequence,

the curve is rectifiable, its length being defined by the number given by the aggregate.

The older analysts regarded the domain of the real variable, or of a set of real variables, as the continuum given by our intuition of space, time, and motion; this continuum was usually accepted uncritically as a notion completely given by intuition and hardly capable of further analysis; however, those points of the continuum which could be represented by number (rational number) formed only a discrete aggregate, and thus the variable had to pass through values which were not definable as numbers. This intuitive notion of the continuum appears to have as its content the notion of unlimited divisibility, the facts that, for instance, in the linear continuum we can within any interval PQ find a smaller one $P'Q'$, that this process may be continued as far as the limits of our perception allow, and that we are unable to conceive that even beyond the limits of our perception the process of divisibility in thought can come to an end. However, the modern discussions as to the nature of the arithmetic continuum have made it clear that this property of unlimited divisibility, or connexity, is only one of the distinguishing characteristics of the continuum, and is insufficient to mark it off from other domains which have the like property. The aggregate of rational numbers, or of points on a straight line corresponding to such numbers, possesses this property of connexity in common with the continuum, and yet is not continuous; between any two rational numbers another pair can be found, and this process may be continued until we obtain an arbitrarily small interval. The other property of an aggregate which is characteristic of a continuum, is that of being, in the technical language of the theory of aggregates (*Mengenlehre*) perfect: the meaning of this is that all the limits of convergent sequences of numbers or points belonging to the aggregate themselves belong to the aggregate; and, conversely, that every number or point of the aggregate can be exhibited as the limit of such a sequence. The aggregate of rational numbers does not possess this property of being perfect, since the limit of a sequence of such numbers does not necessarily belong to the aggregate. That the aggregate of rational numbers is not perfect, or even closed, is the root defect of that aggregate, which led to the difficulty as regards the existence of limits, in the older analysis. The two properties of connexity and of perfection are regarded as the necessary and sufficient characteristics of a continuum; it is remarkable that in analysis the latter property of a continuum, which was not brought

to light by those who took the intuitive continuum as a sufficient basis, is in some respects the more absolutely essential property for the domain of a function which is to be submitted to the operations of the calculus. It has in fact been shown that many of the properties of functions, such as continuity, differentiability, are capable of precise definition when the domain of the variable is not a continuum, provided, however, that domain is perfect; this has appeared clearly in the course of recent investigations of the properties of non-dense perfect aggregates, and of functions of a variable whose domain is such an aggregate.

The arithmetical continuum having been defined and explored, it is then postulated that on a straight line there exists one point, and one only, corresponding to each number of the arithmetical continuum, and that no other points exist on the straight line; this fixing of the point contents of a straight line amounts to an exclusion of the contemplation of fixed infinitesimal lengths. Similarly, it is postulated that in three-dimensional space there exists one point, and one only, corresponding to each specification of three coordinates of the point by means of numbers, and that the points whose existence is thus postulated exhaust the space. The arithmetizing school thus regard the nature of the geometrical continuum as being cleared up and described by means of the previously defined arithmetical continuum; this is, of course, a reversal of the traditional view.

The view I have sketched of the philosophy of the continuum does not meet with the universal acceptance of mathematicians, as an adequate scheme, at the present time. As an example of a rival scheme I may briefly touch upon the one propounded by Veronese. He develops the notion of the abstract linear continuum from the intuitive side, and traces the consequence of supposing that on a straight line two intervals PQ, $P'Q'$ can co-exist such that the smaller $P'Q'$ is so small compared with PQ that no integer n can be found which will make $n.P'Q'$ exceed PQ, thus rejecting what is known as the axiom of Archimedes. This amounts to the affirmation of the existence of fixed infinitesimal lengths, and of fixed infinite lengths, on the straight line. In this scheme, when a unit length is chosen on the straight line, Dedekind's section of rational points is made, not by a single point, as in the Cantor-Dedekind scheme, but by an infinitesimal length, that is, by a length which is infinitesimal relative to the scale of measurement chosen. Veronese contemplates the existence of an indefinite series of scales in the linear continuum, such that each unit is infinite compared with one belonging to a

lower scale, and is infinitesimal compared with a unit belonging to a higher scale; he then proceeds to introduce a scheme of infinite and infinitesimal numbers which will suffice for the complete representation of points of the straight line. On this view, the Dedekind-Cantor continuum, when represented on the straight line, is only a relative continuum, that is, relative to the particular scale employed in the representation; the absolute continuum would require for its representation an indefinite series of infinite and infinitesimal numbers. As to the validity of Veronese's scheme, that is, as to its consistency with a logical theory of magnitude, I do not propose to express any opinion; the matter has been a subject of considerable controversy. Assuming, however, its validity as a possible scheme, it does not affect the validity of the Cantor-Dedekind scheme; the comparative simplicity of the latter would indicate it as the natural basis for analysis, and for the applications to the measurement of magnitude. One of the most interesting results on the speculative side of abstract science, which has been obtained in the nineteenth century, is that it is possible to set up two or more conceptual systems, each self-consistent, but contradictory with one another, each of which provides a sufficient representation of the facts of perception; the most striking example of this has been in geometry, where it has been shown that, under a certain limitation, Euclidean, hyperbolic, and elliptic geometry may each afford a sufficient representation of the properties of figures in perceptual space. We are entitled to postulate the existence of whatever points we choose upon that ideal object, the line of geometry, provided our scheme does not contradict itself, and, further, provided the ideal object thus constituted affords an adequate representation of the concrete lines which we perceive in the external world. Between two such schemes intuition can make no choice, and in abstract science we make that choice between them which is dictated by considerations of simplicity and of suitability for the special purpose on hand.

The question as to the legitimacy of the use of infinite numbers, that is, not merely of the use of a variable which is regarded as becoming indefinitely great, but of numbers which are actually infinite and to be regarded as capable of entering into relations, is a matter which has been discussed by philosophical thinkers from the time of Aristotle onwards. The balance of opinion seems to have been decidedly against the validity of the conception of such numbers; in support of this negative view, Aristotle himself, Locke, Descartes, Spinoza, and Leibniz may be quoted. The grounds of the objection

to the introduction of such numbers may in the main be reduced to three heads. First, it is said that a number is, by its very nature, finite: this is supported by the plea that all actual operations of counting and measuring are performed upon finite aggregates or finite magnitudes; to refute this view, it may be urged that the introduction of infinite numbers, if it can be made at all, will justify itself by a proof of the capability of such numbers for the representation and characterization of non-finite aggregates; in fact, it may be held that the objection contains a *petitio principii*. Secondly, it has been widely held that a scheme of infinite numbers represents an endeavour to make distinctions and determinations within the infinite; whereas the true infinite admits of no determination. If the infinite be identified with the all-embracing absolute of idealistic philosophy, it will probably be admitted that such an absolute admits of no distinctions, for "omnis determinatio est negatio"; however, the question arises whether a domain may not exist which, though not finite, is still not to be regarded as engulfed in the absolute, and which therefore may still in some measure admit of definition and determination, and which may require a special non-finite system of number for the specification of its characteristics; such an intermediate domain has been named by Cantor the "transfinite" or "superfinite." Thirdly, it has been urged that finite numbers would be unable to maintain themselves as against infinite ones; that the finite and its relations would be absorbed in the infinite, and could enter into no relations with it: the value of this objection can be estimated *a posteriori* only, if and when a system of infinite or transfinite numbers has been actually defined and the nature of its connexion with the finite brought to light. That mathematicians still shrink from leaving what they regard as the firm ground of the finite based upon experience is illustrated by a remark in an introductory passage in Tannery's work: *Introduction à la Théorie des Fonctions d'une Variable réelle*. He writes, "On peut constituer entièrement l'analyse avec la notion de nombre entier et les notions relatives à l'addition des nombres entiers; il est inutile de faire appel à aucun autre postulat, à aucune autre donnée de l'expérience, la notion de l'infini dont il ne faut pas faire mystère en mathématique se réduit à ceci, après chaque nombre entier il y'a un autre." However sufficient the restriction to the merely indefinitely great, here indicated, may be for the more ordinary purposes of analysis, provided, however, that an exploration of the properties of the continuum of real numbers is not carried too far, I hope to be able to show, as clearly as possible in the brief

space at my disposal, that the introduction by Cantor of systems of transfinite numbers is justified by the primary necessities of our analytical system; it may be justified in point of utility by the numerous applications which are being made, both in analysis and in geometry, of the conceptions and results of the theory of aggregates, to express the characteristics of which these transfinite numbers are required. No mathematician will wish to make a mystery of the infinite in analysis; mathematics has nothing to do with mysteries except to endeavour to remove them. It is to be remarked that the introduction into analysis of the transfinite numbers was historically by no means the result of a purely speculative tendency to explore the unknown and mysterious, and certainly did not arise from any taste on the part of their inventor for "tricks to show the stretch of human brain"; their introduction arose principally out of the necessities of investigations connected with the peculiarities of Fourier's series and of the functions representable by such series. Cantor writes:—"Zu dem Gedanken, das Unendlichgrosse nicht bloss in der Form des unbegrenzt Wachsenden und in der hiermit eng zusammmenhängenden Form der im siebenzehnten Yahrhundert zuerst eingeführten convergenten unendlichen Reihen zu betrachten, sondern es auch in der bestimmten Form des Vollendetunendlichen mathematisch durch Zahlen zu fixirén, bin ich fast wider meinen Willen, weil im Gegensatz zu mir werthgewordenen Traditionen, durch den Verlauf vieljähriger wissenschafticher Bemühungen und Versuche logisch gezwungen worden."

The first real breach in the infinite—one which established a true line of cleavage—was made when Cantor showed that the aggregate of rational numbers is enumerable, whereas the aggregate of real numbers, rational and irrational, is unenumerable. This denotes that a (1, 1) correspondence can be established between the rational numbers in any given interval, and the aggregate of positive integers, whereas no such correspondence can be established between the numbers of a continuum and the aggregate of integral numbers. Thus the rational numbers can be counted and the irrational numbers cannot be counted. All the rational numbers in any interval can be arranged in a definite order (not of magnitude), so that one of them stands first and each particular number has its assigned place. No such arrangement can be made when the irrational numbers of the interval are taken into account. This far-reaching result brings out in a strong light the difficult nature of the conception of the continuum as a given totality. If it be asked in what sense can the numbers of the continuum be

considered as forming a given or determinate aggregate, we must contrast this aggregate with that of the rational numbers or with that of the integral numbers. These latter are not, of course, given in the sense that we can exhaustively exhibit them by means of symbols on a sheet of paper. We could only do that in the case of a finite aggregate; but they are given in the sense that we can say of any particular number where it is to be found in a regularly arranged scheme. On the other hand, the aggregate of all real numbers is not given in the same sense; no rule, and no set of rules, can be given by which we could obtain successively all the numbers of the aggregate, so that each particular number would necessarily appear in the course of the procedure; and this is a consequence of the unenumerable character of the aggregate. The aggregate of real numbers can be regarded as given only in the sense that every possible real number that we may choose to define by means of an analytical process belongs to the aggregate. This somewhat negative conception of its determinacy is an essential characteristic of the unenumerable aggregate. How far the mathematicians of the future will rest satisfied with this conception of the arithmetic continuum, time alone can decide.

When we count a finite number of objects we take them in some definite order, and establish a correspondence between them and the ordinal numbers. The last ordinal number employed, we call the ordinal number, or simply the number of the collection. When we take into account the fundamental property that this number is independent of the order in which the objects are counted, we identify this number with the cardinal number of the collection. Thus, in dealing with finite aggregates, the distinction between the ordinal and the cardinal number, though logically existent, may be practically disregarded. This, however, is no longer the case when we deal with non-finite aggregates; here cardinal and ordinal numbers must be kept quite distinct, and their properties must be developed on different lines. The theorem that the ordinal number of an aggregate is unaltered by changing the order of the elements of the aggregate no longer holds.

In order to exhibit the way in which transfinite ordinal numbers are required when we deal with non-finite aggregates, I propose to refer to a well known paradox, that of Achilles and the tortoise, which in various forms has afforded an interesting exercise to logicians. Achilles goes ten times as fast as the tortoise, and the latter has ten feet start. When Achilles has gone ten feet the tortoise is one foot in

front of him; when Achilles has gone one foot further the tortoise is $\frac{1}{10}$ ft. in front; when Achilles has gone $\frac{1}{10}$ ft. further the tortoise is $\frac{1}{100}$ ft. in front; and so on, without end; therefore Achilles will never catch the tortoise. The fallacy, of course, lies in the surreptitious transcending of the convergent process when the word "never" is used in the conclusion. Let us indicate the successive positions of Achilles referred to, by the ordinal numbers 1, 2, 3, ... suffixed to the letter A, so that A_1, A_2, A_3, ... represent the positions of Achilles mentioned in the paradox. These points A_1, A_2, A_3, ... have a limiting point, which represents the place

$$\frac{B_2}{A_1} \qquad \frac{B_3\ B_4\ B_\omega}{A_2\ A_3\ A_4\ A_\omega} \qquad \frac{}{A_{\omega+1}\ A_{\omega+2}\ A_{\omega 2}}$$

where Achilles actually catches the tortoise. This limiting point is not contained in the set of points A_1, A_2, A_3, ...; if we wish to represent it, we must introduce a new symbol ω, and denote the point by A_ω. This symbol ω represents Cantor's first transfinite ordinal number. It does not occur in the series 1, 2, 3, ..., but is preceded by all these numbers, and yet there is no number immediately preceding it; it is the first of a new series of numbers.

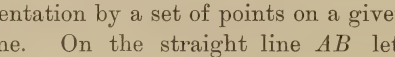
I may now, perhaps, be allowed to tamper with this classical paradox to the extent of supposing that there is a second tortoise moving at the same rate as the first, and ten feet in front of it, and of supposing that we wish to represent the positions of Achilles when he is 1 ft., $\frac{1}{10}$ ft., $\frac{1}{100}$ ft., ... behind the second tortoise. To represent these positions we naturally take $A_{\omega+1}$, $A_{\omega+2}$, $A_{\omega+3}$, ...; the place where Achilles catches the second tortoise will be denoted by $A_{\omega+\omega}$ or $A_{\omega 2}$. These numbers, $\omega+1$, $\omega+2$, $\omega+3$, ..., $\omega 2$, are the transfinite ordinal numbers immediately following ω. It thus appears that, as $\omega+1$ succeeds ω, the two cannot be regarded as identical; thus, $\omega+1 > \omega$. If now we had commenced by denoting the positions of Achilles when he was 100, 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, ... ft. behind the first tortoise by B_1, B_2, B_3 ..., so that A_n and B_{1+n} represent the same point, we see that the place where the tortoise is caught would still have to be represented by B_ω; it is, consequently, necessary to distinguish between $\omega+1$ and $1+\omega$, and to write $1+\omega=\omega$. The two relations $\omega = 1+\omega$, $\omega+1 > \omega$, enable us to illustrate the extent to which a finite number is able to maintain itself against a transfinite one.

The above perennially instructive example of a limiting process at once throws light upon the relation of the unending process to the limit, and upon the necessity for the introduction of transfinite

numbers for the representation of the limit which is not itself contained within the region of the convergent process. If we could imagine that we had no independent knowledge of the position and existence of the point at which Achilles overtakes the tortoise, we should be in the position of the older analyst or geometer in face of most of the problems which he solved. In the above paradox we have, by artificially involving ourselves in a convergent process, placed the limiting point, and all points beyond it, outside the reach of the process itself; for the representation of this point and the points beyond it we have to commence anew with a fresh series of ordinal numbers. The only reason why, in ordinary life, transfinite numbers are unnecessary is because we do not make use of such convergent processes. It would be easy for me to arrange artificially a series of points of time during the delivery of the present address which would be such that the moment of the termination of my address could only be represented by a transfinite number; higher transfinite numbers would be required to denote the times of all subsequent events this evening.

Although, for the reason I have indicated, transfinite numbers are unnecessary for the purposes of ordinary life, this is by no means the case in certain departments of mathematical analysis, where we are in many cases compelled to make use of convergent processes. It appears that a region which from one point of view belongs to the finite, may from another point of view belong to the transfinite, and it is frequently just this latter point of view which the exigencies of analysis compel us to adopt; hence arises the necessity for, and the justification of, the use of transfinite ordinal numbers.

I propose, as an illustration of the use of transfinite ordinal numbers, to give a simple means which I have devised for their systematic representation by a set of points on a given finite segment of a straight line. On the straight line AB let us denote by $P_0, P_1, P_2, P_3, \ldots$ those points at which the expression $\log_k \frac{AB}{PB}$, where k is a fixed number greater than unity, has the values $0, 1, 2, 3, \ldots$



the point P_0 coincides with A, and the point B can only be represented by P_ω. Now take any one of the segments $P_r P_{r+1}$; this we may for convenience represent on an enlarged scale; denote by $Q_{r0}, Q_{r1}, Q_{r2}, \ldots$

the points on $P_r P_{r+1}$, at which $\log_k \frac{P_r P_{r+1}}{Q P_{r+1}}$ takes the values $0, 1, 2, 3, \ldots$;

$$P_r \qquad \overbrace{Q_{r1}\ Q_{r2}\ \ldots\ldots\ Q_{r\omega}} \qquad P_{r+1}$$

thus P_{r+1} can only be represented by $Q_{r\omega}$. Suppose this to have been done with every segment of AB, and now imagine all the points Q to be marked on AB, and to be numbered from left to right;

in $P_0 P_1$ we shall have $0, 1, 2, 3, \ldots, \omega$,

in $P_1 P_2$,, $\omega+1, \omega+2, \ldots, \omega 2$,

in $P_2 P_3$,, $\omega 2+1, \omega 2+2, \ldots, \omega 3$;

the point B can be represented by $\omega\omega$ or ω^2. If now we proceed to take each segment $Q_{rs} Q_{r,s+1}$, and to divide this in a similar manner at points R for which $\log_k \frac{Q_{rs} Q_{r,s+1}}{R Q_{r,s+1}}$ has the values $0, 1, 2, 3, \ldots$, and then imagine all the points R thus obtained to be marked on the original straight line AB, and numbered as before, from left to right, it will be seen that all the numbers $\omega^2 p + \omega q + r$ will be required, and that the point B can be represented by ω^3. The points $P_0, P_1, \ldots, P_\omega$ will have for their ordinal numbers $0, \omega^2, \omega^2 2, \ldots, \omega^3$; the point Q_{rs} will be numbered $\omega^2 r + \omega s$; the finite numbers are all used up in the first sub-segment of AB. By proceeding in a similar manner to further sub-division, we may exhibit on AB the ordinal numbers

$$\omega^n p_n + \omega^{n-1} p_{n-1} + \ldots + p_0,$$

and the point B will then be represented by ω^{n+1}. The distance from A of the point represented by any of these numbers can be easily expressed.

I turn now to the subject of the transfinite cardinal numbers. Every two finite aggregates between the elements of which a (1, 1) correspondence can be established have the same cardinal number, and the cardinal number of an aggregate is independent of the order in which the objects are arranged. The extension of this notion of cardinal number to non-finite aggregates leads to the conception of the power (*Mächtigkeit*, *puissance*) of an infinite aggregate, which power is represented by a transfinite cardinal number. Two infinite aggregates between the elements of which a (1, 1) correspondence can be established have the same power or cardinal number. Thus, the aggregate of all rational numbers, or of rational numbers in a given interval, has the same power or

cardinal number as the aggregate of integral numbers. This may be denoted by a, and is the first transfinite cardinal number. As I have before mentioned, the continuum of real numbers cannot be placed in (1, 1) correspondence with the integral numbers, and has, therefore, a power or cardinal number different from a; this is usually denoted by c. It is a surprising fact that a continuum of two, three, or any number of dimensions has the same power c as that of the linear continuum—*i.e.*, a (1, 1) correspondence exists between all points in an area or in a volume and the points in a straight line, or in a finite segment of a straight line. It was, in fact, until recently supposed that all known aggregates have either the power of the aggregate of natural numbers or else that of the continuum. It has, however, now been shown that the aggregate of all possible functions of a variable x of which the domain is a continuous interval (a, b) has a power higher than that of the continuum; this higher cardinal number is denoted by f. It should, however, be observed that, if the function is restricted to be analytic, the aggregate of such functions then has the power c of the continuum. The numerous attempts which have been made to prove that a and c are consecutive cardinal numbers—that is to say, that no aggregate exists whose power exceeds a and is less than c—have hitherto been unsuccessful. This remains, for the present, as a *hiatus* in the theory of transfinite cardinal numbers. I have here been able to touch only the fringe of the subject of transfinite numbers; but my object has been to indicate, as clearly as is possible in the necessarily brief space I have allotted to them, how they necessarily arise when we try to investigate the peculiarities of non-finite aggregates. The profound study of these numbers and their relations, and especially of their connexion with the theory of the types of ordered aggregates, which has been made by G. Cantor, who, I am proud to remember, has been added to our list of foreign members during my term of office, has resulted in the creation of a veritable arithmetic of the infinite, which seems to be destined to have an ever-increasing range of application in analysis and geometry.

The place which the conception of the infinite occupies in the various schemes of geometry, especially the manner in which infinite elements are adjoined to the finite elements, is a subject on which many interesting remarks might be made. This, however, does not belong to the subject of my discourse this evening, and would, in any case, better be left for treatment by some one more competent to discuss it than I can claim to be.

In the minds of many men who are engaged in the active work of assisting in the progressive development of science, there is a certain impatience with what they are apt to regard as a hypercritical attitude towards fundamental concepts; this feeling of impatience exists perhaps in exceptionally large measure in the English mind, whose genius is in a preponderating degree directed towards the concrete, and upon which, to a considerable extent, purely abstract questions seem to exercise a peculiar repulsion. However, taken on the whole, the impulse towards clear thinking, which leads men to make an ever renewed dissection of the fundamentals of science, and to an ever renewed attempt to state fundamental principles in a form which shall satisfy more nearly the canons of logical thought, is an ineradicable tendency of the human mind, and I, for one, cannot but regard its presence as one of the conditions which are in the long run necessary to render possible the progress of scientific knowledge. Even from the point of view of those who regard mathematics as existing exclusively for the purposes of physical research, it is a short-sighted policy to discourage that free development of mathematics on its abstract side, which is probably a necessary stage in the process of sharpening the tools which mathematics provides for the use of the physicist.

Even in the remarks on one aspect of our science which I have made this evening, I think it has been apparent that criticism, and even erroneous criticism, is not infrequently the parent of construction. It may well be true that perfect intellectual transparency with regard to the fundamentals of any branch of knowledge is an unattainable ideal. May it not even be the case that a perfect comprehension of anything would involve a perfect comprehension of everything? Are there not those with us who assert that an analysis of the abstract creations of the human mind inevitably, when pushed far enough, leads to contradiction, and that this is a necessary consequence of the divorce of these ideal objects from reality? However this may be, the thinkers of each age must do what they can with the possibilities open to them, in faith that, however far short of what is completely satisfactory to the intellect may be the results to which they are led, the efforts of their generation in the direction of criticism, as well as of construction, may be found by those who come after them not to have been entirely fruitless.

APPENDIX TO PRESIDENT'S ADDRESS.

A few references to the literature of the subject are added.

Irrational Numbers and Limits.—Dedekind, *Stetigkeit und irrationale Zahlen*, Brunswick, 1872 (English translation in Dedekind, *Essays on Number*, Chicago, 1901). Heine, *Die Elemente der Functionenlehre*, Crelle, *J. f. Math.*, Bd. lxxiv., 1872. Tannery, J., *Théorie des Fonctions d'une Variable*, Paris, 1886. Pringsheim, "Irrationalzahlen und Konvergenz unendlicher Prozesse," in *Ency. d. math. Wiss.*, Bd. i., Heft 1, Leipzig, 1898. Mathews, "Number," in *Ency. Brit. Supplement*, Vol. xxxi., Edinburgh, 1902. Love, "Functions of Real Variables," in *Ency. Brit. Supplement*, Vol. xxviii.

Arithmetized Analysis.—Dini-Lüroth, *Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse*, Leipzig, 1892. Jordan, *Cours d'Analyse*, Paris, 1893–1896. Klein, *Anwendung der Differential- und Integralrechnung auf Geometrie*, Leipzig, 1902. For Kronecker's complete arithmetization referred to on p. 127, see his memoirs in Crelle, *J. f. Math.*, Bde. xcii., ci., 1882, 1887. For the theory of divergent series referred to on p. 121, see Borel, *Leçons sur les Séries divergentes*, Paris, 1901. For philosophical discussions of the foundations of analysis, see P. du Bois Reymond, *Allgemeine Functionenlehre*, Tübingen, 1882 (French translation by Milhaud and Girot, Nice, 1887); Couturat, *De l'Infini mathématique*, Paris, 1896; Poincaré, *La Science et l'Hypothèse*, Paris [1902].

Theory of Aggregates and Transfinite Numbers.—Cantor's chief memoirs are in *Acta Mathematica*, tt. ii., vii., and *Math. Ann.*, Bde. xvii., xx., xxi., xxiii., xlvi., xlix. A sketch of the theory is given by Schoenflies in *Ency. d. math. Wiss.*, Bd. i., Heft 2, Leipzig, 1899, and a more complete account by the same writer in *Jahresberichte d. Deutschen Math.-Vereinigung*, Bd. viii., 1900. For applications of the theory to analysis, see Jordan's *Cours*, quoted above, and Borel, *Leçons sur la Théorie des Fonctions*, Paris, 1898. For Veronese's conception of the continuum, see his book *Fondamenti di geometria*, Padua, 1891 (German translation, Leipzig, 1894).

On Wave-Propagation in Two Dimensions. *By* HORACE LAMB, F.R.S. Received November 11th, 1902. Read November 13th, 1902. Revised December 22nd, 1902.

1. The chief object of this paper is to work out and illustrate graphically some problems relating to the divergence of waves from a centre of disturbance in a space of two dimensions, the source being of a more or less transient character. The waves in question may be, for example, cylindrical waves of sound, or waves travelling over a uniform and uniformly tense membrane.* In any case they are subject to an equation of the form

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right), \tag{1}$$

or in the case of symmetry about the origin, which is more especially considered,

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right). \tag{2}$$

In a space of one or of three dimensions the solution of such problems is extremely easy, owing to the existence of simple general integrals of the corresponding differential equations. Thus in one dimension we have

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2}, \tag{3}$$

with the solution $$\phi = f\left(t - \frac{x}{c}\right) + F\left(t + \frac{x}{c}\right); \tag{4}$$

whilst in three dimensions we have

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}\right), \tag{5}$$

* The case of waves on a sheet of water of uniform depth may be included, provided the dimensions of the "source" be large compared with the depth.

with the solution*

$$4\pi r\phi = f\left(t-\frac{r}{c}\right)+F\left(t+\frac{r}{c}\right). \qquad (6)$$

It may be well here to remind the reader that the formal resemblance between (4) and (6) is to some extent delusive; the physical interpretation reveals important points of difference, which have been insisted upon by Stokes† and Rayleigh.‡

The theory of solitary waves in *two* dimensions has attracted less attention,§ partly no doubt because of its inferior physical interest, and also because of a certain degree of mathematical obscurity which has attached to it owing to the absence of a simple general formula analogous to (4) and (6). The general integral of (2) which was given by Poisson,‖ viz.,

$$\phi = \int_0^\pi f(ct+r\cos\theta)\,d\theta+\int_0^\pi F(ct+r\cos\theta)\log(r\sin^2\theta)\,d\theta, \qquad (7)$$

is unsuitable for the purpose, as it does not discriminate between the waves which travel inwards and outwards respectively.

The proper analogue of (4) and (6) is, however, easily obtained if we start from the known solution for the case of waves diverging from a simple harmonic source ($e^{i\sigma t}$), viz.,¶

$$2\pi\phi = \int_0^\infty e^{i\sigma\left(t-\frac{r}{c}\cosh u\right)}\,du. \qquad (8)$$

If we generalize this by Fourier's theorem, we have at once

$$2\pi\phi = \int_0^\infty f\left(t-\frac{r}{c}\cosh u\right)du, \qquad (9)$$

corresponding to an arbitrary source $f(t)$. It is of course implied

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* The factor 4π is inserted for the sake of comparison with (9) below. It makes the "strength" of the source at the origin equal to $f(t)+F(t)$.

† *Phil. Mag.*, January, 1849; *Math. and Phys. Papers*, Vol. II., p. 82.

‡ *Theory of Sound*, Vol. II., §§ 274, 279.

§ Except in the case of cylindrical electric waves, which have been discussed to some extent by Heaviside, *Phil. Mag.*, November, 1888; *Electrical Papers*, Vol. II.

‖ *Journal de l'École Polytechnique*, Vol. XII., p. 215 (1821).

¶ *Cf.* Rayleigh, *Sound*, Vol. II., § 342; *Proc. Lond. Math. Soc.*, Vol. XIX., p. 504 (1888); *Scientific Papers*, Vol. III., p. 44.

that the form of $f(t)$ must be such that the integral is convergent. A sufficient (but not an essential) condition for this is

$$\lim_{z=-\infty} z^{1+\epsilon} f(z) = 0, \tag{10}$$

ϵ being any positive quantity; this is fulfilled of course whenever the source has been in action only for a finite time.* The complete formula, embracing both converging and diverging waves, is

$$2\pi\phi = \int_0^\infty f\left(t - \frac{r}{c}\cosh u\right) du + \int_0^\infty F\left(t + \frac{r}{c}\cosh u\right) du, \tag{11}$$

where the convergence of the second integral is assured if

$$\lim_{z=+\infty} z^{1+\epsilon} F(z) = 0. \tag{12}$$

The expression (11) is arrived at independently if we imagine a system of point-sources, such as the origin in (6), to be distributed uniformly along a straight line,† say the axis of z, distance from which is denoted in (11) by r.

The direct verification of (11) by substitution in the differential equation (2) is a matter of some nicety. Taking the first term alone, as in (9), we find, subject to certain conditions,

$$2\pi\left\{c^2\left(\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) - \frac{\partial^2\phi}{\partial t^2}\right\}$$

$$= \int_0^\infty \left\{\sinh^2 u \,.\, f''\left(t - \frac{r}{c}\cosh u\right) - \frac{c}{r}\cosh u \,.\, f'\left(t - \frac{r}{c}\cosh u\right)\right\} du$$

$$= \frac{c^2}{r^2}\int_0^\infty \frac{\partial^2}{\partial u^2} f\left(t - \frac{r}{c}\cosh u\right) du$$

$$= -\frac{c}{r}\left[\sinh u \,.\, f'\left(t - \frac{r}{c}\cosh u\right)\right]_{u=0}^{u=\infty} = 0.$$

The conditions referred to are obviously satisfied whenever $f(z)$ vanishes for negative values of z exceeding a certain limit, as well as

* The referees point out that a result equivalent to (9) was obtained (in a different manner) by T. Levi-Civita, *Nuovo Cimento*, Ser. 4, t. VI. (1897).

† This is the method by which Rayleigh (*loc. cit.*) deduces (8) from the formula for a simple-harmonic point-source in three dimensions.

in other cases less easily defined.* Again,

$$-2\pi r\frac{\partial\phi}{\partial r}=\frac{r}{c}\int_0^\infty \cosh u\,.\,f'\left(t-\frac{r}{c}\cosh u\right)du$$

$$=\frac{r}{c}\int_0^\infty (\sinh u+e^{-u})f'\left(t-\frac{r}{c}\cosh u\right)du$$

$$=-\int_0^\infty \frac{\partial}{\partial u}f\left(t-\frac{r}{c}\cosh u\right)du+\frac{r}{c}\int_0^\infty e^{-u}f'\left(t-\frac{r}{c}\cosh u\right)du$$

$$=-\left[f\left(t-\frac{r}{c}\cosh u\right)\right]_{u=0}^{u=\infty}+\frac{r}{c}\int_0^\infty e^{-u}f'\left(t-\frac{r}{c}\cosh u\right)du$$

$$=f\left(t-\frac{r}{c}\right)+\frac{r}{c}\int_0^\infty e^{-u}f'\left(t-\frac{r}{c}\cosh u\right)du,$$

under similar restrictions. Hence

$$\lim_{r=0}\left(-2\pi r\frac{\partial\phi}{\partial r}\right)=f(t), \tag{13}$$

which verifies the statement previously made, as to the strength of the source in (9).†

A similar process will apply to the second term in (11), subject to certain restrictions which are obviously fulfilled when $F(z)$ vanishes for positive values of z exceeding a certain limit.

It may be remarked that from (11) we can derive the general solution of (1), by a known analogy. In polar coordinates we have

$$\phi=\Sigma_0^\infty r^n\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n\left\{\Phi(r,t)\cos n\theta+\Psi(r,t)\sin n\theta\right\}, \tag{14}$$

where $\Phi(r,t)$ and $\Psi(r,t)$ are functions of the same type as the right-hand member of (11).

2. We may apply the formula (9) to trace the effect of a temporary source at the origin varying according to some simple prescribed law. The most immediate interpretation of our results will be to suppose that ϕ denotes the transverse displacement at any point of an unlimited tense membrane. The waves are then due to a local

* The verification is very similar to that given by Levi-Civita. It is to be remarked that the process may fail, owing to the form of f, even when (9) is really a solution of (2). An instance is supplied by the function (8).

† In the application to sound-waves the definition of ϕ is supposed to be as in the author's *Hydrodynamics*, so that the radial velocity is $-\partial\phi/\partial r$. If s be the "condensation," we have on the same convention $c^2s=\partial\phi/\partial t$.

application of normal force at the origin, whose aggregate amount, reckoned in the direction of ϕ positive, is $f(t)$ multiplied by the tension of the membrane. This appears at once from (13).

If we suppose that everything is quiescent until the time $t = 0$, so that $f(t)$ vanishes for negative values of t, we see from (9), or from the equivalent form

$$2\pi\phi = \int_{-\infty}^{t-r/c} \frac{f(\theta)\, d\theta}{\sqrt{\left\{(t-\theta)^2 - \frac{r^2}{c^2}\right\}}}, \tag{15}$$

that ϕ vanishes at any point so long as $t < r/c$. If, moreover, the source acts only for a finite time τ, so that $f(t) = 0$ for $t > \tau$, we have, for $t > r/c + \tau$,

$$2\pi\phi = \int_{0}^{\tau} \frac{f(\theta)\, d\theta}{\sqrt{\left\{(t-\theta)^2 - \frac{r^2}{c^2}\right\}}}. \tag{16}$$

This expression does not as a rule vanish; the wave accordingly is not sharply defined in the rear, as it is in front, but has, on the contrary, a sort of "tail," whose form, when $t - r/c$ is large compared with τ, is given approximately by the formula*

$$2\pi\phi = \frac{1}{\sqrt{\left(t^2 - \frac{r^2}{c^2}\right)}} \int_0^{\tau} f(\theta)\, d\theta. \tag{17}$$

As a first example, let us suppose that the source is constant during the interval of its existence, say,

$$\begin{aligned} f(t) &= 0, \quad \text{for} \quad t < 0; \\ &= 1, \quad \text{for} \quad 0 < t < \tau; \\ &= 0, \quad \text{for} \quad t > \tau. \end{aligned}$$

We easily find from (9) that

$$\left.\begin{aligned} 2\pi\phi &= 0, && \text{for} \quad t < \frac{r}{c} \\ &= \cosh^{-1}\frac{ct}{r}, && \text{for} \quad \frac{r}{c} < t < \frac{r}{c} + \tau \\ &= \cosh^{-1}\frac{ct}{r} - \cosh^{-1}\frac{c(t-\tau)}{r}, && \text{for} \quad t > \frac{r}{c} + \tau \end{aligned}\right\} \tag{18}$$

* The existence of the "tail," in the case of cylindrical electric waves, was noted by Heaviside, *loc. cit.*

To tabulate the results, we require a series of values of a variable u corresponding to equidistant values of $\cosh u$. For this purpose existing tables of the hyperbolic functions are not very convenient; but for the more important part of the wave it is sufficient to assume

$$\cosh u = 1 + \tfrac{1}{2}z^2, \tag{19}$$

where z^2 is small; this makes

$$u = z\,(1 - \tfrac{1}{24}z^2 + \tfrac{3}{640}z^4 - \ldots). \tag{20}$$

The first two terms of the series are alone sensible within the range of the following table.

$\cosh u$	u	$\cosh u$	u	$\cosh u$	u
1·000	0	1·010	·1413	1·020	·1997
1	·0447	1	·1482	25	·2231
2	·0632	2	·1548	30	·2443
3	·0774	3	·1611	35	·2638
4	·0894	4	·1671	40	·2819
5	·1000	5	·1730	45	·2989
6	·1095	6	·1787	50	·3149
7	·1183	7	·1841		
8	·1264	8	·1895		
9	·1341	9	·1946		

The diagram (Fig. 1), constructed with ϕ as ordinate and t as abscissa, exhibits the variation of ϕ at a particular point ($r = 100c\tau$)

Fig. 1.

as the wave passes over it. Within the range shown, the same curve also indicates very fairly the profile of the wave after a time t which is about equal to 100τ, it being understood, of course (in this interpretation), that the direction of propagation is from right to left. The abrupt front of the wave and the indefinite prolongation in the "tail" are well marked.

The diagram exhibits certain peculiarities due to the discontinuous nature of the source. To obtain a representation of a source of a less abrupt character, we may suppose the strength to rise uniformly from zero to a maximum, and then to fall uniformly to zero again at the same rate. We assume, therefore,

$$\begin{aligned} f(t) &= 0, && \text{for} \quad t < -\tau, \\ &= 1 + \frac{t}{\tau}, && \text{for} \quad -\tau < t < 0; \\ &= 1 - \frac{t}{\tau}, && \text{for} \quad 0 < t < \tau; \\ &= 0, && \text{for} \quad t > \tau. \end{aligned}$$

We find, from (9), or (15), after some reductions,

$$\left.\begin{aligned} 2\pi\phi &= 0, \quad \text{for} \quad t < \frac{r}{c} - \tau \\ &= \frac{r}{c\tau}\chi\left\{\frac{c(t+\tau)}{r}\right\}, \quad \text{for} \quad \frac{r}{c} - \tau < t < \frac{r}{c} \\ &= \frac{r}{c\tau}\left[\chi\left\{\frac{c(t+\tau)}{r}\right\} - 2\chi\left(\frac{ct}{r}\right)\right], \quad \text{for} \quad \frac{r}{c} < t < \frac{r}{c} + \tau \\ &= \frac{r}{c\tau}\left[\chi\left\{\frac{c(t+\tau)}{r}\right\} + \chi\left\{\frac{c(t-\tau)}{r}\right\} - 2\chi\left(\frac{ct}{r}\right)\right], \quad \text{for} \quad t > \frac{r}{c} + \tau \end{aligned}\right\}.$$

where $$\chi(x) = x\cosh^{-1}x - \sqrt{(x^2-1)} \qquad (21)$$

This suggests the tabulation of the function $u\cosh u - \sinh u$ for equidistant values of $\cosh u$. If we put

$$\cosh u = 1 + \tfrac{1}{2}z^2,$$

as before, we find

$$u\cosh u - \sinh u = \tfrac{1}{3}z^3\left(1 - \tfrac{1}{40}z^2 + \tfrac{9}{4480}z^4 - \ldots\right); \qquad (22)$$

the third term is just sensible towards the end of the following table.

$\cosh u$	$u\cosh u - \sinh u$	$\cosh u$	$u\cosh u - \sinh u$	$\cosh u$	$u\cosh u - \sinh u$
1·000	0	1·010	·0009423	1·020	·0026640
1	·0000298	1	·0010871	5	37221
2	843	2	12386	30	48917
3	·0001549	3	13966	5	61627
4	2385	4	15607	40	75275
5	3333	5	17308	5	89799
6	4380	6	19065	50	·0105148
7	5520	7	20880		
8	6743	8	22748		
9	8046	9	24668		

The relation between ϕ and t at a distance $r = 100c\tau$ is shown in Fig. 2, which corresponds in scale with Fig. 1, the time-integral of

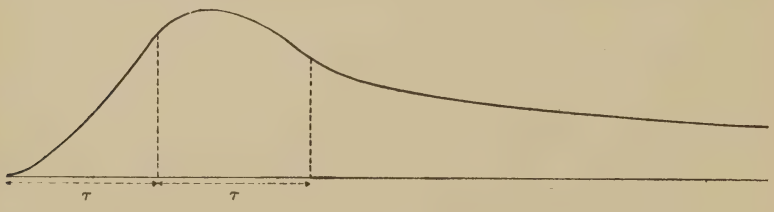

Fig. 2.

the source being the same in each case. It will be noticed that there is now no abrupt change of direction in the curve—only changes of curvature.

3. A solitary wave, free from every degree of discontinuity, is obtained if we assume a source

$$f(t) = \frac{\tau}{t^2 + \tau^2}. \qquad (23)$$

This has, it is true, no definite beginning or ending, but the preceding examples enable us to understand what features in the result are to

be attributed solely to this cause. For purposes of calculation it is a little simpler to assume

$$f(t) = \frac{1}{t - i\tau}, \tag{24}$$

and retain, in the end, only the imaginary part.* We have then, from (9),

$$2\pi\phi = \int_0^\infty \frac{du}{t - \frac{r}{c}\cosh u - i\tau}$$

$$= 2\int_0^1 \frac{dz}{t - \frac{r}{c} - i\tau - \left(t + \frac{r}{c} - i\tau\right) z^2}, \tag{25}$$

where

$$z = \tanh \tfrac{1}{2}u.$$

We now write

$$t - \frac{r}{c} - i\tau = a^2 e^{-2i\alpha}, \quad t + \frac{r}{c} - i\tau = b^2 e^{-2i\beta}, \tag{26}$$

where we may suppose that a, b are positive, and that the angles 2α, 2β lie between 0 and π. We have

$$\left.\begin{aligned} a^4 = \left(t - \frac{r}{c}\right)^2 + \tau^2, \quad b^4 = \left(t + \frac{r}{c}\right)^2 + \tau^2 \\ \tan 2\alpha = \frac{c\tau}{ct - r}, \quad \tan 2\beta = \frac{c\tau}{ct + r} \end{aligned}\right\}. \tag{27}$$

It appears that $a \lessgtr b$ according as $t \gtrless 0$, and that $\alpha - \beta$ lies between 0 and $\frac{1}{2}\pi$. With this notation

$$2\pi\phi = 2\int_0^1 \frac{dz}{a^2 e^{-2i\alpha} - b^2 e^{-2i\beta} z^2}$$

$$= \frac{1}{ab} e^{i(\alpha+\beta)} \left(\int_0^1 \frac{dz}{z + \frac{a}{b} e^{-i(\alpha-\beta)}} - \int_0^1 \frac{dz}{z - \frac{a}{b} e^{-i(\alpha-\beta)}} \right)$$

$$= \frac{1}{ab} e^{i(\alpha+\beta)} \left\{ \log\left(z + \frac{a}{b} e^{-i(\alpha-\beta)}\right) - \log\left(z - \frac{a}{b} e^{-i(\alpha-\beta)}\right) \right\}. \tag{28}$$

---

* *Cf.* Lord Kelvin, *Phil. Mag.*, Feb., 1899, p. 189. The graph of the function (23) is the curve marked *A* in Fig. 6, p. 157.

To interpret the logarithms, let us mark, in the plane of a complex variable z, the points

$$I=+1, \quad P=-\frac{a}{b}e^{-i(\alpha-\beta)}, \quad Q=\frac{a}{b}e^{-i(\alpha-\beta)},$$

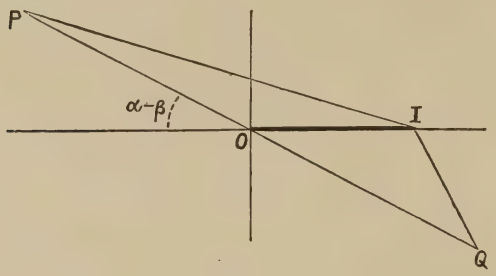

FIG. 3.

Since the integrals in the second line of (28) are to be taken along the path OI, the proper value of the third line is

$$\frac{1}{ab}e^{i(\alpha+\beta)}\left\{\left(\log\frac{IP}{OP}+i\,.\,OPI\right)-\left(\log\frac{IQ}{OQ}-i\,.\,OQI\right)\right\}, \qquad (29)$$

where real logarithms and positive values of the angles are to be understood. Hence, rejecting all but the imaginary part, we find

$$2\pi\phi=\frac{1}{ab}\sin(\alpha+\beta)\log\frac{IP}{IQ}+\frac{1}{ab}\cos(\alpha+\beta)(OPI+OQI), \qquad (30)$$

as the expression of the disturbance due to the source (23). Since

$$\left.\begin{aligned}\tan OPI=\frac{b\sin(\alpha-\beta)}{a+b\cos(\alpha-\beta)}, \quad \tan OQI=\frac{b\sin(\alpha-\beta)}{a-b\cos(\alpha-\beta)},\\ \frac{IP}{IQ}=\frac{\sin OQI}{\sin OPI}\end{aligned}\right\}, \qquad (31)$$

ϕ is uniquely determined as a function of a, b, α, β, and thence, by (27), of r and t.

Some special points may be noted. When $t=0$, we have

$$a=b=\left(\frac{r^2}{c^2}+r^2\right)^{\frac{1}{2}}, \quad \alpha+\beta=\tfrac{1}{2}\pi, \quad \tan 2\beta=\frac{cr}{r};$$

and therefore

$$2\pi\phi=\frac{c}{\sqrt{(r^2+c^2r^2)}}\log\frac{1+\tan\beta}{1-\tan\beta}. \qquad (32)$$

This may be confirmed by an independent calculation from (9), with $t = 0$ throughout. If r be large compared with $c\tau$, the formula makes $2\pi\phi = c^2\tau/r^2$, approximately.

Again, when t is large compared with r/c (as well as with τ), and negative, we find

$$\alpha+\beta = \pi+\frac{\tau}{t}, \quad \alpha-\beta = \frac{\tau}{t}\cdot\frac{r}{ct},$$

$$a = \sqrt{\left(-t+\frac{r}{c}\right)}, \quad b = \sqrt{\left(-t-\frac{r}{c}\right)},$$

approximately, whence

$$OPI = \tfrac{1}{2}\frac{\tau}{t}\cdot\frac{r}{ct}, \quad OQI = -\frac{\tau}{t}.$$

Keeping only the most important term in (30), we obtain

$$2\pi\phi = -\frac{1}{\sqrt{\left(t^2-\frac{r^2}{c^2}\right)}}\cdot\frac{\tau}{t}\log\left(-\frac{2ct}{r}\right). \tag{33}$$

On the other hand, when t is large compared with r/c, and positive,

$$\alpha+\beta = \frac{\tau}{t}, \quad \alpha-\beta = \frac{\tau}{t}\cdot\frac{r}{ct},$$

$$a = \sqrt{\left(t-\frac{r}{c}\right)}, \quad b = \sqrt{\left(t+\frac{r}{c}\right)},$$

approximately, and

$$OPI = \tfrac{1}{2}\frac{\tau}{t}\cdot\frac{r}{ct}, \quad OQI = \pi-\frac{\tau}{t}.$$

The most important part of (30) is then

$$2\pi\phi = \frac{\pi}{\sqrt{\left(t^2-\frac{r^2}{c^2}\right)}}, \tag{34}$$

which agrees with (17) when the limits of integration are suitably modified.

The crest of the wave at any instant will be in the neighbourhood of $r = ct$. If we put $r = ct$, exactly, we have

$$\alpha = \tfrac{1}{4}\pi, \quad \tan 2\beta = \tfrac{1}{2}\frac{c\tau}{r}, \quad a = \sqrt{\tau}, \quad b = \sqrt[4]{\left(\frac{4r^2}{c^2}+\tau^2\right)}.$$

If, moreover, r be large compared with $c\tau$, the angle PIQ will be small, and

$$2\pi\phi = \frac{\pi}{2\tau}\sqrt{\left(\frac{c\tau}{r}\right)}, \tag{35}$$

approximately.

To examine more closely the progress of the disturbance at a distance r which is large compared with $c\tau$, as the crest of the wave passes, we put

$$t = \frac{r}{c} + \tau\tan\eta.$$

This makes $\quad \alpha = \tfrac{1}{4}\pi - \tfrac{1}{2}\eta, \quad a = \surd(\tau\sec\eta),$

whilst $\quad \beta = \tfrac{1}{4}\frac{c\tau}{r}, \quad b = \sqrt{\left(\frac{2r}{c}\right)},$

approximately, provided $r/c\tau$ be large compared with the greatest value of $\tan\eta$ considered. The ratio a/b will then be small, so that $IP/IQ = 1$, nearly, whilst PIQ is a small angle. The most important part of (30) is then

$$2\pi\phi = \frac{\pi}{ab}\cos\alpha = \frac{\pi}{\surd 2\,\tau}\sqrt{\left(\frac{c\tau}{r}\right)}\cos(\tfrac{1}{4}\pi - \tfrac{1}{2}\eta)\surd(\cos\eta). \tag{36}$$

The following table gives a series of values of the functions

$$x = \tan\eta, \quad y = \cos(\tfrac{1}{4}\pi - \tfrac{1}{2}\eta)\surd(\cos\eta).$$

$2\eta/\pi$	x	y	$2\eta/\pi$	x	y
0	0	·707	±·5	±1·000	{·777 ·322
±·1	±·158	{·756 ·645	±·6	±1·376	{·729 ·237
±·2	±·325	{·789 ·573	±·7	±1·963	{·655 ·157
±·3	±·510	{·805 ·493	±·8	±3·078	{·549 ·087
±·4	±·727	{·801 ·408	±·9	±6·314	{·394 ·031

The maximum value of y is ·806, corresponding to $\eta = \tfrac{1}{6}\pi$, or

$t = r/c + \cdot 577\tau$. The curve in Fig. 4, constructed with x, y as co-ordinates, shows the variation of ϕ with t.

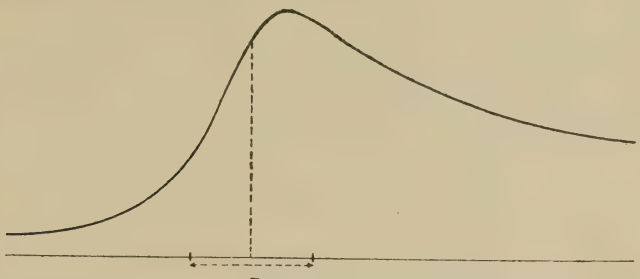

FIG. 4.

4. The most interesting feature in the preceding diagrams is, of course, the unsymmetrical character of the wave, and the prolongation of its rear in the form of the "tail" already referred to. If, however, the source change its sign, and especially if its time-integral be zero, the positive and negative tails will tend to obliterate one another, although they are not without influence on the residual wave-form. The almost complete cancelling in the special case mentioned is evident from the approximate formula (17). To illustrate the matter further, we may suppose the temporary constant source of § 2 to be followed immediately by a negative source of equal strength and duration. The resulting curve, easily constructed from the numerical table given on p. 146, is shown in Fig. 5. The tail is now insignificant,

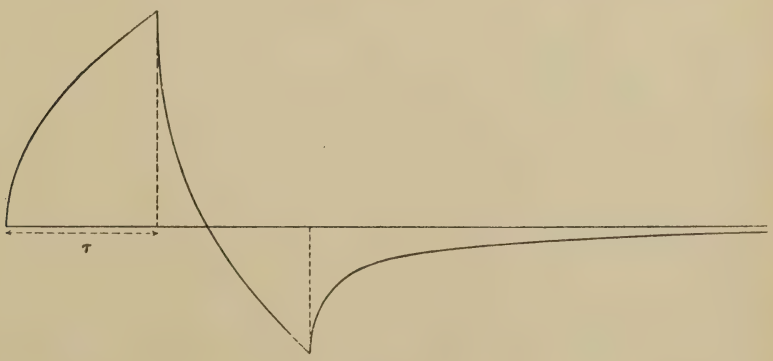

FIG. 5.

but it will be noticed that the depression which follows the primary

elevation has only about one-half its amplitude.* It is plain, moreover, that in any case of a periodic source suddenly beginning to act the first wave will be much higher than those which follow it.

Now that we are in possession of a number of examples (which might easily be multiplied) of the propagation of a solitary wave in two dimensions, it is of interest to examine how they are related to the corresponding results in one and in three dimensions.

The equation (1) is included in the general form

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \ldots + \frac{\partial^2\phi}{\partial x_n^2}\right), \tag{37}$$

which has been called the equation of wave-motion in n dimensions. In the case of symmetry about the origin, this reduces to

$$\frac{\partial^2\phi}{\partial t^2} = c^2\left(\frac{\partial^2\phi}{\partial r^2} + \frac{n-1}{r}\frac{\partial\phi}{\partial r}\right), \tag{38}$$

which includes (2) and (5) as particular cases. The equation (37) has been very fully discussed by Duhem,† who has insisted on the exceptional position occupied by the cases $n = 1$, $n = 3$; in particular he has shown, amongst other things, that it is only in these cases that (38) admits of a solution of the type

$$\phi = \psi(r) f\left(t - \frac{r}{c}\right). \tag{39}$$

However interesting and important such investigations may be from an analytical point of view, we must recognize that what they chiefly determine is the comparative facility of mathematical expression in the various cases. From a physical standpoint, the idea that the cases $n = 1$, $n = 3$ are closely analogous, whilst that of $n = 2$ occupies a sort of exceptional and outlying position, would be wholly misleading. A truer account of the matter would be to say that the cases $n = 1$, $n = 2$, $n = 3$ form a sequence, with a regular gradation of properties due to the increasing degree of mobility of the medium.‡

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* A further illustration is supplied accidentally by the diagrams on p. 157. The curve B shows the relation between ϕ and t for a source whose graph has the form C.

† *Hydrodynamique, Élasticité, Acoustique*, Paris, 1891, Vol. II., p. 138.

‡ The one common characteristic is that the boundary of a disturbed region spreads everywhere normally with the constant velocity c. It would be interesting to have a proof of this property direct from the differential equation (37). Duhem, adopting a method of Hugoniot, endeavours to supply this. The argument is interesting, but I do not think that it amounts to a demonstration. It appears to assume tacitly that the disturbance spreads *at some finite rate*; if this be granted, the conclusion that the rate is constant appears to be unexceptionable.

For the sake of a definite comparison between the three cases, we may take the case of air-waves, and examine the effect (A) of a plane-source, (B) of a line-source, and (C) of a point-source, whose "strength" is given in each case by (23). The results may be conveniently expressed in terms of the condensation s, which is given by the formula

$$c^2 s = \frac{\partial \phi}{\partial t}. \tag{40}$$

(A) In the case $n = 1$, we find, for $x > 0$,

$$s = \frac{\tau}{2c} \frac{1}{\left(t - \frac{x}{c}\right)^2 + \tau^2}. \tag{41}$$

(B) When $n = 2$, the value of s is to be deduced from (30). Near the "crest" of the wave a sufficient approximation is obtained by differentiating (36) with respect to $\tau \tan \eta$. We thus find

$$s = \frac{1}{4\sqrt{2}\, c^2 \tau^2} \sqrt{\left(\frac{c\tau}{r}\right)} \sin\left(\tfrac{1}{4}\pi - \tfrac{3}{2}\eta\right) \cos^{\frac{3}{2}} \eta. \tag{42}$$

The values of the functions

$$x = \tan \eta, \quad y = \sin\left(\tfrac{1}{4}\pi - \tfrac{3}{2}\eta\right) \cos^{\frac{3}{2}} \eta$$

are tabulated below.

$2\eta/\pi$	x	y	$2\eta/\pi$	x	y
0	0	+·707	±·5	±1·000	−·228 +·549
±·1	±·158	+·513 +·837	±·6	±1·376	−·265† +·365
±·2	±·325	+·287 +·882*	±·7	±1·963	−·233 +·199
±·3	±·510	+·066 +·838	±·8	±3·078	−·153 +·078
±·4	±·727	−·114 +·719	±·9	±6·314	−·060 +·014

* Maximum. † Minimum.

(C) In three dimensions we have

$$4\pi r\phi = \frac{\tau}{\left(t-\frac{r}{c}\right)^2+\tau^2}, \tag{43}$$

$$s = \frac{\tau}{2\pi c^2}\,\frac{\frac{r}{c}-t}{r\left\{\left(t-\frac{r}{c}\right)^2+\tau^2\right\}^2}. \tag{44}$$

The three cases are represented, with s as ordinate and t as abscissa, on p. 157. The scale of t is the same in each case, but there is, of course, no relation between the vertical scales. In (A) we have a wave of pure condensation; in (B) the primary condensation is followed by a rarefaction of less amount, but lasting for a longer time; whilst in (C) the condensation and rarefaction are anti-symmetrical. In (B) and (C) alike we necessarily have, at any point,

$$\int_{-\infty}^{\infty} s\,dt = 0, \tag{45}$$

in virtue of (40). If the source had been strictly limited in duration, the medium, in the case of three dimensions, would have remained absolutely* at rest after the passage of the wave, as in the case of one dimension, although for a different reason. In the intermediate case of two dimensions, there is only an asymptotic approach to rest.

5. If in a two-dimensional medium free from singular points we wish to calculate the disturbance consequent on arbitrary initial (symmetrical) conditions, it is probably simplest to adopt the method

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* The statement has reference to a *point*-source. If a source $f(t)$ be uniformly distributed over the surface of a sphere of radius a, we have

$$4\pi r\phi = \frac{c}{a}\,e^{-[ct-(r-a)]/a}\int_{-\infty}^{t-(r-a)/c} f(\lambda)\,e^{c\lambda/a}\,d\lambda$$

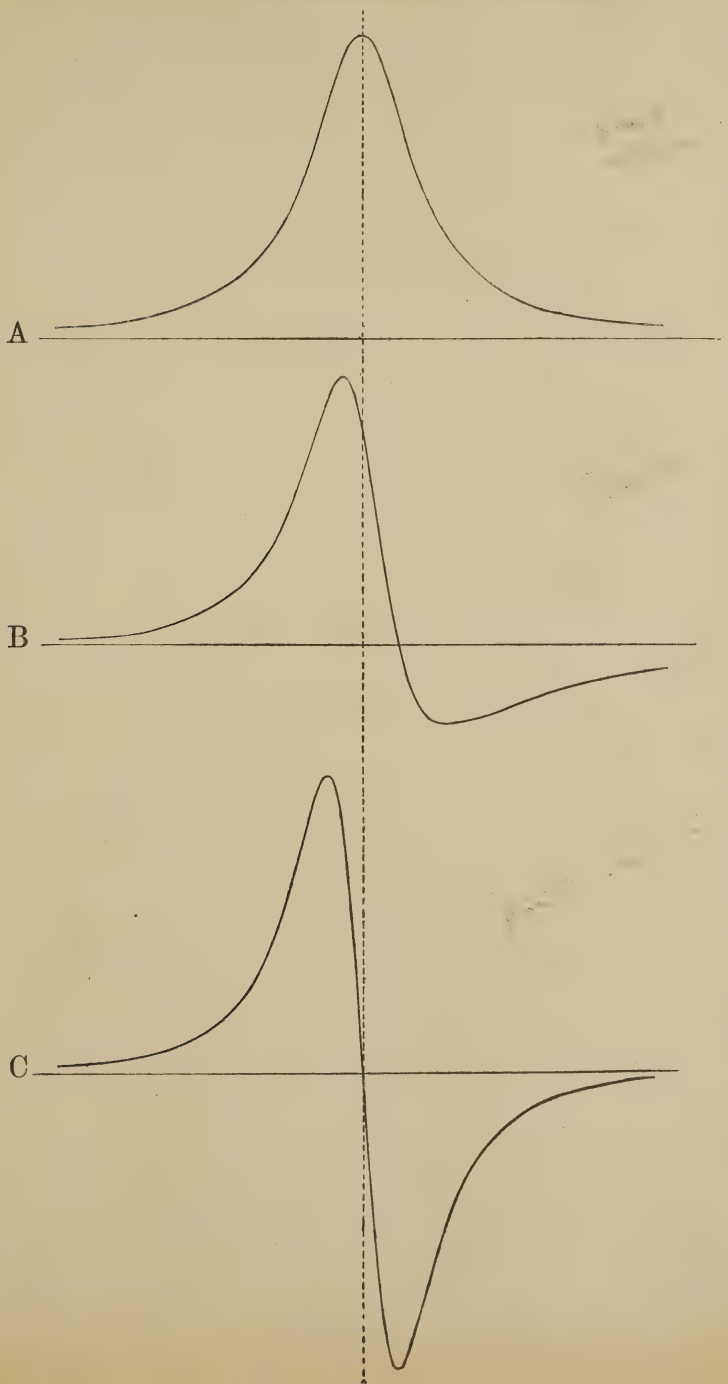
$$= \frac{c}{a}\int_0^{\infty} f\left(t-\frac{r-a}{c}-\theta\right)e^{-c\theta/a}\,d\theta,$$

as may be easily verified. If $f(t)=0$ for $t<0$ and for $t>\tau$, we find, for $t>(r-a)/c+\tau$,

$$4\pi r\phi = \frac{c}{a}\,e^{-[ct-(r-a)]/a}\int_0^{\tau} f(\lambda)\,e^{c\lambda/a}\,d\lambda,$$

so that ϕ falls only asymptotically to 0. But when $ct-r$ is only a moderate multiple of a the residual disturbance is already very small.

This point was noticed by Stokes (*l.c. ante*) in connection with the same problem. Reference may also be made to Kirchhoff, *Mechanik*, c. xxiii.


A
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indicated by Rayleigh,* treating the problem as a particular case of motion in three dimensions. If, however, we wish to follow a more analytical procedure, we require, in the first instance, the solution of the following subsidiary problem:—Given a function f, to find a function F, such that

$$\int_0^\infty F(r\cosh u)\,du = f(r). \tag{46}$$

Provided $f(\infty) = 0$, a condition naturally fulfilled in the applications we have in view, the solution is

$$F(z) = -\frac{2}{\pi} z \int_0^\infty f'(z\cosh v)\,dv. \tag{47}$$

For this makes

$$\int_0^\infty F(r\cosh u)\,du = -\frac{2}{\pi}\int_0^\infty r\cosh u\,du \int_0^\infty f'(r\cosh u\cosh v)\,dv. \tag{48}$$

Let us now write

$$x = r\cosh u\cosh v, \quad y = r\cosh u\sinh v,$$

so that

$$\frac{\partial(x, y)}{\partial(u, v)} = r^2\sinh u\cosh u.$$

Transforming to x, y as independent variables, we have

$$\int_0^\infty F(r\cosh u)\,du = -\frac{2}{\pi}\iint \frac{f'(x)\,dx\,dy}{\sqrt{(x^2-y^2-r^2)}}, \tag{49}$$

where the integrations extend over that part of the plane xy which lies above the axis of x and to the right of the positive branch of the hyperbola $x^2-y^2=r^2$. If we integrate first with respect to y, we find

$$\int_0^\infty F(r\cosh u)\,du = -\int_r^\infty f'(x)\,dx; \tag{50}$$

so that (46) is satisfied, subject to the condition stated.†

Writing (11) in the now slightly more convenient form

$$\phi = \int_0^\infty f(ct-r\cosh u)\,du + \int_0^\infty F(ct+r\cosh u)\,du, \tag{51}$$

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* *Sound*, § 275.

† This determination of the unknown function F in (46) is suggested by an investigation of Schlömilch, which is reproduced in Todhunter's *Functions of Laplace*, &c., § 441.

we have to determine the arbitrary functions so that

$$\varphi = \psi(r), \quad \frac{\partial \varphi}{\partial t} = \chi(r), \tag{52}$$

say, for $t = 0$; that is, we must have

$$\left.\begin{aligned} \int_0^\infty \{f(-r\cosh u) + F(r\cosh u)\}\, du &= \psi(r) \\ \int_0^\infty \{f'(-r\cosh u) + F'(r\cosh u)\}\, du &= \frac{1}{c}\chi(r) \end{aligned}\right\}. \tag{53}$$

These conditions will be satisfied, in virtue of (47), provided

$$f(-r) + F(r) = -\frac{2r}{\pi}\int_0^\infty \psi'(r\cosh v)\, dv, \tag{54}$$

and

$$f'(-r) + F'(r) = -\frac{2r}{\pi c}\int_0^\infty \chi'(r\cosh v)\, dv, \tag{55}$$

it being assumed that $\psi(r)$ and $\chi(r)$ vanish for $r = \infty$. The condition (55) may be replaced by

$$f(-r) - F(r) = \frac{2}{\pi c}\int r\, dr \int_0^\infty \chi'(r\cosh v)\, dv, \tag{56}$$

where the lower limit of the integration with respect to r is indifferent. The solution of our problem is now virtually complete; for (54) and (56) determine f for positive and F for negative values of the argument, whilst the determination of f for positive arguments is supplied by

$$f(ct) + F(ct) = 0, \tag{57}$$

which is the condition of "continuity" at the origin.*

As an example, suppose that initially $\varphi = 0$, everywhere, whilst $\partial\varphi/\partial t = 1$ for $r < a$, and $= 0$ for $r > a$.† We have, from (54),

$$f(-r) + F(r) = 0. \tag{58}$$

Again, in (56) we have, denoting by ϵ an infinitely small positive constant,

$$\int_0^\infty \chi'(r\cosh v)\, dv = \int_{a-\epsilon}^{a+\epsilon} \frac{\chi'(z)\, dz}{\sqrt{(z^2 - r^2)}} = -\frac{1}{\sqrt{(a^2 - r^2)}},$$

* *Cf.* Rayleigh, *Sound*, § 279.

† The corresponding problem in three dimensions is solved in *Hydrodynamics*, § 264.

provided $r<a$, whilst for $r>a$ the integrand vanishes. Hence, taking a for the unindicated lower limit in (56), we find

$$\left.\begin{aligned} f(-r)-F(r) &= \frac{2}{\pi c}\surd(a^2-r^2), \quad \text{for} \quad r<a; \\ &= 0, \qquad\qquad\quad \text{for} \quad r>a. \end{aligned}\right\} \tag{59}$$

From (57), (58), (59) we infer

$$\left.\begin{aligned} f(r) = f(-r) &= -F(r) \\ &= \frac{1}{\pi c}\surd(a^2-r^2), \quad \text{for} \quad r<a; \\ &= 0, \qquad\qquad\quad \text{for} \quad r>a. \end{aligned}\right\} \tag{60}$$

To trace the course of the disturbance at a distance r which is $>a$, we remark, in the first place, that, since $ct+r\cosh u>a$, the second term in (51) disappears, and we have accordingly to deal with a diverging wave only. Again, so long as $ct<r-a$, we have $|ct-r\cosh u|>a$, and therefore $\phi=0$. When $r-a<ct<r+a$, we find

$$\phi = \frac{1}{\pi c}\int_0^{\cosh^{-1}(ct+a)/r} \surd\{a^2-(ct-r\cosh u)^2\}\,du\,; \tag{61}$$

whilst, for $ct>r+a$,

$$\phi = \frac{1}{\pi c}\int_{\cosh^{-1}(ct-a)/r}^{\cosh^{-1}(ct+a)/r} \surd\{a^2-(ct-r\cosh u)^2\}\,du. \tag{62}$$

The last result may also be written

$$\phi = \frac{a^2}{\pi c}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\cos^2\theta\,d\theta}{\surd\{(ct+a\sin\theta)^2-r^2\}}, \tag{63}$$

or, for large values of $ct-r$,*

$$\phi = \frac{a^2}{2c\surd(c^2t^2-r^2)}. \tag{64}$$

6. It would be of interest to examine whether the energy of a free progressive wave in two dimensions is equally divided between the kinetic and the potential forms. That this is the case in one dimension is well known, and the proof for spherical waves is easily supplied. For we have, identically,

$$r^2\left(\frac{\partial\phi}{\partial r}\right)^2 = \left\{\frac{\partial\,(r\phi)}{\partial r}\right\}^2 - \frac{\partial}{\partial r}(r\phi^2).$$

* This may be compared with Rayleigh, *Sound*, § 275, equation (1).

If we assume
$$r\phi = f(ct - r), \tag{65}$$

and put
$$u = -\frac{\partial \phi}{\partial r}, \quad c^2 s = \frac{\partial \phi}{\partial t},$$

this may be written
$$r^2 u^2 = c^2 r^2 s^2 - \frac{\partial (r\phi^2)}{\partial r}. \tag{66}$$

Hence
$$\int_0^\infty \tfrac{1}{2}\rho u^2 . 4\pi r^2 dr = \int_0^\infty \tfrac{1}{2}\rho c^2 s^2 . 4\pi r^2 dr, \tag{67}$$

if we suppose that $r\phi^2$ vanishes at the inner and outer boundaries of the wave.

The direct examination of the question for the case of cylindrical waves, starting from the formula (9), would not appear to be easy. If we try to adapt the indirect method given by Rayleigh* for the case of plane waves, the argument would run somewhat as follows :— Consider at any instant (which we may conveniently take as origin of t) an arbitrary diverging wave A, and also the converging wave B which is obtained from A by reversing the *velocity* of every particle . It is plain that an initial disturbance $(A+B)$ constructed by superposing these would split up into the two waves A and B, and that, *if in the subsequent motion these become wholly separate*, the sum of the energies of A and B can be asserted to be equal to the initial energy of the disturbance $A+B$. Now the energies of A and B alone are constant, and are initially equal, and the initial energy of A is therefore half that of $A+B$. Since the latter is wholly potential, and equal to four times the initial potential energy of A, the desired conclusion follows.

The proviso in italics would seem to be essential to the argument. It is satisfied in the case of spherical waves, but not in the case of cylindrical waves, at all events such as result from a limited original disturbance.

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* *Sound*, Vol. II., § 245.

Expansions by means of Lamé's Functions. *By* A. C. Dixon. Received September 19th, 1902. Read November 13th, 1902.

I have found myself hampered lately by the want of strict proofs of the validity of expansions by means of Lamé's functions. The present paper is meant to supply this want, and to give the reader a fairly connected idea of the analytical theory, so far as it relates to the determination of a potential or harmonic function whose values on an ellipsoidal surface are given. The better known parts of the theory have only been sketched.

An important development, which I have not met with before, is the use of Lamé's functions to express functions in a double space, analogous to a Riemann surface.

The notation used for elliptic functions will, I think, be found to have advantages in working with ellipsoidal coordinates.

The works that I have chiefly consulted are the papers of Lamé and Liouville, and the treatises of Heine (*Kugelfunctionen*) and Halphen (*Fonctions elliptiques*, Vol. II.).*

1. In working with ellipsoidal coordinates we shall use the following notation. We shall write au, bu, cu for the square roots of $a^2+\lambda$, $b^2+\lambda$, $c^2+\lambda$, where λ, u are connected by the relation

$$2u = \int_{\lambda}^{\infty} \{(a^2+\theta)(b^2+\theta)(c^2+\theta)\}^{-\frac{1}{2}} d\theta. \qquad (1)$$

When u is real, positive, and small the values of au, bu, cu will be taken positive. In formulæ there will be complete symmetry as to a, b, c. For instance, to differentiate we have

$$\left.\begin{aligned} a'u &= -bu\,cu \\ b'u &= -cu\,au \\ c'u &= -au\,bu \end{aligned}\right\}. \qquad (2)$$

* The paragraphs that I recognize as due in substance to earlier writers are 2–7, 14, 20 and in part 10, 15.

The addition theorem is

$$a(u+v) = (au\,bv\,cv - av\,bu\,cu)/(a^2v - a^2u), \tag{3}$$

with two similar equations.

Let α, β, γ be the half periods, so that

$$a\alpha = b\beta = c\gamma = 0, \quad a^2\beta = a^2 - b^2, \quad a^2\gamma = a^2 - c^2, \ \&c. \tag{4}$$

The formulæ for the addition of half periods are of the types

$$\left.\begin{aligned} a(\alpha+u) &= \epsilon' \alpha/au \\ a(\beta+u) &= a\beta\, cu/bu \end{aligned}\right\}. \tag{5}$$

The three functions au, bu, cu are odd and for small values of u approach the limiting form u^{-1}.

We shall suppose $a^2 - b^2$ and $b^2 - c^2$ positive; thus γ may be taken real and positive, α purely imaginary, and of the same sign as ι; β will be fixed by the equation

$$\alpha + \beta + \gamma = 0. \tag{6}$$

Then $a\gamma$, $b\gamma$, $\iota b\alpha$, $\iota c\alpha$, $-a\beta$, $-\iota c\beta$ are all positive.

The relations between the Cartesian coordinates x, y, z and the ellipsoidal coordinates u, v, w are

$$\left.\begin{aligned} \frac{x^2}{a^2u} + \frac{y^2}{b^2u} + \frac{z^2}{c^2u} &= 1 \\ \frac{x^2}{a^2v} + \frac{y^2}{b^2v} + \frac{z^2}{c^2v} &= 1 \\ \frac{x^2}{a^2w} + \frac{y^2}{b^2w} + \frac{z^2}{c^2w} &= 1 \end{aligned}\right\}, \tag{7}$$

or

$$\left.\begin{aligned} x &= au\,av\,aw/a\beta\,a\gamma \\ y &= bu\,bv\,bw/b\gamma\,b\alpha \\ z &= cu\,cv\,cw/c\alpha\,c\beta \end{aligned}\right\}. \tag{8}$$

Of the three arguments, u is purely real, the real part of v is an odd multiple of γ, and the imaginary part of w an odd multiple of α; we shall suppose v to range between $\beta \pm 2\alpha$ and w between $\beta \pm 2\gamma$, and then for any fixed value of u the corresponding ellipsoid will be covered twice, the points u, v, w and u, $2\beta - v$, $2\beta - w$ coinciding. We have

$$a^2u > a^2\gamma > a^2v > a^2\beta > a^2w > 0. \tag{9}$$

Double integrals of the form

$$\iint \phi \frac{\partial \psi}{\partial \nu}\, dS$$

often occur, dS being an element of area on the surface of the ellipsoid $u = \text{const.}$, and $d\nu$ an element of the outward normal to that surface. Since

$$\left.\begin{aligned} d\nu &= -(a^2u-a^2v)^{\frac{1}{2}}(a^2u-a^2w)^{\frac{1}{2}}\, du \\ dS &= -\iota\,(a^2u-a^2v)^{\frac{1}{2}}(a^2u-a^2w)^{\frac{1}{2}}(a^2v-a^2w)\, dv\, dw \end{aligned}\right\}, \qquad (10)$$

this double integral becomes

$$\iint \phi \frac{\partial \psi}{\partial u}\, \iota\,(a^2v-a^2w)\, dv\, dw,$$

and we shall write $d\omega$ for the positive quantity

$$-\iota\,(a^2v-a^2w)\, dv\, dw.$$

2. The transformation of Laplace's equation

$$\Delta\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad (11)$$

is

$$(a^2v-a^2w)\frac{\partial^2\phi}{\partial u^2} + (a^2w-a^2u)\frac{\partial^2\phi}{\partial v^2} + (a^2u-a^2v)\frac{\partial^2\phi}{\partial w^2} = 0, \qquad (12)$$

and Lamé pointed out that, if this equation was satisfied by a value for ϕ of the form $fu\,fv\,fw$, then the function f must be a solution of the equation

$$f''u = fu\,(Aa^2u + B), \qquad (13)$$

A, B being constants.

Lamé's functions are rational integral algebraic expressions in au, bu, cu, satisfying equations of the type (13). It is readily seen that such expressions will contain terms of odd degree only or else terms of even degree only, and that, if n is the degree of the highest term, $A = n(n+1)$. The general expression of the n-th degree that is to be considered may thus be written—

$$\begin{aligned} &p_0a^nu + p_1a^{n-2}u + p_2a^{n-4}u + \ldots + bu\,(q_0a^{n-1}u + q_1a^{n-3}u + \ldots) \\ &\qquad + cu\,(r_0a^{n-1}u + r_1a^{n-3}u + \ldots) \\ &\qquad + bu\,cu\,(s_0a^{n-2}u + s_1a^{n-4}u + \ldots), \end{aligned}$$

and it contains $2n+1$ terms. If it is substituted for fu in the equation (13), the terms of the degree $n+2$ will disappear on account of the value

chosen for A, and there will be $2n+1$ linear homogeneous equations that must be satisfied by the $2n+1$ coefficients p, q, r, s. These equations are also linear in B, and thus the result of eliminating p, q, r, s is an equation for B of the degree of $2n+1$. A value satisfying this equation having been taken for B, the ratios of the coefficients p, q, r, s are determinate,* and thus the function fu is found. A term in fu consists of a power of au multiplied by one or other, or both, or neither of the factors bu, cu, and a term of any one of these four forms gives terms of the same form only in $f''u$. Hence it appears that Lamé's functions are of four kinds, in each of which terms occur of one of the four forms only, and the equation for B accordingly breaks up into four factors.

3. Now let fu, f_1u denote two of Lamé's functions. The products $fufvfw$, or ϕ, and $f_1uf_1vf_1w$, or ϕ_1, satisfy the equation (11), and have no singularity except at infinity, since they are, as is readily proved, rational integral algebraic expressions in x, y, z. Hence

$$\int \left(\phi \frac{\partial \phi_1}{\partial u} - \phi_1 \frac{\partial \phi}{\partial u}\right) d\omega = 0, \tag{14}$$

by Green's equation, if the integral is taken over any ellipsoid u. The integral is doubled in value if we take $\beta \pm 2\alpha$ as the limits for v and $\beta \pm 2\gamma$ for w, and these limits will always be understood when no others are stated or implied. Substituting for ϕ, ϕ_1, we have

$$(fuf'_1u - {}_1uf'u) \int fvfwf_1vf_1w\, d\omega = 0$$

for all real values of u between 0 and γ.

The first factor does not vanish unless fu, f_1u are practically the same, their ratio being constant. Hence, if f, f_1 are different Lamé functions

$$\int fvfwf_1vf_1w\, d\omega = 0. \tag{15}$$

4. From (15) it follows that all the roots of the equation for B are real. For, if B, B_1 were two conjugate complex roots, they would give conjugate complex values to the coefficients in the corresponding

* It is found later (§ 4) that the equation for B has no double root, and hence this statement is justified.

functions f, f_1. Hence fvf_1v and fwf_1w would be of constant sign over the surface, and the equation (15) could not therefore be true.

In somewhat the same way it may be proved that the equation for B cannot have equal roots. If a root B were repeated, then a quantity C and the coefficients in an expression gu of the same form as fu could be so chosen that the coefficient of ϵ in

$$\frac{d^2}{du^2}(fu+\epsilon gu)-(Aa^2u+B+\epsilon C)(fu+\epsilon gu)$$

should vanish. That is, we should have

$$g''u = (Aa^2u+B)\,gu+Cfu. \tag{16}$$

Then $gu\,fv\,fw+fu\,gv\,fw+fu\,fv\,gw$ would satisfy Laplace's equation (11) and could be put in the place of ϕ_1 in equation (14), since it would be a rational integral algebraic function of x, y, z. The result would be

$$(g'u\,fu-f'u\,gu)\int f^2v\,f^2w\,d\omega = 0,$$

which is impossible, as gu is not a mere constant multiple of fu, and the subject of integration is of constant sign.

If C were zero, this argument would fail, since gu need not then be of the same form as fu, but only of the same degree, and then the expression put for ϕ_1 would not be rational in x, y, z. We should, however, have from the differential equation

$$g''u\,fu-gu\,f''u = 0,$$

and therefore $\qquad g'u\,fu-gu\,f'u = h$, a constant.

Thus $\qquad 2h\,fu\,gu = f^2u\,\dfrac{d}{du}g^2u-g^2u\,\dfrac{d}{du}f^2u$

$$= au\,bu\,cu\times\text{an expression rational in } a^2u.$$

This is impossible, unless $h=0$, since $fu\,gu$ is of even degree, $2n$, in au, bu, cu; hence, again,

$$g'u\,fu-gu\,f'u = 0;$$

so that f, g are the same function practically.

Hence all the roots of the equation for B are real and distinct, and for any value of the order n there are $2n+1$ distinct Lamé functions with real coefficients.

5. The next matters to be discussed are superior and inferior

limits to the value of B, and the distribution of the values of u for which fu vanishes.

Now f^2u is a quantic in a^2u, and hence its derivative $2fuf'u$ contains the factor $au\,bu\,cu$, and vanishes at α, β, γ; thus, for each of these values of u, either fu or $f'u$ is zero. They cannot both vanish for any value of u; otherwise, from the equation (13) and its derivatives, it would follow that all the successive derivatives of fu would vanish, which is impossible.

The limits between which B must lie are 0 and $-n(n+1)a^2\gamma$. For suppose the case otherwise; then the sign of $f''u/fu$ is constant when u lies between γ and $\gamma+\alpha$ or between $\gamma+\alpha$ and α. Take this sign to be positive; then the curve $y=f(\alpha+x)$ is convex to the axis of x for values of the abscissa between 0 and γ, and yet either y or dy/dx vanishes for each of these extreme values. The figure shows that this is absurd. If the sign of $f''u/fu$ is taken negative, it will, in like manner, be found impossible to draw the curve $y=f(\gamma+\iota x)$ for values of x between 0 and $-\iota\alpha$.

Since B lies between 0 and $-n(n+1)a^2\gamma$, it follows that $f''u/fu$ is positive for all real values of u. Hence fu and $f'u$ are constant in sign and increase in numerical value as u decreases from γ to 0. The same is true of $f\iota u$ and $f'\iota u$ as u decreases from $-\iota\alpha$ to 0. When $u=0$, fu and $f'u$ become infinite.

Now fu contains as a factor a quantic in a^2u, and, since fu, $f'u$ cannot vanish together, no root of this quantic can be repeated and none can be equal to 0 or $a^3\beta$ or $a^3\gamma$; also no root can be negative or greater than $a^2\gamma$. The coefficients in the polynomial are algebraic functions of a^2, b^2, c^2, and no variation in a^2, b^2, c^2 can alter the number of roots of the polynomial that lie between 0 and $a^2\beta$ or between $a^2\beta$ and $a^2\gamma$; for this would imply a transition stage in which two roots would be equal, or a root would be equal to 0 or $a^2\beta$ or $a^2\gamma$. Hence the number of roots between 0 and $a^2\beta$, say, can be found by considering a special case: for instance, that in which $b=c$.

6. The elliptic functions then degenerate, and, writing κ for the value of $a\beta$ or $a\gamma$, we have

$$\left.\begin{aligned} au &= \kappa\coth\kappa u \\ bu = cu &= \kappa\operatorname{cosech}\kappa u \end{aligned}\right\}. \tag{17}$$

Putting $\qquad v=\gamma+\iota\phi, \quad w=-\alpha+\psi,$

so that ϕ, ψ are real, and using the formulæ (5), we have

$$\left.\begin{aligned} av &= a\gamma\, b\iota\phi/c\iota\phi = \kappa \\ bv/b\gamma &= a\iota\phi/c\iota\phi = \cos\kappa\phi \\ cv/b\gamma &= c'\gamma/b\gamma\, c\iota\phi = -\iota\sin\kappa\phi \end{aligned}\right\}, \tag{18}$$

and, similarly,

$$\left.\begin{aligned} aw &= \kappa\tanh\kappa\psi \\ bw &= cw = \iota\kappa\,\mathrm{sech}\,\kappa\psi \end{aligned}\right\}. \tag{19}$$

Lamé's equation (13) becomes, for fu,

$$\frac{d^2U}{du^2} = \left\{n(n+1)\kappa^2\coth^2\kappa u + B\right\}U; \tag{20}$$

for fv,

$$\frac{d^2V}{d\phi^2} = -\left\{n(n+1)\kappa^2 + B\right\}V; \tag{21}$$

for fw,

$$\frac{d^2W}{d\psi^2} = \left\{n(n+1)\kappa^2\tanh^2\kappa\psi + B\right\}W. \tag{22}$$

Now V, or fv, is a polynomial in $\sin\kappa\phi$, $\cos\kappa\phi$ of degree not higher than n, and, if it is to satisfy the equation (21), it must be a multiple of $\sin m\kappa\phi$ ($m = 1, 2, \ldots, n$) or $\cos m\kappa\phi$ ($m = 0, 1, 2, \ldots, n$), and the value of B will be $m^2\kappa^2 - n(n+1)\kappa^2$.

Thus when μ is put for $\tanh\kappa\psi$ the equation (22) becomes that which is satisfied by the zonal factor in a tesseral harmonic, and W is a constant multiple of

$$(\mu^2-1)^{\frac{1}{2}m}\frac{d^{m+n}}{d\mu^{m+n}}(\mu^2-1)^n.$$

Hence W vanishes for $n-m$ real values of ψ of which one is zero when $n-m$ is odd; thus the number of roots of the polynomial in a^2u that lie between 0 and $a^2\beta$ is determined.

The real values of $\kappa\phi$ for which V vanishes are, when $V = \sin m\kappa\phi$, 0, $\pm\pi/m$, $\pm 2\pi/m$, ..., and when $V = \cos m\kappa\phi$, $\pm\pi/2m$, $\pm 3\pi/2m$, ... To these correspond roots of the polynomial in a^2u between $a^2\beta$ and $a^2\gamma$.

Thus a Lamé function of any order may be distinguished from others of the same order by the distribution of its zeros.* Let C_m,

* Klein, *Math. Ann.*, Vol. XVIII., pp. 237–244.

S_m denote those for which, in the limit when $b = c$, V becomes $\cos m\kappa\phi$, $\sin m\kappa\phi$ respectively. Then the distribution is as follows :—

	n even				n odd			
Zeros of	C_{2m-1}	C_{2m}	S_{2m-1}	S_{2m}	C_{2m-1}	C_{2m}	S_{2m-1}	S_{2m}
At α	1	0	1	0	0	1	0	1
From α to $\alpha + 2\gamma$*	$n-2m$	$n-2m$	$n-2m$	$n-2m$	$n-2m+1$	$n-2m-1$	$n-2m+1$	$n-2m-1$
At $\alpha+\gamma$	1	0	0	1	1	0	0	1
From $2\alpha+\gamma$ to γ*	$2m-2$	$2m$	$2m-2$	$2m-2$	$2m-2$	$2m$	$2m-2$	$2m-2$
At γ	0	0	1	1	0	0	1	1

* Exclusive of the half periods.

It follows that whenever fu vanishes a^2u is real. The classification might have been arrived at by considering the other extreme case, $a = b$, instead.

The process that we have used for finding the fourth row in the table amounts to this : that we write $a^2-b^2\sin^2\kappa\phi - c^2\cos^2\kappa\phi$ for a^2u in fu, divide out by as high a power as possible of b^2-c^2, and then put $b^2 = c^2$. This reduces C_m, S_m to multiples of $\cos m\kappa\phi$, $\sin m\kappa\phi$; so that their zeros are known.

7. In discussing expansions by means of Lamé's functions it seems necessary to consider another limiting case—namely, that in which a^2-b^2, b^2-c^2 are made infinitesimal in some given ratio. The ellipsoids of the confocal system thus become concentric spheres, and the hyperboloids confocal cones with vertex at the centre. The real argument u must be taken as infinitesimal, and, since in the limit it becomes r^{-1}, we are led to the following equations :—

$$x = r\frac{av\,aw}{a\beta\,a\gamma}, \quad y = r\frac{bv\,bw}{b\gamma\,ba}, \quad z = r\frac{cv\,cw}{ca\,c\beta}. \tag{23}$$

These only contain the ratios of a^2-b^2, b^2-c^2, a^2v, a^2w, and so henceforth these quantities need not be supposed infinitesimal. Laplace's equation (11) becomes

$$(a^2v-a^2w)\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) - \frac{\partial^2\phi}{\partial v^2} + \frac{\partial^2\phi}{\partial w^2} = 0. \tag{24}$$

Any spherical surface harmonic of order n is a function of v, w, and the equation it satisfies is

$$\frac{\partial^2\phi}{\partial v^2}-\frac{\partial^2\phi}{\partial w^2}=n\,(n+1)(a^2v-a^2w)\,\phi. \qquad (25)$$

This is satisfied by the $2n+1$ products $fvfw$; so that these are surface harmonics. Let constant multiples of them be denoted by $\theta_0, \theta_1, \theta_2, \ldots, \theta_{2n}$. The element of surface on the sphere is $r^2d\omega$, and we proved above that

$$\int \theta_p\theta_q\,d\omega=0 \quad (p\neq q). \qquad (26)$$

In each of the functions θ there is an arbitrary constant factor. Let this be so chosen that

$$\int \theta_p^2\,d\omega=8\pi/(2n+1) \quad (p=0, 1, 2, \ldots, 2n). \qquad (27)$$

This would only be impossible if $\int f^2vf^2w\,d\omega$ were zero, which cannot be, since the subject of integration is of constant sign. It follows that the harmonics $\theta_0, \theta_1, \theta_2, \ldots, \theta_{2n}$ are linearly independent; for, if θ_{2n}, say, could be expressed in the form

$$a_0\theta_0+a_1\theta_1+\ldots+a_{2n-1}\theta_{2n-1},$$

we should have, since

$$\int \theta_{2n}\theta_p\,d\omega=0 \quad (p\neq 2n),$$

that

$$\int \theta_{2n}^2\,d\omega=0,$$

which is untrue.

8. A set of $2n+1$ complete spherical surface harmonics, $\sigma_0, \sigma_1, \ldots, \sigma_{2n}$, of order n, such that

$$\int \sigma_p\sigma_q\,d\omega=0 \quad (p\neq q), \qquad (28)$$

$$\int \sigma_p^2\,d\omega=8\pi/(2n+1), \qquad (29)$$

has one or two important properties. Since the functions are linearly independent, any other complete harmonic of order n can be expressed as a linear function of them. Take, for instance, $P_n(\cos PP')$, where P is the current point, P' a fixed point on the sphere. Thus we have

$$P_n(\cos PP')=a_0\sigma_0+a_1\sigma_1+\ldots,$$

where

$$8\pi a_p=(2n+1)\int P_n(\cos PP')\,\sigma_p\,d\omega,$$

as we find on multiplying both sides by σ_p and integrating. Now,

by a known theorem, since σ_p is a complete surface harmonic of order n, and the sphere is covered twice in the integration,

$$\int P_n(\cos PP')\,\sigma_p\,d\omega = 8\pi\sigma_p'/(2n+1),$$

where σ_p' denotes the value of σ_p at P'. Hence

$$\alpha_p = \sigma_p',$$

and $$P_n(\cos PP') = \sigma_0\sigma_0' + \sigma_1\sigma_1' + \sigma_2\sigma_2' + \ldots \text{ to } 2n+1 \text{ terms.} \qquad (30)$$

Taking P' to coincide with P, we have

$$\sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_{2n}^2 = 1. \qquad (31)$$

We may call $\sigma_0, \sigma_1, \ldots, \sigma_{2n}$ a canonical set of surface harmonics of order n, and thus it follows from (31) that two canonical sets are connected by an orthogonal substitution. The converse also holds that a set derived from a canonical one by an orthogonal substitution is itself canonical.

Now the products $\theta_0, \theta_1, \ldots, \theta_{2n}$ form a canonical set, and thus it follows from (31) that no one of them can have a greater value than unity at any point of the surface.

9. We can now conveniently fix the constant factor in Lamé's function fu. The function will be taken positive for real values of u between 0 and γ. Then $f^2v\,f^2w$ is negative or positive according as $f\beta$ does or does not vanish. Let j stand for -1 when $f\beta = 0$, and for $+1$ when $f\beta \neq 0$; then we shall suppose the constant factor* so chosen that

$$\int f^2v\,f^2w\,d\omega = 8j\pi/(2n+1). \qquad (32)$$

So then $|fv\,fw|$ cannot exceed 1 at any point of the surface.

10. The chief problem to be solved is that of finding ϕ, a function of x, y, z, without singularity within, or else without, a given ellipsoid, say u_0, which shall be a potential, that is, shall satisfy Laplace's equation $\Delta\phi = 0$, and which shall have assigned values on the surface of the ellipsoid u_0.

The first step is to express the assigned values by means of a series of the form $\Sigma k\,fv\,fw$. Let this value at a point (u_0, v, w) be denoted

* Prof. G. H. Darwin (*Roy. Soc. Phil. Trans.*, Vol. CXCVII.) has found another choice of this factor more convenient for purposes of computation. The factors could be compared by comparing the values of the integral in (32) in the two notations.

by $F(v, w)$; then we may consider it as belonging to the corresponding point on the unit sphere, that is, to the point

$$\left(\frac{av\,aw}{a\beta\;a\gamma},\ \frac{bv\,bw}{b\gamma\;ba},\ \frac{cv\,cw}{ca\;c\beta}\right),$$

and expand it in a series of spherical surface harmonics

$$Y_0+Y_1+Y_2+\ldots+Y_n+\ldots.$$

The term Y_n can then be expressed linearly in terms of the harmonics $\theta_0, \theta_1, \theta_2, \ldots, \theta_{2n}$, and thus we have the expansion desired.

For the present we shall assume that $F(v, w)$ is an analytical function of the Cartesian coordinates over the whole surface of the ellipsoid; then the same will hold over the surface of the sphere, and the series ΣY_n will be convergent by ratio; that is, the numerical value of Y_n will everywhere be less than pq^n where p, q are two positive quantities and $q<1$. (See *Camb. Phil. Trans.*, Vol. XIX., Part 2, pp. 205–7.)

Let $$Y_n = k_0\theta_0+k_1\theta_1+\ldots+k_{2n}\theta_{2n}.$$

Then $$\int Y_n^2\,d\omega = 8\pi\,(k_0^2+k_1^2+\ldots+k_{2n}^2)/(2n+1).$$

But $$\int d\omega = 8\pi \quad \text{and} \quad Y_n^2 < p^2q^{2n};$$

therefore $$k_0^2+k_1^2+\ldots+k_{2n}^2 < (2n+1)\,p^2q^{2n}.$$

Thus each of the coefficients k is numerically less than $pq^n\sqrt{2n+1}$, and, if we write $\bar{Y}_n$ for $\sum_{m=0}^{2n} |k_m\theta_m|$, the series $\bar{Y}_0+\bar{Y}_1+\ldots$ is convergent by ratio, since $\bar{Y}_n$ does not exceed $pq^n(2n+1)^{\frac{1}{2}}$.* The series whose n-th term is $\sum_{m=0}^{2n} |k_m|$ is also convergent by ratio.

Since $fu\,fv\,fw$ is a potential, it is natural to suppose that the solution of our problem is given by the series $\Sigma\, k\,fu\,fv\,fw/fu_0$; for this is a series of terms each of which is a potential without singularity except at infinity, and when $u=u_0$ it takes the form $\Sigma\, k\,fv\,fw$, which is by hypothesis the expansion of $F(v, w)$. It is necessary, however, to prove (1) that the new series converges for all internal points, and that its sum is a potential; (2) that the limit of the sum when u approaches u_0 is the sum of the series $\Sigma\, k\,fv\,fw$.

* In using a result of this kind we may drop such a factor as $(2n+1)^{\frac{3}{2}}$ on condition of supposing q slightly increased, and possibly p increased also.

The expression not being a power series, the theorem of Abel's that is generally applied cannot be used to prove (2).

11. In dealing with series of this type we shall take the $(n+1)$-th term to mean the sum of the $2n+1$ partial terms of order n. The products $fvfw$ are all numerically $\not> 1$, and they may therefore be neglected in discussing absolute convergency. We need to prove that, if the series $\Sigma\,|\,kfu_0\,|$ is convergent by ratio, the same is true for each of the series $\Sigma\,|\,kfu\,|$, $\Sigma\,|\,kf'u\,|$ for a continuous range of real values of u, including u_0 and γ.

More generally, if λu is a function of u satisfying one of Lamé's equations

$$\lambda'' u = \{n(n+1)a^2u + B\}\lambda u,$$

and such that λu_0 or $\lambda' u_0$ vanishes, where u_0 is a fixed real quantity between 0 and 2γ inclusive, and if either of the series $\Sigma\,|\,k\lambda u_1\,|$, $\Sigma\,|\,k\lambda' u_1\,|$ is convergent by ratio, u_1 being a quantity between 0 and 2γ exclusive, then the two series $\Sigma\,|\,k\lambda u\,|$, $\Sigma\,|\,k\lambda' u\,|$ converge by ratio for a continuous range of values of u, including both u_0 and u_1.

Let $\mu(\theta)$ denote λu where $u = u_0 + \theta(u_1 - u_0)$, and take θ in the first place to be between 0 and 1. Then

$$\mu''(\theta) = (u_1 - u_0)^2 \{n(n+1)a^2u + B\}\,\mu(\theta).$$

Thus $\mu''(\theta)$, $\mu(\theta)$ have the same sign, and the curve $y = \mu(x)$ is always convex to the axis of x. We may, without loss of generality, take $\mu(\theta)$ and k to be positive. As θ increases $\mu'(\theta)$ and $\mu(\theta)$ increase.

Now first let $\Sigma k\mu(1)$ be convergent by ratio. Since $\mu(\theta) < \mu(1)$, the series $\Sigma k\mu(\theta)$ is also convergent by ratio.

Also $$\mu(1) - \mu(\theta) = \int_\theta^1 \mu'(\theta)\,d\theta > (1-\theta)\,\mu'(\theta),$$

since $\mu'(\theta)$ increases with θ. The series $\Sigma k\{\mu(1) - \mu(\theta)\}$ is convergent by ratio; so therefore is $\Sigma k\mu'(\theta)$.

Also $$\mu''(\theta)/\mu(\theta) < (u_1 - u_0)^2\, n(n+1)\, a^2u,$$

and thus $\Sigma k\mu''(\theta)$ also converges by ratio for values of θ up to 1 inclusive. Integrating this from θ to 1, we find that

$$\Sigma k\{\mu'(1) - \mu'(\theta)\}$$

is convergent by ratio; and so therefore is $\Sigma k\mu'(1)$ since $\Sigma k\mu'(\theta)$ has been proved to be.

Again, let $\Sigma k\mu'(1)$ be supposed convergent by ratio. The same is true of $\Sigma k\mu'(\theta)$, which is less term by term. Let θ' be a quantity between θ and 1. Then

$$\mu'(\theta')-\mu'(\theta) = (u_1-u_0)\int_{u_0+\theta(u_1-u_0)}^{u_0+\theta'(u_1-u_0)} \{n(n+1)a^2u+B\}\lambda u\,du$$

$$> n(n+1)(u_1-u_0)\mu(\theta)\int_{u_0+\theta(u_1-u_0)}^{u_0+\theta'(u_1-u_0)} c^2u\,du,$$

since $B>-n(n+1)a^2\gamma$ and $\lambda u>\mu(\theta)$ throughout the range. Now $\Sigma k\{\mu'(\theta')-\mu'(\theta)\}$ is convergent by ratio; so then is $\Sigma k\mu(\theta)$, and, since

$$\mu(1)-\mu(\theta) = \int_\theta^1 \mu'(\theta)\,d\theta,$$

so is $\Sigma k\mu(1)$.

12. The theorems are then proved for values of θ between 0 and 1 inclusive. We are to show that they hold for a wider range of values. Take $\theta>1$.

Let u_2, u_1, u_3 be in order of magnitude, and take a quantity η that is greater than au for any value of u between u_2 and u_3. Let $\nu(\theta)$ stand for

$$\mu(1)\cosh\{(n+1)\eta(u_1-u_0)(\theta-1)\}$$

$$+\frac{\mu'(1)}{(n+1)\eta(u_1-u_0)}\sinh\{(n+1)\eta(u_1-u_0)(\theta-1)\}.$$

Then $$\nu''(\theta) = (n+1)^2\eta^2(u_1-u_0)^2\nu(\theta),$$

while $$\mu''(\theta) = [n(n+1)a^2u+B](u_1-u_0)^2\mu(\theta);$$

thus $$\mu(\theta)\nu''(\theta)-\mu''(\theta)\nu(\theta) = \mu(\theta)\nu(\theta)\times\text{a positive quantity},$$

so long as u is between u_2, u_3.

Also, when $\theta>1$, $\nu(\theta)$ is positive and increases with θ. By integration we have then

$$\{\mu(\theta)\nu'(\theta)-\mu'(\theta)\nu(\theta)\}-\{\mu(1)\nu'(1)-\mu'(1)\nu(1)\}$$

$$=\text{a positive quantity}.$$

Now $$\nu(1) = \mu(1),\quad \nu'(1) = \mu'(1);$$

hence $\mu(\theta)\nu'(\theta)-\mu'(\theta)\nu(\theta)$ is positive, and $\nu(\theta)/\mu(\theta)$ increases with θ.

Thus $\mu(\theta)<\nu(\theta)$, since we are taking $\theta<1$. Now $\mu(1)$, $\mu'(1)$ are both $<pq^n$ where p, q are independent of n and of $q<1$. If we take

$\theta > 1$, but $< 1 - (\log q)/\eta\ (u_1 \sim u_0)$, the series $\Sigma k\nu\ (\theta)$ will be convergent by ratio, and therefore also the series $\Sigma k\mu\ (\theta)$, $\Sigma k\mu'(\theta)$. In the same way these series converge by ratio for negative values of θ that are not too great. The proof does not apply to this case when u_0 is 0 or 2γ; but, since λu is then an even or odd function of u or $u - 2\gamma$, as the case may be, the theorem is otherwise evident.

13. In the present case $u_0 = \gamma$ and λu is fu; the coefficient that has been called k in §§ 11, 12 will now be called k/fu_0, u_0 being put for u_1.

Now in Green's equation

$$\int \left(\phi \frac{\partial \phi_1}{\partial \nu} - \phi_1 \frac{\partial \phi}{\partial \nu}\right) dS = \int\int\int (\phi\, \Delta\phi_1 - \phi_1\, \Delta\phi)\, dx\, dy\, dz \qquad (33)$$

let $$\phi = fu\, fv\, fw,$$

and let ϕ_1 be the reciprocal of the distance of P or (u, v, w) from an internal point (u', v', w'), the integration being over the surface of the ellipsoid u twice. We thus find

$$\int \left(fu \frac{\partial}{\partial u} \frac{1}{PP'} - f'u \frac{1}{PP'}\right) fv\, fw\, d\omega = 8\pi\, fu'\, fv'\, fw'. \qquad (34)$$

Multiply this by k/fu and sum, changing the notation to agree with that used in § 10; thus

$$\Sigma k\, fu\, fv\, fw/fu_0 = \frac{1}{8\pi} \int \left(\Sigma k fv_0 fw_0 \frac{\partial}{\partial u_0} \frac{1}{PP_0} - \Sigma k \frac{f'u_0}{fu_0} fv_0 fw_0 \frac{1}{PP_0}\right) d\omega_0, \qquad (35)$$

an expression which from its form represents a potential for the point P, having no singularity within the ellipsoid u_0.

We conclude that, if, in the series $\Sigma\, k\, fu\, fv\, fw/fu_0$, $|\, k\,|$ does not exceed pq^n, where p, q are positive, $q < 1$, and n is the order of Lamé's function fu, then the series is convergent when $\gamma \geq u > u_0$, and represents a potential without singularity in the space enclosed by the ellipsoid u_0.*

The series is also convergent by ratio in a region outside the ellipsoid u_0. Take an ellipsoid u_0' in this region; in (34) write

* [The integration on the right-hand side of (35) gives a definite result, since the infinite series in the subject of integration are uniformly convergent over the surface, and their sums are therefore continuous.—*December 16th*, 1902.]

u, v, w for u', v', w', and u_0' for u, multiply by k/fu_0, and sum. Thus

$$\Sigma kfu\,fv\,fw/fu_0 = \frac{1}{8\pi}\int\left(\Sigma k\frac{fu_0'}{fu_0}fv_0'fw_0'\frac{\partial}{\partial u_0'}\frac{1}{PP_0'} - \Sigma k\frac{f'u_0'}{fu_0}fv_0'fw_0'\frac{1}{PP_0'}\right)d\omega_0',$$

which expression represents a potential without singularity within the ellipsoid u_0'.

Hence the series $\Sigma kfu\,fv\,fw/fu_0$ is valid, and represents a potential without singularity, throughout the volume of an ellipsoid of the confocal system outside the ellipsoid u_0, and the surface u_0 is included in its region of continuity. The value of this potential when $u = u_0$ is then $\Sigma kfv\,fw$; that is, it fulfils the assigned boundary conditions, and the problem is completely solved.

14. The value of k is easily found as a definite integral. Multiply the equation

$$F(v, w) = \Sigma kfv\,fw$$

by $fv\,fw\,d\omega$ and integrate with the usual limits; thus

$$8\pi jk = (2n+1)\int F(v, w)fv\,fw\,d\omega. \tag{36}$$

The expansion is then

$$8\pi F(v', w') = \Sigma(2n+1)jfv'\,fw'\int F(v, w)fv\,fw\,d\omega. \tag{37}$$

This has been proved when $F(v, w)$ is an analytical function of x, y, z all over the surface; the difficulty in extending the solution to other cases is that the series $\Sigma(2n+1)jfv'\,fw'\,fv\,fw$ is not convergent.

15. With a view to this extension we shall next investigate Green's function for the ellipsoid; that is, the potential that vanishes over the surface u_0 and has no singularity but that of $1/PP_1$ within this surface, P being the current point and P_1 a fixed internal point. To do this we must find the expansion of $1/PP_1$. Let (u_1, v_1, w_1) be the coordinates of P_1, and suppose the expansion

$$1/PP_1 = \Sigma Afv\,fw$$

to hold when P lies on the ellipsoid u_0. It follows that

$$8\pi A = (2n+1)\,j\int\frac{1}{P_0P_1}fv_0fw_0\,d\omega_0. \tag{38}$$

A is a function of u_0, u_1, v_1, w_1, and thus

$$8\pi \frac{\partial A}{\partial u_0} = (2n+1)j \int \frac{\partial}{\partial u_0} \frac{1}{P_0 P_1} fv_0 fw_0 d\omega_0. \tag{39}$$

We have then

$$8\pi \left(\frac{\partial A}{\partial u_0} fu_0 - Af'u_0\right) = (2n+1)j \int \left(fu_0 \frac{\partial}{\partial u_0} \frac{1}{P_0 P_1} - f'u_0 \frac{1}{P_0 P_1}\right) fv_0 fw_0 d\omega_0$$

$$= (2n+1)j \,.\, 8\pi fu_1 fv_1 fw_1 \quad [\text{by (34)}].$$

Hence
$$A = j \,.\, gu_0 fu_1 fv_1 fw_1, \tag{40}$$

where gu is a function of u such that

$$g'u\, fu - gu\, f'u = 2n+1, \tag{41}$$

and $g0 = 0$, since when P_0 is at infinity $A = 0$. From (41), by differentiation

$$g''u\, fu - gu\, f''u = 0\,;$$

so that
$$g''u/gu = f''u/fu,$$

and gu is the second solution of Lamé's equation (13). Its value is $(2n+1)fu \int_0^u \frac{du}{f^2 u}$, and when u is small it is approximately a constant multiple of u^{n+1}. Thus when $u < u_1$ we have

$$1/PP_1 = \Sigma j\, gu fv fw fu_1 fv_1 fw_1. \tag{42}$$

Similarly, $\frac{\partial}{\partial u_0} \frac{1}{P_0 P_1}$ may be expanded, and we find from (39) that $\frac{\partial A}{\partial u_0}$ is the coefficient of $fv_0 fw_0$ in its expansion. Thus

$$\frac{\partial}{\partial u} \frac{1}{PP_1} = \Sigma j g'u\, fv fw fu_1 fv_1 fw_1. \tag{43}$$

The higher derivatives and those with respect to u_1 may be found similarly.

Now let $\varpi(P, P_1)$ denote $\Sigma j\, gu_0 fu fv fw fu_1 fv_1 fw_1 / fu_0$. This series is valid so long as P, P_1 are both within the ellipsoid u_0, and also when one lies on the surface and the other within. It is a potential without singularity for either point, and when one point lies on the surface its value is $1/PP_1$. Hence $1/PP_1 - \varpi(P, P_1)$ is Green's function, say, $G(P, P_1)$, and, substituting it for ϕ_1 in (33), we have the value of a potential ϕ at an internal point P_1 given in terms

of its values on the surface by the equation

$$8\pi\phi(P_1) = \int \phi(P) \frac{\partial}{\partial u} G(P, P_1)\, d\omega, \tag{44}$$

which agrees with our former result, since

$$\frac{\partial}{\partial u} G(P, P_1) = \Sigma j \left(g'u - \frac{g u_0 f'u}{f u_0} \right) fv\, fw\, fu_1\, fv_1\, fw_1,$$

which becomes $\Sigma(2n+1) j fv\, fw\, fu_1\, fv_1 fw_1 / fu_0$ when $u = u_0$.

16. When $u_1 = u_0$, $\qquad G(P, P_1) = 0,$

and therefore $\qquad \dfrac{\partial}{\partial u} G(P, P_1) = 0.$

Our next step is to prove that the last result holds in the limit when $u = u_0$ and u_1 approaches the limit u_0. Let Q, Q_1 denote the limiting positions of P, P_1, that is, the points (u_0, v, w) (u_0, v_1, w_1), and R_1 the image of P_1 in the tangent plane at Q_1. Then $PP_1 < PR_1$ at every point of the surface except Q_1, and thus $1/PP_1 - 1/PR_1 - G(P, P_1)$ is a potential without singularity throughout the region and positive throughout, since it is positive on the boundary; that is,

$$G(P, P_1) < 1/PP_1 - 1/PR_1.$$

Now let a sphere be described about Q, of radius r less than QQ_1; let S be the region common to this sphere and the given ellipsoid, S_1 the hemisphere cut off by the tangent plane at Q and including S, S_2 the rest of the sphere. Let λ be the potential which has no singularity within this sphere and is equal to -1 over the curved surface of S_2, $+1$ on that of S_1. On the plane dividing S_1 from S_2, $\lambda = 0$, and, at Q, $\dfrac{\partial \lambda}{\partial \nu} = -\dfrac{3}{2r}$, $d\nu$ being an element of the outward normal. Let κ be the greatest value of $G(P, P_1)$ on the spherical boundary of S. Then, on the whole boundary of S, $\kappa\lambda - G(P, P_1)$ is positive; this expression is also a potential without singularity within S and vanishing at Q. Hence it is positive throughout, and must increase as P travels inwards from Q; that is, $\dfrac{\partial}{\partial \nu} \{\kappa\lambda - G(P, P_1)\}$ is negative at Q or $\left[-\dfrac{\partial}{\partial \nu} G(P, P_1) \right]_{u = u_0} < 3\kappa/2r$, where κ is less than the greatest value taken by $1/PP_1 - 1/PR_1$ on the spherical boundary of S.

By taking $u_1 - u_0$ small enough, we can make κ as small as we please, and thus the limit when u_1 approaches u_0 of $\left[\frac{\partial}{\partial n} G(P, P_1)\right]_{u=u_0}$ is zero, unless $r = 0$, that is, unless Q, Q_1 coincide. This is the result desired.* We may write it

$$\lim_{u=u_0} \Sigma (2n+1) j fu fv fw fv_0 fw_0 / fu_0 = 0, \tag{45}$$

which holds unless $v = v_0$, $w = w_0$.

17. Now let $\psi(u, v, w, u_0, v_0, w_0)$ stand for

$$\Sigma (2n+1) j fu fv fw fv_0 fw_0 / fu_0.$$

We have the following facts :—

(*a*) $\int \psi(u, v, w, u_0, v_0, w_0)\, d\omega_0 = 8\pi$,

(*b*) $\lim_{u=u_0} \psi(u, v, w, u_0, v_0, w_0) = 0$, unless $v = v_0$, $w = w_0$,

(*c*) ψ is always positive when $u > u_0$ and $\not> \gamma$;

for $G(P, P_1)$ can have no minimum within the ellipsoid, and cannot therefore be negative; it must then increase when P or P_1 moves inwards from the surface.

Let ϕ stand for $\frac{1}{8\pi}\int F(v_0, w_0)\, \psi(u, v, w, u_0, v_0, w_0)\, d\omega_0$, F being one-valued on the surface, but otherwise as little restricted as possible. We have

$$\phi - F(v, w) = \frac{1}{8\pi}\int \{ F(v_0, w_0) - F(v, w) \}\, \psi(u, v, w, u_0, v_0, w_0)\, d\omega_0.$$

Suppose that when $v \sim v_0$ and $w \sim w_0$ do not exceed ϵ, then $F(v_0, w_0) \sim F(v, w)$ does not exceed η; then the part of this integral contributed by values of v_0, w_0 between $v \pm \epsilon$, $w \pm \epsilon$ and by values between $2\beta - v \pm \epsilon$, $2\beta - w \pm \epsilon$ lies between

$$\pm \eta \frac{1}{8\pi}\int \psi(u, v, w, u_0, v_0, w_0)\, d\omega_0,$$

since ψ is always positive; that is, it lies between $\pm \eta$. The rest of

* The method here used applies to any ordinary kind of bounding surface unless the tangent plane meets the surface again. In such a case we only need to describe a sphere touching the surface externally and not meeting it in any other point, and then to invert from a point on the surface of this sphere.

the integral vanishes in the limit when $u = u_0$, since $\psi = 0$ in the limit by (45). Hence, if F is continuous near v, w, so that η can be made as small as we please by diminishing ϵ, we have

$$\lim_{u = u_0} \phi = F(v, w).$$

This result is not affected by a discontinuity in F elsewhere on the surface. Hence the following theorem:—If $F(v, w)$ is a one-valued function of position on the surface u_0, and is such that the integral $\int F(v_0, w_0)\, \psi(u, v, w, u_0, v_0, w_0)\, d\omega_0$ has a definite value $\phi(u, v, w)$ and may be differentiated twice by the ordinary rule, then ϕ is a potential without singularity within the ellipsoid u_0, and when u approaches u_0 the limiting value of ϕ is $F(v, w)$, provided that $F(v, w)$ is continuous in the neighbourhood of the point (v, w) on the surface u_0.

[It is clearly necessary that the boundary values $F(v, w)$ should be integrable over the surface, and this condition is also sufficient to ensure the validity of the solution of the problem.

For let the whole surface be divided into small elements of which σ is any one, and let ϵ be the difference between the greatest and least values of F at the points of σ, and η the difference between the greatest and least values of another function $G(v, w)$ at the points of σ; let M, N be the superior limits of $\pm F, \pm G$ over the whole surface. Then the greatest and least values of the product FG, upon σ, differ by less than $M\eta + N\epsilon$.

Now, if F, G are integrable separately, $\Sigma\epsilon\sigma$ and $\Sigma\eta\sigma$ diminish indefinitely when all the elements σ do so, and therefore $\Sigma(M\eta + N\epsilon)\sigma$ diminishes indefinitely, and the product FG is also integrable.* If G is everywhere infinitesimal, the integral of FG will also be infinitesimal.

From these results it follows that when G is, as here, an analytical function of u, v, w, v_0, w_0 and F is an integrable function of v_0, w_0, finite and one-valued everywhere, the function represented by $\iint FG\, dv_0\, dw_0$ has a definite meaning, and can be differentiated any number of times with respect to u, v, w by the ordinary rule, that is, by differentiating the factor G in the subject.—*Feb. 20th*, 1903.]

The expansion of ϕ in Lamé's functions is convergent by ratio within the ellipsoid; but the question whether this expansion is valid on the surface is not affected; this question amounts to

* Compare du Bois Reymond, *Math. Ann.*, Vol. xx., pp. 122–24.

exactly the same as the corresponding one in the theory of spherical harmonics, and need not be treated here.

18. In order to prove that

$$\int \{F(v_0, w_0) - F(v, w)\} \psi(u, v, w, u_0, v_0, w_0)\, d\omega_0$$

was zero in the limit if (v, w) was outside the limits of the integration, we proved that the limit of ψ was zero when u approached u_0. A somewhat different proof may be given.

Let a sphere A be described touching the boundary at (u_0, v, w) and having no other point in common with it. Let a sphere B be described concentric with A and lying outside it, but not containing any of the points over which the integration extends. Let Q be any point on the sphere B, and let us write $A(P)$ for the square of the tangent from P to A. Consider the potential without singularity within or on the boundary, whose value on the boundary is $KA(P)/A(Q)$, where K is the greatest numerical value of $F(v_0, w_0) - F(v, w)$. This potential at (u, v, w) is $\frac{1}{8\pi}$ of

$$\int KA(P_0)/A(Q) \cdot \psi(u, v, w, u_0, v_0, w_0)\, d\omega_0;$$

every element in the integral is positive; and when P_0 is outside the sphere B the element at P_0 is numerically greater than the corresponding element in

$$\int \{F(v_0, w_0) - F(v, w)\} \psi(u, v, w, u_0, v_0, w_0)\, d\omega_0.$$

But, since $KA(P)/A(Q)$ is analytical on the surface and (u_0, v, w) lies on the sphere A, the former integral tends to the limit zero when u approaches u_0; the latter integral over the smaller range must then also vanish in the limit, since it is numerically less, element by element.

19. If a one-valued potential is to be constructed with given boundary values and given singularities within the region, it is possible, by differentiating or integrating Green's function, to form a potential vanishing along the boundary and having these singularities if they are such as a one-valued potential can have; adding to this a potential without singularities and with the given boundary values, we have the required result.

20. A potential having given values on the ellipsoid u_0 and given singularities in the region outside may be found in like manner.

The typical term of the expansions is a multiple of $gu\,fv\,fw$. Since gu vanishes with u, we may use the results of §§ 11, 12, putting 0 for u_0 and gu for λu.

If the boundary values are denoted by $F(v, w)$, and there are no singularities, and the potential vanishes at infinity, its value is

$$\frac{1}{8\pi}\int F(v_0, w_0)\left\{\Sigma j\,(2n+1)\,gu\,fv\,fw\,fv_0\,fw_0/gu_0\right\}d\omega_0.$$

Green's function $G(P, P_1)$ is $1/PP_1 - \Sigma j\,fu_0\,gu\,fv\,fw\,gu_1\,fv_1\,fw_1/gu_0$. The potential without singularities that has the value 1 at infinity and 0 at the bounding surface is $1-u/u_0$.

21. Lamé's functions may also be used to solve the same problem for the region bounded by two confocal ellipsoids. Before discussing this I shall indicate a remarkable extension of which it is capable, and which may be illustrated by reference to an analogous problem in the theory of the logarithmic potential. Take a two-sheeted Riemann surface with two branch points. With the branch points as foci describe an ellipse in each sheet, and consider the boundary value problem (*Randwerthaufgabe*) for the part of the surface bounded by these ellipses. Taking the axes of the ellipses as axes of coordinates, we may put

$$x = c\cosh u\cos v, \quad y = c\sinh u\sin v\,;$$

so that the curves $u = \text{const.}$ are ellipses of the confocal system and the curves $v = \text{const.}$ hyperbolas of the same system. The points (u, v) and $(-u, -v)$ are the same in position, but belong to different sheets.

The functions corresponding to Lamé's are $\cosh nu$, $\sinh nu$,* $\cos nv$, $\sin nv$, and the problem may be solved by using expansions in multiples of $\cosh nu\cos nv$, $\sinh nu\sin nv$, $\cosh nu\sin nv$, $\sinh nu\cos nv$. In the simpler case when the potential is to be uniform over the area of an ellipse and there is only one sheet the products $\cosh nu\sin nv$ and $\sinh nu\cos nv$ do not come in, since they are not one-valued, but have opposite signs at (u, v), $(-u, -v)$; but all four products are to be used in the case of the two-sheeted surface.

22. In dealing with space of three dimensions we shall speak of a change of colour instead of a change of sheet, and conceive of two

* Instead of $2n+1$ of the order n there are only two; whether we take $\cosh nu$ or $\sinh nu$ to correspond to fu, we have e^{-nu} as corresponding to gu.

infinite spaces, red and blue, coextensive and connected by one or more doors which will be surfaces bounded by curves. In the present case there will be one such door—namely, the plane area that is bounded by the focal ellipse, and a point passing through this door in either direction changes its colour. The two points (u, v, w), $(2\gamma-u, 2\gamma-v, w)$ are the same in position, but differ in colour; for we may pass from the first to the second by travelling along the intersection of the hyperboloids v, w to the point $(2\gamma-u, v, w)$, passing through the door at (γ, v, w), and then along the intersection of the surfaces $2\gamma-u, w$ to $(2\gamma-u, 2\gamma-v, w)$, passing round the edge of the door.

Let two ellipsoids of the system be described, one in each colour; then the problem is to find a potential for the space they enclose, having given boundary values and no singularity within the space except the branching: the potential will be an analytical function of x, y, z at all internal points except those on the focal ellipse, and even there it will be an analytical function of u, v, w, symmetrical as regards v, w.

23. Take the two bounding ellipsoids to be u_0, u_1 where

$$0 < u_0 < \gamma < u_1 < 2\gamma.$$

The discussion will, for the most part, apply to the case of an ordinary shell where u_0, u_1 are both between 0 and γ.

Let F_0, F_1 denote the assigned values on the surfaces u_0, u_1 respectively, and assume for the solution the series $\Sigma(kfu+lgu)fvfw$. Of the two potentials $fufvfw$, $gufvfw$, the former is unaffected by a change of colour, but the latter is not; so that the whole expression is two-valued in ordinary space, but one-valued in the space we are considering.

The values that must be given to k, l are easily found by substituting u_0, u_1 for u and integrating over the two ellipsoids respectively. Thus

$$8\pi(kfu_0+lgu_0) = (2n+1)j\int F_0 fv\,fw\,d\omega,$$

$$8\pi(kfu_1+lgu_1) = (2n+1)j\int F_1 fv\,fw\,d\omega.$$

Let us first assume that $F_0 = 0$ and F_1 is an analytical function of x, y, z everywhere. Thus

$$kfu_0+lgu_0 = 0,$$

and the series may be written $\Sigma k \lambda u fv fw$, where λu stands for $fu - fu_0 gu/gu_0$, the solution of Lamé's equation which vanishes when $u = u_0$. Also the series $\Sigma |k \lambda u_1|$ is convergent by ratio when the $2n+1$ terms of the same order n are grouped together to make one term. The series $\Sigma k \lambda u fv fw$, $\Sigma k \lambda' u fv fw$, $\Sigma k \lambda'' u fv fw$ are therefore all convergent by ratio (§§ 11, 12). Thus, integrating this last series twice between u and u_1, we find

$$\frac{\partial}{\partial u} \Sigma k \lambda u fv fw = \Sigma k \lambda' u fv fw,$$

$$\frac{\partial^2}{\partial u^2} \Sigma k \lambda u fv fw = \Sigma k \lambda'' u fv fw.$$

Again, the series $\Sigma k \lambda u f''v fw$ and $\Sigma k \lambda u fv f''w$ are also convergent by ratio, since $f''v/fv$ and $f''w/fw$ lie between $\pm n(n+1)(a^2 - c^2)$. Integrate the first twice between the limits v, v_0 where v_0 is an arbitrary argument between $\beta \pm 2\alpha$. The result is

$$\Sigma k \lambda u fw \{fv - fv_0 - (v - v_0) f'v_0\},$$

which is therefore a series convergent by ratio. The series $\Sigma k \lambda u fv fw$ and $\Sigma k \lambda u fv_0 fw$ converge separately by ratio, and therefore the same is true of $(v - v_0) \Sigma k \lambda u f'v_0 fw$ and of $\Sigma k \lambda u f'v fw$, since v_0 is as arbitrary as v. Thus

$$\frac{\partial}{\partial v} \Sigma k \lambda u fv fw = \Sigma k \lambda u f'v fw$$

and

$$\frac{\partial^2}{\partial v^2} \Sigma k \lambda u fv fw = \Sigma k \lambda u f''v fw.$$

Similarly for the derivatives with respect to w. Hence $\Sigma k \lambda u fv fw$ satisfies the equation (12), that is, it is a potential. Its value when $u = u_0$ is zero, and, since we may write it

$$\int_{u_0}^{u} \Sigma k \lambda' u fv fw \, du,$$

it varies continuously with u from $u = u_0$ to $u = u_1$ inclusive. The limit it approaches when u approaches u_1 is therefore $\Sigma k \lambda u_1 fv fw$ or F_1. It can be proved as in the former case that neither of the original bounding surfaces u_0, u_1 is a true boundary of the region within which this potential exists and is validly represented by the series, having no singularity except the branching.

Let ϕ_1 denote this potential; in like manner a potential ϕ_0 could

be found having the value zero on the surface u_1 and F_0, if this is analytical, on the surface u_0, and with no singularity except the branching. Then $\phi_1+\phi_0$ solves the original problem in which boundary values F_0, F_1, other than zero, were given on both surfaces.

24. To complete the discussion we must find Green's function, whose only singularity, besides the branching, which will not be counted a singularity in future, is that of $1/PP_2$ at the point P_2 of a specified colour, and which vanishes over the whole boundary.

The expression for this will vary according to the relative magnitudes of u, u_0, u_1, u_2. Suppose first that $0 < u_0 < u_2 < 2\gamma - u_1 < \gamma$. Then the point P_2 of the other colour is outside the region, and all we need to do is to find the coefficients k, l, so that $\Sigma\,(k\,fu+l\,gu)\,fv\,fw$ shall have the value $1/PP_2$ when $u = u_0$ or u_1. This gives

$$k\,fu_0+l\,gu_0 = j\,gu_0\,fu_2\,fv_2\,fw_2,$$

$$k\,fu_1+l\,gu_1 = j\,gu_2\,fu_1\,fv_2\,fw_2,$$

and Green's function is $\dfrac{1}{PP_2}-\Sigma\,(k\,fu+l\,gu)\,fv\,fw$ or

$$\frac{1}{PP_2}-\Sigma\,j\,fv\,fw\,fv_2\,fw_2\{fu_1\,gu\,(fu_0\,gu_2-fu_2\,gu_0) + gu_0\,fu\,(fu_2\,gu_1-fu_1\,gu_2)\} \div (fu_0\,gu_1-fu_1\,gu_0). \quad (47)$$

When $u_0 > 2\gamma-u_1$ and $2\gamma-u_2$ lies between them the expression for Green's function is derived from (47) by putting $2\gamma-u_1$, $2\gamma-u_0$, $2\gamma-u_2$, $2\gamma-u$ for u_0, u_1, u_2, u respectively, and using the relations

$$f\,(2\gamma-u) = \pm fu,$$

$$g\,(2\gamma-u) = \mp\,(gu-e\,fu),$$

where $$e = (2n+1)\int_0^{2\gamma} du/f^2u.$$

The value of e is finite and unambiguous even if $f\gamma = 0$.

25. When u_2 is between u_1 and $2\gamma-u_1$ the matter is not so simple; for $1/PP_2$ has an infinity at each of the points P_2, and both lie within the region. We need to prove the existence of a potential with only one of these singularities. This can be done by means of the alternating method of combining volumes, adapted so as to avoid the usual infinite series of operations.

Suppose, to fix the ideas, that $0 < u_0 \leqslant 2\gamma-u_1 < u_2 < \gamma < u_1$.

Take u_4 between u_2 and $2\gamma - u_2$, u_3 between u_2 and u_4. Then in (u_0, u_4), that is, the region bounded by the ellipsoids u_0, u_4, the expression

$$\phi_1(P) \equiv \frac{1}{PP_2} - \Sigma j\, fv\, fw\, fv_2 fw_2 \{gu\, gu_2 fu_0 fu_4 - (fu\, gu_2 + fu_2\, gu)\, gu_0 fu_4 + fu\, fu_2\, gu_0\, gu_4\} \div (fu_0\, gu_4 - fu_4\, gu_0)$$

represents a potential vanishing on the boundary and having only the singularity of its first term; for this is a case in which (47) may be used.

The potential without singularity in the region (u_3, u_1) vanishing at the boundary u_1 and having the same value as $\phi_1(P)$ at the boundary u_3 is

$$\phi_2(P) \equiv \Sigma\, j\, fv\, fw\, fv_2 fw_2 \frac{fu\, gu_1 - fu_1\, gu}{fu_3\, gu_1 - fu_1\, gu_3} \times \left[gu_2 fu_3 - \frac{gu_3\, gu_2 fu_0 fu_4 - (fu_3\, gu_2 + fu_2\, gu_3)\, gu_0\, fu_4 + fu_2 fu_3\, gu_0\, gu_4}{fu_0\, gu_4 - fu_4\, gu_0} \right],$$

the first term in the bracket arising from the known expansion of $1/PP_2$, since $u_3 > u_2$.

On the surface u_4 we have then, after reduction,

$$\phi_2(P_4) - \phi_1(P_4) = \Sigma j\, fv_4 fw_4 fv_2 fw_2 \frac{(fu_4\, gu_1 - fu_1\, gu_4)(fu_0\, gu_2 - fu_2\, gu_0)(fu_3\, gu_4 - fu_4\, gu_3)}{(fu_3\, gu_1 - fu_1\, gu_3)(fu_0\, gu_4 - fu_4\, gu_0)}.$$

Let us apply the same process to a potential $\chi_1(P)$, having no singularity in the region (u_0, u_4), vanishing on the surface u_0, and having on the surface u_4 values represented by $\Sigma\, \rho\, fv\, fw$. Thus

$$\chi_1(P) = \Sigma \frac{fu_0\, gu - fu\, gu_0}{fu_0\, gu_4 - fu_4\, gu_0} \rho\, fv\, fw.$$

The potential without singularity in (u_1, u_3) vanishing on u_1 and equal to $\chi_1(P)$ on u_3 is

$$\chi_2(P) = \Sigma \frac{fu_0\, gu_3 - fu_3\, gu_0}{fu_0\, gu_4 - fu_4\, gu_0} \frac{fu\, gu_1 - fu_1\, gu}{fu_3\, gu_1 - fu_1\, gu_3} \rho\, fv\, fw,$$

and on the surface u_4 we have, after reduction,

$$\chi_2(P_4) - \chi_1(P_4) = \Sigma \frac{fu_0\, gu_1 - fu_1\, gu_0}{fu_0\, gu_4 - fu_4\, gu_0} \frac{fu_4\, gu_3 - fu_3\, gu_4}{fu_3\, gu_1 - fu_1\, gu_3} \rho\, fv_4 fw_4.$$

This series is the same as that for $\phi_2(P_4) - \phi_1(P_4)$ if

$$\rho = -j\, fv_2 fw_2\, (fu_0\, gu_2 - fu_2\, gu_0)(fu_4 gu_1 - fu_1\, gu_4)/(fu_0\, gu_1 - fu_1\, gu_0).$$

Let us assume for the moment that the series $\Sigma \rho f v f w$ is convergent when this value is taken for ρ, and consider the potentials $\phi_1 - \chi_1$, $\phi_2 - \chi_2$. In the region (u_3, u_4) neither of these has any singularity, and on the boundaries of this region they have the same values. Hence throughout the region they are the same potential, say ω. This potential therefore exists throughout (u_0, u_1), and its only singularity is that of $1/PP_2$ at the point (u_2, v_2, w_2). There is no singularity at P_2 in the other colour, and, as ϕ_1, χ_1 vanish on u_0 and ϕ_2, χ_2 on u_1, the problem is solved, if we can prove the series $\Sigma \rho f v f w$ convergent.

Now from its formation the expansion of $\phi_2 - \phi_1$ is convergent in (u_3, u_4), and this is the same as the series for $\chi_2 - \chi_1$. We therefore need to examine the fraction

$$\frac{fu_0\, gu_1 - fu_1\, gu_0}{fu_0\, gu_4 - fu_4\, gu_0} \frac{fu_3\, gu_4 - fu_4\, gu_3}{fu_3\, gu_1 - fu_1\, gu_3} (= \kappa, \text{ say}),$$

with the view of finding an inferior limit to its value. We have

$$1 - \kappa = \frac{fu_0\, gu_3 - fu_3\, gu_0}{fu_0\, gu_4 - fu_4\, gu_0} \frac{fu_4\, gu_1 - fu_1\, gu_4}{fu_3\, gu_1 - fu_1\, gu_3}.$$

Now, if λu stands for $fu_0\, gu - fu\, gu_0$, we have seen that λu, $\lambda' u$ increase numerically with $u - u_0$, and thus $\lambda u_3 / \lambda u_4$ is positive but less than $(u_3 - u_0)/(u_4 - u_0)$, since $u_0 < u_3 < u_4$. Now $\lambda u_3 / \lambda u_4$ is the first factor in $1 - \kappa$; in the same way the second factor is positive, but $< (u_1 - u_4)/(u_1 - u_3)$, since $u_1 > u_4 > u_3$. Thus $1 - \kappa$ is positive and $< \frac{u_3 - u_0}{u_4 - u_0} \frac{u_1 - u_4}{u_1 - u_3}$; $\kappa < 1$ and $> \frac{u_1 - u_0}{u_4 - u_0} \frac{u_4 - u_3}{u_1 - u_3}$, a quantity independent of n. Now the series $\Sigma |\, \kappa\rho\, |$ is convergent by ratio, and the same is therefore true for $\Sigma |\, \rho\, |$ and $\Sigma \rho f v f w$. The solution ω is accordingly valid.

26. The expression $\phi_1 - \chi_1$ thus found for the potential is exactly the same as (47). This holds in the region (u_0, u_4), and u_4 may be as near as we please to $2\gamma - u_2$. Hence Green's function is given by the expression (47) for values of u between u_0 and $2\gamma - u_2$ when u_2 is between u_0 and γ.

If in (47) we write for $1/PP_2$ the expansion $\Sigma j\, gu_2 fu fv fw fv_2 fw_2$, which holds when $u > u_2$, we arrive at another expression for Green's function, which agrees with $\phi_2 - \chi_2$ and holds when u is between u_3 and u_1, that is, between u_2 and u_1, since u_3 may be as near u_2 as we

please. This expression is

$$\Sigma j\, fv\, fw\, fv_2\, fw_2\, (fu\, gu_1 - fu_1\, gu)(fu_0\, gu_2 - fu_2\, gu_0)/(fu_0\, gu_1 - fu_1\, gu_0). \tag{48}$$

To pass to the case when $u_2 > \gamma$ we need only write $2\gamma - u_1$, $2\gamma - u$, $2\gamma - u_2$, $2\gamma - u_0$ for u_0, u, u_2, u_1.

27.* There is a difficulty in the case when $u_2 = \gamma$, since we cannot take u_3, u_4 between u_2 and $2\gamma - u_2$.

The expressions (47), (48) still represent potentials without singularity in the regions (u_0, γ), (γ, u_1) respectively, but, as these regions have no part in common, we need to prove that the two potentials are the same and to find what singularities they have on the surface $u = \gamma$.

Let $\omega(P, P_2)$ denote the value of the function represented by (47), (48) so long as $u_2 < \gamma$. Describe two spheres A, B with a common centre O lying within the focal ellipse. Let ρ, σ be their radii and $\sigma > \rho$; suppose also that B does not reach to any point on the bounding ellipsoids or the focal ellipse or to the point P_2. Then throughout B the potential $\omega(P, P_2)$ can be expressed by an expansion in spherical harmonics, say $\Sigma\, \alpha_{n^2+m}\, \eta_{n^2+m} \left(\frac{r}{\rho}\right)^n$, where $r = OP$, $m = 1, 2, \ldots, 2n+1$ and $\eta_{n^2+1}, \eta_{n^2+2}, \ldots, \eta_{n^2+2n+1}$ are a canonical system of surface harmonics of order n, the quantities α being constant coefficients, depending on P_2. We shall examine the effect on α of gradually changing u_2 into u_2' where $u_2 < u_2' < \gamma$.

Now, throughout the whole region $\omega(P, P_2) < 1/PP_2$, for $1/PP_2 - \omega(P, P_2)$ is a potential whose values on the boundary are positive, and whose only singularity within the region is a positive infinity at P_2', the point at P_2 in the other colour. Hence $1/PP_2 - \omega(P, P_2)$ is positive throughout the region. It follows that over the surface B the value of $\omega(P, P_2)$ is less than h, the reciprocal of the least distance from P_2 in its different positions to B.

Let us take $\gamma - u_2 < \delta$ and describe the ellipsoid $u = \gamma - \delta$. This will cut out from the surface of B a belt B'. Now we have

$$\alpha_{n^2+m} = \left(\frac{\rho}{\sigma}\right)^n \frac{2n+1}{4\pi\sigma^2} \int_B \omega(P, P_2)\, \eta_{n^2+m}\, dS.$$

* Rewritten December, 1902.

The contribution of B' to the integral here is a quantity of the same order of magnitude as the area of B', that is, as δ, since the subject of integration $< h$ everywhere. Hence the change in value of this part of the integral is of the same order of magnitude as δ, or a lower order.

Now consider the change in the contribution of $B-B'$. This depends on the change in $\omega(P, P_2)$. Let a sphere C of finite radius ρ' be described about P_2, so as not to reach any point of B, and let $1/h'$ be the least distance between B and the various positions of C. Then $\omega(P, P_2)$ is a potential for P_2, without singularity within C, and its values on C are less than h'. Hence its space derivative at P_2 in any direction cannot exceed the finite quantity $3h'/\rho'$, so that the change in $\omega(P, P_2)$ does not exceed this quantity multiplied by the length of the path; it is therefore of the same order of magnitude as the length of path, that is to say, as δ. The same is therefore true of the integral in the expression for α_{n^2+m}, when taken over $B-B'$. Hence on the whole the change in α_{n^2+m} is of the same order of magnitude as δ. Diminishing δ without limit, we see that α_{n^2+m} approaches a definite limit when u_2 approaches γ. Let this limit be β_{n^2+m}; then $\beta_{n^2+m} < \left(\frac{\rho}{\sigma}\right)^n (2n+1)\, h$, since α_{n^2+m} is always less than this quantity. Hence the series $\Sigma \beta_{n^2+m} \eta_{n^2+m} \left(\frac{r}{\rho}\right)^n$ is convergent by ratio throughout the sphere A, and represents a potential without singularity in that sphere. The sum of this series is the limit of the sum of $\Sigma \alpha_{n^2+m} \eta_{n^2+m} \left(\frac{r}{\rho}\right)^n$, since this latter is uniformly convergent up to the limit. Hence $\lim_{u_2 = \gamma} \omega(P, P_2)$ is a potential without singularity within A.

The above proof holds good even if P_2' lies within the sphere A, since it depends on the values of $\omega(P, P_2)$ on the surface of B, and a superior limit of these is still h.

If P_2 itself lies within A, then $\omega(P, P_2')$ has no singularity within A, and, as $1/PP_2 - \omega(P, P_2) - \omega(P, P_2')$ has no singularity within the whole region, it follows that $\omega(P, P_2)$ has none but that of $1/PP_2$ at P_2.

If P_2 lies on the focal ellipse, it coincides with P_2', and the difficulty of isolating the singularity does not arise. We have

$$\omega(P, P_2) = \omega(P, P_2');$$

the function $\omega(P, P_2)$ vanishes on the surface and has the singularity of $1/2PP_2$ at P_2.

28. In all there are four expressions for Green's function in the region (u_0, u_1), where $u_1 > u_0$, namely: (47), valid for the region common to (u_0, u_1) and $(u_0, 2\gamma - u_2)$; (47) with u_0, u_1, u_2, u changed into $2\gamma - u_1$, $2\gamma - u_0$, $2\gamma - u_2$, $2\gamma - u$—this is valid for the region common to (u_0, u_1) and $(2\gamma - u_2, u_1)$; (48), valid for the region (u_2, u_1); (48) with u, u_2 interchanged, valid for the region (u_0, u_2). Of these the first, third, and fourth serve for an ordinary shell, in which u_0, u_1 are both $< \gamma$.

By combination of volumes we may pass to the extreme case $u_0 = 0$, $u_1 = 2\gamma$ in which the region includes the whole of the two-coloured space. Writing again e for $(2n+1)\int_0^{2\gamma} du/f^2u$, we have the following expressions for the potential which vanishes at infinity in each colour and has no singularity but that of $1/PP_2$ at (u_2, v_2, w_2):—

$$1/PP_2 - \Sigma j e^{-1} gu\, gu_2 fv\, fw\, fv_2 fw_2 \quad (\text{when } u < 2\gamma - u_2),$$

$$1/PP_2 - \Sigma j e^{-1} (gu - e fu)(gu_2 - e fu_2) fv\, fw\, fv_2 fw_2 \quad (\text{when } u > 2\gamma - u_2),$$

$$\Sigma j (fu - e^{-1} gu)\, gu_2 fv\, fw\, fv_2 fw_2 \quad (\text{when } u > u_2),$$

$$\Sigma j (fu_2 - e^{-1} gu_2)\, gu\, fv\, fw\, fv_2 fw_2 \quad (\text{when } u < u_2).$$

It is supposed of course that u, u_2 lie between 0 and 2γ.

29. The problem for the shell in one or two colours, when the boundary values are not analytical, may be discussed exactly as in the simpler case, except in one point. We have to prove that $\dfrac{\partial}{\partial u} G(P, P_2)$ diminishes indefinitely when $u = u_0$ or u_1 and u_2 approaches either u_0 or u_1, but P_2 does not coincide with P in the limit. The former proof fails when $u = u_0$ and u_2 approaches u_0, if $2\gamma - u_0$ lies between u_0 and u_1, but another is given by the combination method. Let us use the former notation (§§ 25, 26), but suppose $0 < 2\gamma - u_1 \leqslant u_0 < u_2 < u_3 < u_4 < \gamma < u_1$, which will not affect the former work, and write ω^3, ω^4, ... for the greatest numerical value of ω, ... on the surfaces u_3, u_4 respectively.

The potential which has no singularity in (u_0, u_4) vanishes when $u = u_0$ and $= 1$ when $u = u_4$ is $(u - u_0)/(u_4 - u_0)$. The value of this when $u = u_3$ is $(u_3 - u_0)/(u_4 - u_0)$; hence, if a potential θ has no singularity in (u_0, u_4), has the value zero on u_0 and values not exceeding M on u_4, its value on u_3 cannot exceed $M(u_3 - u_0)/(u_4 - u_0)$, and $\partial\theta/\partial u$ on u_0 cannot exceed $M/(u_4 - u_0)$. Similarly in other cases. We wish to find a limit to ω^4.

Now $\omega = \phi_2 - \chi_2$

in (u_3, u_1), and thus $\omega^4 \gtrless \phi_2^4 + \chi_2^4$.

Also $\phi_2^4 \gtrless \phi_2^3 (u_1-u_4)/(u_1-u_3)$, that is, $\phi_1^3 (u_1-u_4)/(u_1-u_3)$,

and $\chi_2^4 \gtrless \chi_2^3 (u_1-u_4)/(u_1-u_3)$, that is, $\chi_1^3 (u_1-u_4)/(u_1-u_3)$,

$\chi_1^3 \gtrless \chi_1^4 (u_3-u_0)/(u_4-u_0)$ or $\omega^4 (u_3-u_0)/(u_4-u_0)$,

since $\omega = -\chi_1$

on u_4. Thus $\omega \gtrless \phi_1^3 \dfrac{u_1-u_4}{u_1-u_3} + \omega^4 \dfrac{u_1-u_4}{u_1-u_3} \dfrac{u_3-u_0}{u_4-u_0}$,

and after reduction

$$\chi_1^4 \quad \text{or} \quad \omega^4 \gtrless \phi_1^3 \frac{u_1-u_4}{u_1-u_0} \frac{u_4-u_0}{u_4-u_3}.$$

The greatest value on u_0 of $\partial\chi_1/\partial u$ does not exceed $\dfrac{1}{u_4-u_0}$ of this, or $\phi_1^3 (u_1-u_4)/(u_1-u_0)(u_4-u_3)$, which diminishes indefinitely when u_2 approaches u_0, since ϕ_1 is Green's function for (u_0, u_4). Since then, in (u_0, u_4),

$$\frac{\partial\omega}{\partial u} = \frac{\partial\phi_1}{\partial u} - \frac{\partial\chi_1}{\partial u},$$

and each of the terms on the right in this equation tends to the limit 0 when $u = u_0$ and u_2 approaches u_0, the limit of $\partial\omega/\partial u$ is also 0, unless of course $v_2 = v_0$, $w_2 = w_0$.

30. This method fails if $u_0 = \gamma$, but in this case Green's function may be put in a special form. Let $G(P, P_2, u_0, u_1)$ denote Green's function for the points P, P_2 in the region bounded by the ellipsoids u_0, u_1, and denote the point $(2\gamma-u_2, v_2, w_2)$ by P_2''. Then, if P_2 lies in the region (γ, u_1), P_2'' does not, and hence

$$G(P, P_2, 2\gamma-u_1, u_1) - G(P, P_2'', 2\gamma-u_1, u_1)$$

has only one singularity in (γ, u_1), namely, that at P_2. This expression vanishes when $u = u_1$, $2\gamma-u_1$, or γ; the last follows since its sign changes when u is changed into $2\gamma-u$. Hence the expression is $G(P, P_2, \gamma, u_1)$, and

$$\frac{\partial}{\partial u} G(P, P_2, \gamma, u_1) = \frac{\partial}{\partial u} G(P, P_2, 2\gamma-u_1, u_1) - \frac{\partial}{\partial u} G(P, P_2'', 2\gamma-u_1, u_1)$$

$$= 0,$$

in limit when $u = \gamma$ and u_2 approaches γ, unless P, P_2 coincide in the limit. Here the continuation of Green's function across the boundary is known, so that the usual difficulty does not arise.

31.* Let us now discuss the problem of finding a potential without singularity within a given ellipsoid when the values at the boundary are given of its differential coefficient along the normal; that is, we are to find ϕ so that, when $u = u_0$,

$$\frac{\partial \phi}{\partial u} = F(v, w),$$

a given function, and that within the ellipsoid u_0

$$\Delta \phi = 0.$$

It is necessary that the condition

$$\int F(v, w)\, d\omega = 0$$

should be satisfied.

There is no difficulty in proving, when the given function F is analytical everywhere on the surface, that the solution of the problem is

$$\phi = \frac{1}{8\pi} \Sigma (2n+1)\, j \frac{fu}{f'u_0} fv\, fw \int F(v_0, w_0)\, fv_0 fw_0\, d\omega_0.$$

It is not so easy to prove this when F is not everywhere analytical. The idea of the following proof is taken from C. Neumann (*Untersuchungen über das . . . Potential*, Leipzig, 1877).

Take $d\nu$ to be an element of the outward normal to the ellipsoid u_0, and suppose an electrical distribution of surface density $-F(v, w) \dfrac{\partial u}{\partial \nu}$ upon this surface. The potential of the distribution at an outside point (u, v, w) is

$$\tfrac{1}{2} \Sigma j\, gu\, fv\, fw\, fu_0 \int F(v_0, w_0)\, fv_0 fw_0\, d\omega_0,$$

say $\Sigma \lambda\, gu\, fv\, fw$. This potential is continuous everywhere, even at the surface, if $F(v, w)$ is everywhere finite. Let us denote it by ψ inside and by ψ' outside.

In the expansion of ψ' the term of order 0 is wanting, since by hypothesis

$$\int F(v_0, w_0)\, d\omega_0 = 0\,;$$

hence, since ψ' is continuous over the surface, Neumann's method of the arithmetic mean may be used to prove that the ellipsoid u_0 can

* §§ 31 to end have been added to the paper as originally written.—*Feb. 20th*, 1903.

be considered as a magnetic shell of such strength that its potential at an outside point will be ψ'. Let the potential of this shell be denoted by χ inside and χ' outside, so that $\chi' = \psi'$.

Since ψ, ψ' are the internal and external potentials of the same electric distribution, we have at the surface u_0

$$\frac{\partial \psi'}{\partial \nu} - \frac{\partial \psi}{\partial \nu} = 4\pi F(v, w) \frac{\partial u}{\partial \nu}$$

or

$$\frac{\partial \psi'}{\partial u} - \frac{\partial \psi}{\partial u} = 4\pi F(v, w).$$

Similarly

$$\frac{\partial \chi'}{\partial u} - \frac{\partial \chi}{\partial u} = 0,$$

since χ, χ' are the internal and external potentials of a magnetic shell. Since

$$\psi' - \chi' = 0$$

at all points outside, we have by subtraction

$$\frac{\partial}{\partial u}(\chi - \psi) = 4\pi F(v, w).$$

Hence the potential sought is $\frac{1}{4\pi}(\chi - \psi)$. It remains to expand this in terms of Lamé's functions. The expansion will be absolutely convergent, and valid, everywhere inside the ellipsoid u_0, since the potential has no singularity inside this surface.

32. We have

$$\psi = \Sigma \lambda \, g u_0 f u \, f v \, f w / f u_0,$$

while

$$\chi = \Sigma \nu \, f u \, f v \, f w,$$

where

$$8\pi \nu f' u = (2n+1) j \int \frac{\partial \chi}{\partial u} f v \, f w \, d\omega,$$

taken upon the ellipsoid u $(u > u_0)$. But

$$8\pi \lambda g' u = (2n+1) j \int \frac{\partial \chi'}{\partial u} f v \, f w \, d\omega,$$

taken upon the ellipsoid u $(u < u_0)$, and the quantities $\frac{\partial \chi}{\partial u}$, $\frac{\partial \chi'}{\partial u}$ approach the same limit at the surface. Hence, passing to the limit in each case, we have

$$\nu f' u_0 = \lambda g' u_0$$

and $$\chi-\psi = \Sigma\lambda fu\, fv\, fw \left(\frac{g'u_0}{f'u_0} - \frac{gu_0}{fu_0}\right)$$

$$= \Sigma(2n+1)\,\lambda\, fu\, fv\, fw/fu_0 f'u_0,$$

$$\frac{1}{4\pi}(\chi-\psi) = \frac{1}{8\pi}\Sigma(2n+1)\,j\,\frac{fu}{f'u_0}\,fv\,fw\int F(v_0, w_0)\,fv_0\,fw_0\,d\omega_0.$$

The theorem is therefore proved, and the solution holds good if the function F is integrable all over the surface and satisfies the condition

$$\int F(v, w)\,d\omega = 0.$$

Exactly the same method applies when the potential is to be found for external space. The expansion is

$$\frac{1}{8\pi}\Sigma(2n+1)\,j\,\frac{gu}{g'u_0}\,fv\,fw\int F(v_0, w_0)\,fv_0 fw_0\,d\omega_0,$$

and the condition $$\int F(v, w)\,d\omega = 0$$

is unnecessary.

33. Lastly, take the case of a region bounded by the two ellipsoids u_0, u_1, whether this is a shell in the ordinary sense of the word or a two-coloured region. Let $F_0(v, w)$ and $F_1(v, w)$ represent the assigned values of $\dfrac{\partial\varphi}{\partial u}$ on the two surfaces. Then it is necessary that

$$\int F_0(v, w)\,d\omega = \int F_1(v, w)\,d\omega.$$

If this condition is satisfied and the value of each side of the equation is $8\pi A$, we have

$$\varphi = Au + \phi_0 + \phi_1,$$

where ϕ_0, ϕ_1 are potentials without singularity within the region, and,

when $u = u_0$, $$\frac{\partial\phi_0}{\partial u} = F_0(v, w) - A, \quad \frac{\partial\phi_1}{\partial u} = 0;$$

when $u = u_1$, $$\frac{\partial\phi_0}{\partial u} = 0, \quad \frac{\partial\phi_1}{\partial u} = F_1(v, w) - A.$$

If these potentials ϕ_0, ϕ_1 can be found, the problem can be solved; the same kind of process serves to find both.

Now the expression for ϕ_0, which is certainly valid when $F_0(v, w)$

is analytical, is

$$\frac{1}{8\pi}\Sigma(2n+1)\,j\,\frac{fu\,g'u_1-gu\,f'u_1}{f'u_0\,g'u_1-g'u_0\,f'u_1}\,fv\,fw\int\{F_0(v_0,w_0)-A\}\,fv_0\,fw_0\,d\omega_0,$$

or, say,

$$\Sigma k\,\frac{fu\,g'u_1-gu\,f'u_1}{f'u_0\,g'u_1-g'u_0\,f'u_1}.$$

We need to show that even when F_0 is not assumed to be analytical, but only integrable, the limit of

$$\Sigma k\,\frac{f'u\,g'u_1-g'u\,f'u_1}{f'u_0\,g'u_1-g'u_0\,f'u_1}$$

when u approaches u_0 is $F_0(v,w)-A$.

Suppose first that $u_0<\gamma$ and $<u_1$; then the limit of u is u_0+0, and we know that $\Sigma k f'u/f'u_0$ approaches the desired limit. Now

$$\frac{f'u\,g'u_1-g'u\,f'u_1}{f'u_0\,g'u_1-g'u_0\,f'u_1}-\frac{f'u}{f'u_0}=\frac{f'u_1\,(f'u\,g'u_0-f'u_0\,g'u)}{f'u_0\,(f'u_0\,g'u_1-g'u_0\,f'u_1)}$$

$$=\frac{f'u\,g'u_0-f'u_0\,g'u}{f'u_2\,g'u_0-f'u_0\,g'u_2}\left[\frac{f'u_2\,g'u_1-f'u_1\,g'u_2}{f'u_0\,g'u_1-g'u_0\,f'u_1}-\frac{f'u_2}{f'u_0}\right],$$

where u_2 has any value. If now we take $u_2>u_0$, but $<u_1$ and $<\gamma$, and take u between u_0 and u_2, each of the three fractions

$$\frac{f'u\,g'u_0-f'u_0\,g'u}{f'u_2\,g'u_0-f'u_0\,g'u_2},\quad\frac{f'u_2\,g'u_1-f'u_1\,g'u_2}{f'u_0\,g'u_1-g'u_0\,f'u_1},\quad\frac{f'u_2}{f'u_0}$$

does not exceed pq^n, where p, q are certain positive constants and $q<1$. Hence the series

$$\Sigma k\left[\frac{f'u\,g'u_1-g'u\,f'u_1}{f'u_0\,g'u_1-g'u_0\,f'u_1}-\frac{f'u}{f'u_0}\right]$$

converges uniformly when u is between u_0 and u_2. Its limit when u approaches u_0 is therefore found by putting $u=u_0$ and is zero, which was to be proved.

This method applies when $u_0-\gamma$ and u_0-u_1 have the same sign. When $u_1<u_0<\gamma$ the series to be used for comparison is $\Sigma k\,g'u/g'u_0$, and when $u_1>u_0>\gamma$ it is $\Sigma k\,g'(2\gamma-u)/g'(2\gamma-u_0)$. When $u_0=\gamma$ the method fails, but the gap may be filled up as follows.

34. In any region $(u_0,\ u_1)$ bounded by the ellipsoids u_0, u_1 we have constructed a potential ω (§ 25), vanishing at the boundary and having no singularity except that of $1/PP_2$ at the single point (u_2, v_2, w_2).

When $u_0 < u_1$, and both differ from γ, we can construct a potential ψ without singularity within the region, and such that,

when $u = u_0$, $$\frac{\partial \psi}{\partial u} = \frac{\partial \omega}{\partial u} - \tfrac{1}{2},$$

when $u = u_1$, $$\frac{\partial \psi}{\partial u} = \frac{\partial \omega}{\partial u} + \tfrac{1}{2};$$

the terms $\pm\frac{1}{2}$ are added on account of the condition to be satisfied by the values of $\dfrac{\partial \psi}{\partial u}$ at the boundary. Then $\omega - \psi$ is a potential with the singularity at (u_2, v_2, w_2) and no other in the region, and its derivative with respect to u is constant over each of the bounding surfaces. Let this function be denoted by $\Omega(P, P_2, u_0, u_1)$. Let P'', P_2'' denote the points $(2\gamma - u, v, w)$, $(2\gamma - u_2, v_2, w_2)$. We then have, if $u_1 > \gamma$,

$$\Omega(P, P_2'', 2\gamma - u_1, u_1) = \Omega(P'', P_2, 2\gamma - u_1, u_1).$$

Thus $$\frac{\partial}{\partial u}\Omega(P, P_2'', 2\gamma - u_1, u_1) = \frac{\partial}{\partial u}\Omega(P'', P_2, 2\gamma - u_1, u_1).$$

But, when $u = \gamma$, P and P'' coincide, and

$$\frac{\partial}{\partial u}\Omega(P'', P_2, 2\gamma - u_1, u_1) = -\frac{\partial}{\partial u}\Omega(P, P_2, 2\gamma - u_1, u_1).$$

Thus $$\frac{\partial}{\partial u}\{\Omega(P, P_2, 2\gamma - u_1, u_1) + \Omega(P, P_2'', u_1, 2\gamma - u_1, u_1)\} = 0$$

when $u = \gamma$. It is therefore possible to form the function $\Omega(P, P_2, u_0, u_1)$ even when $u_0 = \gamma$, and its value is in fact

$$\Omega(P, P_2, 2\gamma - u_1, u_1) + \Omega(P, P_2'', 2\gamma - u_1, u_1) + \tfrac{1}{2}u.$$

Its region of existence is certainly not bounded by the ellipsoid γ even when $u_2 = \gamma$.

Now consider the expression

$$\phi_0 = -\frac{1}{8\pi}\int \Omega(P, P_0, u_0, u_1)\{F_0(v_0, w_0) - A\}\, d\omega_0,$$

in which P_2 is replaced by a point $P_0(u_0, v_0 w_0)$ on the surface u_0. As a function of the point P this is a potential without singularity within the region. If we take $u = u_1$, we have, since the ellipsoid u_1 does not bound the region within which Ω, ϕ_0 are analytical,

$$\frac{\partial \phi_0}{\partial u} = -\frac{1}{8\pi}\int (-\tfrac{1}{2})\{F(v_0, w_0) - A\}\, d\omega_0 = 0$$

for $\dfrac{\partial\Omega}{\partial u} = -\frac{1}{2}$ at this surface, and by hypothesis

$$\int F'(v_0, w_0)\, d\omega_0 = 8\pi A.$$

If $u_0 = \gamma$, we have that $\Omega - \frac{1}{2}u$, and therefore also φ_0, is unchanged by putting $2\gamma - u$ for u; so that the function φ_0 exists in both the regions (γ, u_1) and $(2\gamma - u_1, \gamma)$, and is continuous at the surface γ. On the other hand, $\dfrac{\partial\varphi_0}{\partial u}$ is an odd function of $\gamma - u$ and is discontinuous at this surface. There is no discontinuity in the part of the definite integral contributed by those regions of the surface which are at a finite distance from P or which are near it in position, but in the other colour. In considering the part of the surface which is near P we may write

$$\varphi_0 = -\frac{1}{8\pi}\int\left(\frac{1}{PP_0} + \frac{1}{P'P_0}\right)\{F_0(v_0, w_0) - A\}\, d\omega_0$$

$$-\frac{1}{8\pi}\int\left\{\Omega - \frac{1}{PP_0} - \frac{1}{P'P_0}\right\}\{F_0(v_0, w_0) - A\}\, d\omega_0,$$

and the discontinuity is wholly due to the first term, since $\Omega - \dfrac{1}{PP_0} - \dfrac{1}{P'P_0}$ has no singularity at P_0. But the discontinuity in the first term of $\dfrac{\partial\varphi_0}{\partial u}$ is exactly $2F_0(v, w) - 2A$, that is,

$$\left(\frac{\partial\varphi_0}{\partial u}\right)_{\gamma+0} - \left(\frac{\partial\varphi_0}{\partial u}\right)_{\gamma-0} = 2F_0(v, w) - 2A,$$

and, since $\dfrac{\partial\varphi_0}{\partial u}$ is an odd function of $\gamma - u$, its two limiting values when $u = \gamma \pm 0$ must be $\pm F(v, w) \mp A$.

The expression $\pm\dfrac{1}{8\pi}\displaystyle\int\Omega(P, P_0, u_0, u_1)\{F(v_0, w_0) - A\}\, d\omega_0$ gives then the desired value of φ_0 when $u_0 = \gamma$, and there is no difficulty in proving that this agrees in form with the general solution. The upper or lower sign is to be taken according as $u_1 <$ or $> \gamma$.

On a New Connexion of Bessel Functions with Legendre Functions. *By* E. T. Whittaker. Received and Read November 13th, 1902.

1. *Introduction.*

The object of this communication is to establish a new connexion between the Legendre functions $P_n(z)$ and $Q_n(z)$ and the Bessel functions $J_n(z)$ and $Y_n(z)$. Hitherto the functions have been known to be connected in two ways, namely, (1) by Heine's expression of the Bessel functions of the first kind and integral order as limiting cases of the associated Legendre functions of infinite order and finite degree, and (2) by a set of formulæ which express Legendre functions in terms of definite integrals involving Bessel functions. The most general form of these latter integrals was given by Schafheitlin (*Math. Ann.*, Vol. xxx.). The integral

$$P_{2n}(z) = (-1)^n \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty J_{2n+\frac{1}{2}}(x) \cos xz \,.\, x^{-\frac{1}{2}} dx,$$

which is valid for a certain range of values of n, may be given as typical of this set of formulæ.*

The new connexion developed in this paper between the Legendre functions $P_n(z)$, $Q_n(z)$ and the Bessel functions $J_n(z)$, $Y_n(z)$ is not restricted to integer values of the order n of the functions, but can be expressed in various forms applicable to all real or complex values of n. The connexion is expressed by the results numbered (I.) to (V.) below.

It may be noted here that a certain kind of reciprocity exists between the theorems of Schafheitlin and those of this paper; this is in agreement with a remarkable theorem due to Retzval, namely, that, if two functions $u(z)$ and $v(z)$ are connected by a relation of the form

$$u(z) = \int e^{zt} v(t)\, dt,$$

then they are also in general connected by a relation of the form

$$v(z) = \text{constant} \times \int e^{-zt} u(t)\, dt.$$

* An interesting family of integrals of this type is given by Steinthal, *Quart. Jour.*, Vol. xviii., pp. 330–345.

2. *A Formula which embodies a Simple Case of the Connexion.*

We shall first recall a formula which is due to Lord Rayleigh,* namely,

$$\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^n J_{n+\frac{1}{2}}(z) = P_n\left(\frac{d}{i\,dz}\right)\left(\frac{\sin z}{z}\right);$$

this formula may be regarded as embodying the simplest special case of the connexion which is the subject of this paper.

For the sake of completeness a short proof will be given of this formula. It is true when $n = 0$ and $n = 1$, since it then reduces to the well known results

$$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin z$$

and

$$J_{\frac{3}{2}}(z) = -\left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \frac{d}{dz}\left(\frac{\sin z}{z}\right).$$

We shall now show by induction that it is true in general. For suppose it true for $n = k-2$ and $n = k-1$. Then, from the recurrence formula

$$kP_k(z) - (2k-1)\,zP_{k-1}(z) + (k-1)\,P_{k-2}(z) = 0,$$

we have

$$kP_k\left(\frac{d}{i\,dz}\right)\left(\frac{\sin z}{z}\right)$$

$$= \left\{(2k-1)\frac{d}{i\,dz}P_{k-1}\left(\frac{d}{i\,dz}\right) - (k-1)\,P_{k-2}\left(\frac{d}{i\,dz}\right)\right\}\left(\frac{\sin z}{z}\right),$$

and by hypothesis the right-hand member is equal to

$$(2k-1)\frac{d}{i\,dz}\left\{\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^{k-1} J_{k-\frac{1}{2}}(z)\right\} - (k-1)\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^{k-2} J_{k-\frac{3}{2}}(z)$$

or

$$\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^k \left\{\frac{2k-1}{2z} J_{k-\frac{1}{2}}(z) - (2k-1)\frac{dJ_{k-\frac{1}{2}}(z)}{dz} + (k-1)\,J_{k-\frac{3}{2}}(z)\right\}$$

or

$$\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^k \left\{-\frac{(k-1)(2k-1)}{z} J_{k-\frac{1}{2}}(z) + (2k-1)\,J_{k+\frac{1}{2}}(z) + (k-1)\,J_{k-\frac{3}{2}}(z)\right\}$$

* *Theory of Sound*, Vol. II., p. 263.

or

$$\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^k k J_{k+\frac{1}{2}}(z) + \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^k (k-1) \left\{ -\frac{2k-1}{z} J_{k-\frac{1}{2}}(z) + J_{k+\frac{1}{2}}(z) + J_{k-\frac{3}{2}}(z) \right\},$$

and the quantity in the last bracket is zero, in consequence of the recurrence-formulæ for the Bessel functions. We have therefore

$$P_k\left(\frac{d}{i\,dz}\right)\left(\frac{\sin z}{z}\right) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} i^k J_{k+\frac{1}{2}}(z),$$

and consequently the formula is true for $n = k$ if it is true for $n = k-1$ and $n = k-2$. But it was shown to be true for $n = 0$ and $n = 1$. It is therefore true universally.

It will be noticed that this result connects the case in which the Legendre function $P_n(z)$ can be expressed in a finite form in terms of elementary functions (namely, the case of $n =$ an integer) with the case in which the Bessel function $J_m(z)$ can be expressed in a finite form in terms of elementary functions (namely, m equal to half an odd integer).

3. *Transition to a Definite-Integral form of the Connexion.*

It is obvious that Rayleigh's equation has a meaning only when n is an integer. In order to extend it to the case in which n is not an integer, we must therefore express the result in a form which is capable of this generalization.

For this purpose we observe that, if r be a postive integer, we have

$$\int_{-1}^{1} e^{izt} t^r dt = (-i)^r \frac{d^r}{dz^r} \int_{-1}^{1} e^{izt} dt = 2\,\frac{d^r}{i^r dz^r}\left(\frac{\sin z}{z}\right).$$

Since, when n is an integer, $P_n(z)$ is a sum of positive integral powers of z, we have therefore

$$\int_{-1}^{1} e^{izt} P_n(t)\,dt = 2P_n\left(\frac{d}{i\,dz}\right)\left(\frac{\sin z}{z}\right);$$

and therefore, by (I.), we have

$$J_{n+\frac{1}{2}}(z) = (-i)^n \left(\frac{z}{2\pi}\right)^{\frac{1}{2}} \int_{-1}^{1} e^{izt} P_n(t)\,dt,$$

and on combining the two parts of the range of integration, namely,

from -1 to 0 and from 0 to 1, this becomes

$$J_{n+\frac{1}{2}}(z) = \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \int_0^1 \cos\left(zt - \frac{n\pi}{2}\right) P_n(t)\, dt.$$

This equation has so far been proved for the case in which n is an integer. But each member of it has a meaning, even when n is not an integer, and we may therefore now inquire how far the result is still true when the restriction to integral values of n is removed.

4. *General Case in which n is unrestricted: a Family of Integrals which satisfy Bessel's Differential Equation.*

It will now appear that the integral last obtained is merely one member of a large family of definite integrals by which the Bessel functions can be represented for all values of z and of n, real or complex. To show this, let u denote the integral

$$z^{\frac{1}{2}} \int_\gamma e^{zti} P_{n-\frac{1}{2}}(t)\, dt,$$

where $P_{n-\frac{1}{2}}(t)$ is the Legendre function of order $(n-\frac{1}{2})$, and where the path of integration, γ, is left for the present unspecified.

Then, on performing the differentiations, we have

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{n^2}{z^2}\right) u$$

$$= (\tfrac{1}{4} - n^2)\, z^{-\frac{3}{2}} \int_\gamma e^{zti} P_{n-\frac{1}{2}}(t)\, dt + 2z^{-\frac{1}{2}} i \int_\gamma e^{zti} t P_{n-\frac{1}{2}}(t)\, dt$$

$$+ z^{\frac{1}{2}} \int_\gamma e^{zti} (1-t^2) P_{n-\frac{1}{2}}(t)\, dt,$$

and, since $P_{n-\frac{1}{2}}(t)$ satisfies Legendre's equation, namely,

$$\frac{d}{dt}\left\{(1-t^2)\frac{dP_{n-\frac{1}{2}}(t)}{dt}\right\} + (n^2 - \tfrac{1}{4}) P_{n-\frac{1}{2}}(t) = 0,$$

it follows that the right-hand member can be written in the form

$$z^{-\frac{3}{2}} \int_\gamma e^{zti} d\left\{(1-t^2)\frac{dP_{n-\frac{1}{2}}(t)}{dt}\right\} + 2z^{-\frac{1}{2}} i \int_\gamma e^{zti} t P_{n-\frac{1}{2}}(t)\, dt$$

$$+ z^{\frac{1}{2}} \int_\gamma e^{zti} (1-t^2) P_{n-\frac{1}{2}}(t)\, dt,$$

or

$$\int_\gamma d\left\{z^{-\frac{3}{2}} e^{zti} (1-t^2) \frac{dP_{n-\frac{1}{2}}(t)}{dt} - iz^{-\frac{1}{2}} (1-t^2)\, e^{zti} P_{n-\frac{1}{2}}(t)\right\}.$$

This result is clearly equally true if the Legendre function of the second kind, $Q_{n-\frac{1}{2}}(t)$, is used instead of $P_{n-\frac{1}{2}}(t)$, since no property of $P_{n-\frac{1}{2}}(t)$ has been required beyond the fact that it satisfies Legendre's equation. We have therefore the result

The integrals
$$z^{\frac{1}{2}}\int_{\gamma} e^{zti}P_{n-\frac{1}{2}}(t)\,dt$$
and
$$z^{\frac{1}{2}}\int_{\gamma} e^{zti}Q_{n-\frac{1}{2}}(t)\,dt \qquad \text{(I.)}$$

satisfy Bessel's differential equation of order n, namely,

$$\frac{d^2u}{dz^2}+\frac{1}{z}\frac{du}{dz}+\left(1-\frac{n^2}{z^2}\right)u=0,$$

provided γ *is any path of integration (it may be either an open path or a closed circuit) which fulfils the condition that the quantity*

$$e^{zti}(1-t^2)\left(\frac{dP_{n-\frac{1}{2}}(t)}{dt}-izP_{n-\frac{1}{2}}(t)\right)$$

[or, in the second case, the corresponding expression in $Q_{n-\frac{1}{2}}(t)$*] resumes its initial value after describing* γ.

Suitable contours γ can be found in plenty to satisfy this condition. Thus it is evident that an arc joining the points $t=-1$ and $t=+1$ is a possible path γ for the integrals (II.), p. 203; that is, the expression

$$z^{\frac{1}{2}}\int_{-1}^{1} e^{zti}P_{n-\frac{1}{2}}(t)\,dt$$

is a solution of Bessel's differential equation of order n.

5. *Determination of some Notable Members of the Family of Integrals found in* § 4.

The theorem of § 4 enables us to express the connexion between the Legendre and Bessel functions in a variety of forms; for, if we write

$$I=z^{\frac{1}{2}}\int_{-1}^{1} e^{zti}P_{n-\frac{1}{2}}(t)\,dt,$$

it has just been shown that the quantity I is a solution of Bessel's differential equation of order n, for all values of n, real or complex. To find its precise relation to the known solutions $J_n(z)$, we need only find the first term of the asymptotic expansion of I for large

positive values of z. This is obtained in the following way:—We have, integrating by parts,

$$I = z^{\frac{1}{2}} \left[\frac{e^{zti}}{zi} P_{n-\frac{1}{2}}(t) \right]_{-1}^{1} - \frac{1}{iz^{\frac{1}{2}}} \int_{-1}^{1} e^{zti} \frac{dP_{n-\frac{1}{2}}(t)}{dt} dt.$$

The first term of the asymptotic expansion of I is therefore

$$\frac{1}{iz^{\frac{1}{2}}} \{ e^{zi} P_{n-\frac{1}{2}}(1) - e^{-zi} P_{n-\frac{1}{2}}(-1) \}$$

or

$$\frac{1}{iz^{\frac{1}{2}}} \{ e^{zi} - e^{-zi + (n-\frac{1}{2})i\pi} \},$$

or

$$2z^{-\frac{1}{2}} e^{(n-\frac{1}{2})\frac{1}{2}i\pi} \cos \left\{ z - (n+\tfrac{1}{2}) \frac{\pi}{2} \right\}.$$

Since the first term of the asymptotic expansion of $J_n(z)$ is

$$\left(\frac{2}{\pi z} \right)^{\frac{1}{2}} \cos \left\{ z - (n+\tfrac{1}{2}) \frac{\pi}{2} \right\},$$

it follows that

$$I = (2\pi)^{\frac{1}{2}} e^{(n-\frac{1}{2})\frac{1}{2}i\pi} J_n(z),$$

so

$$J_n(z) = \left(\frac{z}{2\pi} \right)^{\frac{1}{2}} e^{-(n-\frac{1}{2})\frac{1}{2}i\pi} \int_{-1}^{1} e^{zti} P_{n-\frac{1}{2}}(t)\, dt.$$

Combining the parts of the integral which arise from the ranges -1 to 0 and 0 to 1 respectively, *we have a connexion between the Bessel and Legendre functions expressed by the formula*

$$J_n(z) = \left(\frac{2z}{\pi} \right)^{\frac{1}{2}} \int_{0}^{1} \cos \left\{ zt - (n-\tfrac{1}{2}) \frac{\pi}{2} \right\} P_{n-\frac{1}{2}}(t)\, dt \qquad \text{(II.)}$$

which is valid for all values of z and of n, real or complex. This result was previously (in § 3) shown to be true for the particular case in which n is half an odd integer.

The general theorem of § 4 enables us not only to express $J_n(z)$ in terms of Legendre functions of the first kind, but also in terms of Legendre functions of the second kind—namely, the Q-functions which are defined by the equation

$$Q_n(z) = \frac{1}{2^{n+1}} \int_{-1}^{+1} (1-t^2)^n (z-t)^{-n-1}\, dt.$$

For, in the plane of a complex variable s, let S denote a contour which encircles the part of the real axis between the points $s = 1$ and $s = -1$, and which begins and ends at infinity. The direction in which S tends to infinity will be supposed to be such as to make

the real part of szi negative. Then, if t be a real quantity lying between 1 and -1, we have

$$\int_S (s-t)^{-n-\frac{1}{2}} e^{zsi}\, ds = z^{n-\frac{1}{2}} e^{zti} \int_V v^{-n-\frac{1}{2}}\, e^{iv}\, dv,$$

where $v = z(s-t)$, and where V denotes a contour in the v-plane which encircles the origin and begins and ends at infinity in the v-plane.

But, from Heine's expression for the Gamma-function, we have

$$\frac{1}{\Gamma(n+\frac{1}{2})} = \frac{1}{2\pi} e^{-(n+\frac{1}{2})\frac{1}{2}i\pi} \int_V v^{-n-\frac{1}{2}} e^{iv}\, dv,$$

and therefore

$$\int_S (s-t)^{-n-\frac{1}{2}} e^{zsi}\, ds = \frac{2\pi}{\Gamma(n+\frac{1}{2})} z^{n-\frac{1}{2}} e^{zti+(n+\frac{1}{2})\frac{1}{2}i\pi}.$$

If we substitute this result in Bessel's integral

$$J_n(z) = \frac{z^n}{2^n \pi^{\frac{1}{2}}\, \Gamma(n+\frac{1}{2})} \int_{-1}^{1} e^{zti} (1-t^2)^{n-\frac{1}{2}}\, dt,$$

which is valid when the real part of $(n+\frac{1}{2})$ is positive, we have

$$J_n(z) = \frac{z^{\frac{1}{2}} e^{-(n+\frac{1}{2})\frac{1}{4}i\pi}}{2^{n+1}\pi^{\frac{3}{2}}} \int_S \int_{-1}^{1} (1-t^2)^{n-\frac{1}{2}} (s-t)^{-n-\frac{1}{2}}\, e^{zsi}\, ds\, dt,$$

and, on substituting the expression given above for the Legendre function of the second kind, *we obtain the formula*

$$J_n(z) = \frac{z^{\frac{1}{2}} e^{-(n+\frac{1}{2})\frac{1}{4}i\pi}}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}} \int_S e^{zsi} Q_{n-\frac{1}{2}}(s)\, ds. \qquad \text{(III.)}$$

This integral is of the type found in § 4*; it connects the Bessel function* $J_n(z)$ *with the Legendre functions of the second kind.* It is valid for all values of z, and for all values of n whose real part is greater than $-\frac{1}{2}$.

6. *The Second Solution* $Y_n(z)$ *of Bessel's Differential Equation; and the General Solution.*

We now proceed to consider the second solution of Bessel's differential equation. When the order n of the equation is not an integer, it is known that $J_n(z)$ and $J_{-n}(z)$ are two independent solutions; and therefore the second solution when n is not an integer is

$$J_{-n}(z),$$

or, by (II.), $$\left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \int_0^1 \cos\left(zt + \frac{n\pi}{2} + \frac{\pi}{4}\right) P_{-n-\frac{1}{2}}(t)\, dt.$$

But the Legendre function of the first kind satisfies the relation

$$P_k(z) = P_{-k-1}(z)$$

for all values of k, real or complex; and therefore we have

$$J_{-n}(z) = \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \int_0^1 \cos\left\{zt + \frac{n\pi}{2} + \frac{\pi}{4}\right\} P_{n-\frac{1}{2}}(t)\, dt.$$

Since the general solution of Bessel's equation (n being still supposed to be not an integer) consists of any linear combination of $J_n(z)$ and $J_{-n}(z)$, it follows that this general solution is represented by any linear combination of

$$z^{\frac{1}{2}} \int_0^1 \cos zt\, P_{n-\frac{1}{2}}(t)\, dt$$

and

$$z^{\frac{1}{2}} \int_0^1 \sin zt\, P_{n-\frac{1}{2}}(t)\, dt.$$

Consider now the case in which n is an integer. In this case we have

$$J_{-n}(z) = (-1)^n J_n(z);$$

and therefore the function $J_{-n}(z)$ no longer provides an independent second solution of the differential equation. We take therefore the second solution $Y_n(z)$ introduced by Hankel, which is defined to be

$$\text{limit}_{\epsilon=0} \frac{(-1)^n J_{-(n-\epsilon)}(z) - J_{(n-\epsilon)}(z)}{\epsilon}.$$

Substituting the values found above for $J_{-(n-\epsilon)}(z)$ and $J_{(n-\epsilon)}(z)$, we have

$$Y_n(z) = \text{limit}_{\epsilon=0} \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \int_0^1 \frac{1}{\epsilon} \left\{(-1)^n \cos\left(zt + \frac{\pi}{4} + \frac{n\pi}{2} - \frac{\epsilon\pi}{2}\right) - \cos\left(zt + \frac{\pi}{4} - \frac{n\pi}{2} + \frac{\epsilon\pi}{2}\right)\right\} P_{n-\frac{1}{2}}(t)\, dt$$

$$= \text{limit}_{\epsilon=0} \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \int_0^1 \frac{2\sin\frac{\epsilon\pi}{2}}{\epsilon} \sin\left(zt + \frac{\pi}{4} - \frac{n\pi}{2}\right) P_{n-\frac{1}{2}}(t)\, dt;$$

and therefore *we have the formula*

$$Y_n(z) = (2\pi z)^{\frac{1}{2}} \int_0^1 \sin\left\{zt - (n - \tfrac{1}{2})\frac{\pi}{2}\right\} P_{n-\frac{1}{2}}(t)\, dt, \qquad \text{(IV.)}$$

which expresses the Bessel function of the second kind in terms of the Legendre function.

Since the most general solution of Bessel's differential equation of order n consists of a linear combination of $J_n(z)$ and $Y_n(z)$, it follows that this can be expressed as a linear combination of

$$z^{\frac{1}{2}} \int_0^1 \cos zt \, P_{n-\frac{1}{2}}(t) \, dt$$

and

$$z^{\frac{1}{2}} \int_0^1 \sin zt \, P_{n-\frac{1}{2}}(t) \, dt.$$

But this result, which is thus obtained for the case in which n is integral, is the same as the result already obtained for the case in which n is not integral; and therefore we see that *in all cases the general solution of Bessel's equation is a linear combination of*

$$\left.\begin{array}{l} z^{\frac{1}{2}} \int_0^1 \cos zt \, P_{n-\frac{1}{2}}(t) \, dt \\ \text{and} \quad z^{\frac{1}{2}} \int_0^1 \sin zt \, P_{n-\frac{1}{2}}(t) \, dt \end{array}\right\}. \qquad \text{(V.)}$$

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*On Groups which are Linear and Homogeneous in both Variables and Parameters.** *By* W. Burnside. Received and Read November 13th, 1902.

In this paper I propose to discuss the nature of the characteristic determinant, first of any simply transitive linear homogeneous group, and secondly of any transitive linear homogeneous group. The result of this discussion, leading to a quite general form for the

* Prof. L. E. Dickson has called my attention to two errors in my paper "On the Continuous Group defined by any given Group of Finite Order" (*Proc. Lond. Math. Soc.*, Vol. xxix., pp. 546–565). These occur in §§ 5, 6, dealing with particular cases of the characteristic determinant of a simply transitive linear homogeneous group. The induction in § 5 is faulty; and, as Prof. Dickson has shown, it is *not* the case that, if the characteristic determinant is the power of a single linear factor, the group is necessarily Abelian. The error in § 6 occurs in the third line from the bottom of p. 553. The correct inference from the previous reasoning, there, should obviously be not $\nu = \mu$, but that ν is a multiple of μ. The results of §§ 5, 6 are used in § 10 in establishing one of the main results of the paper. In order that this should rest on a sound basis it is essential that these errors should be corrected.

determinant in each case, is given in the first two sections of the paper. A short third section is added to apply the results so obtained to giving a materially simpler and shorter proof of the main propositions contained in the second section of my paper "On the Continuous Group defined by any given Group of Finite Order" (*Proc. Lond. Math. Soc.*, Vol. XXIX., pp. 546–565).

The question considered in the first section of this paper has been dealt with at considerable length by Herr Molien in his memoir "Ueber Systeme höherer complexer Zahlen" (*Math. Ann.*, Vol. XLI., pp. 83–156, 1892); and most of the results arrived at in that section are contained in, or may be deduced from, those given by Herr Molien. I do not think, however, that this is the case with all of them. For instance, Herr Molien nowhere shows that the characteristic determinants of two reciprocal simply transitive groups are identical. It is not, in fact, very easy to find exactly what is and what is not contained in Herr Molien's memoir; for he treats the different parts of what is one subject from the point of view first of complex numbers, then of linear groups, and lastly of matrices. The methods used here are, in any case, quite distinct from those of Herr Molien: they depend solely (i) on the existence of the reciprocal group, and (ii) on the fact that the only self-conjugate subgroup of the general linear group is the cyclical group multiplying each variable by the same factor.

I.

By a linear homogeneous group in the following pages is meant a group whose finite equations are linear and homogeneous both in the parameters and in the variables. When, of the parameters entering in the equations, the number that are linearly independent is equal to the number of variables, the group is called simply transitive. Corresponding to any simply transitive group there is always a second linear homogeneous group (identical with the first, if the first is Abelian, but not otherwise) simply transitive in the same variables, every one of whose operations is permutable with every operation of the first group. This is called the reciprocal group.

Let G and G' be reciprocal simply transitive groups in n variables. An arbitrarily chosen substitution of G' may be reduced to the canonical form

$$\begin{aligned} &y_1' = \lambda_1 y_1, \quad y_2' = a_2 y_1 + \lambda_1 y_2, \quad \ldots, \quad y_p' = a_p y_{p-1} + \lambda_1 y_p, \\ &z_1' = \lambda_2 z_1, \quad z_2' = b_2 z_1 + \lambda_2 z_2, \quad \ldots, \quad z_q' = b_q z_{q-1} + \lambda_2 z_q, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots, \end{aligned}$$

where $\lambda_1, \lambda_2, \ldots$, are the different roots of the characteristic equation of the substitution and $p, q, \ldots$, their multiplicities. Every substitution which is permutable with this substitution, and therefore every substitution of G, must transform the y's, z's, ..., separately among themselves. Hence G is certainly reducible. The linear transformation of the y's effected by G will constitute a group which, since G is transitive, is certainly transitive, but is not necessarily simply transitive. This may or may not be reducible. If it is so, similar considerations will apply to the transitive group of transformations on the $p' (< p)$ linear functions of the y's which are transformed among themselves. Continuing thus, we must eventually arrive at a set of independent linear functions

$$x_{11}, \ x_{12}, \ \ldots, \ x_{1r}, \quad (r \geq 1)$$

of the original variables, which are transformed among themselves by every operation of G such that it is impossible to form $r' (< r)$ linear functions of them which are transformed among themselves. In other words, the group of transformations on the r functions is irreducible.

Suppose now that some substitution S' of the reciprocal group G

changes $\quad x_{11}, \ x_{12}, \ \ldots, \ x_{1r},$

into $\quad x_{21}, \ x_{22}, \ \ldots, \ x_{2r}.$

Since the first set are linearly independent among themselves, so also must the second set be; but the $2r$ symbols may possibly be connected by one or more linear equations. If the linear equation

$$\Sigma b_i x_{2i} = \Sigma a_i x_{1i}$$

holds, S' changes the linear function $\Sigma b_i x_{1i}$ of the first set into another linear function $\Sigma a_i x_{1i}$ of the first set. Now, S being any operation of G, $\quad S' S S'^{-1} = S.$

Hence, if S changes $\Sigma a_i x_{1i}$ into $\Sigma c_i x_{1i}$, and $\Sigma b_i x_{1i}$ into $\Sigma d_i x_{1i}$, then S' changes $\Sigma d_i x_{1i}$ into $\Sigma c_i x_{1i}$. Now G contains operations which severally change $\Sigma b_i x_{1i}$ into r linearly independent functions of the first set. Hence, since S' changes each of these into another function of the first set, S' must transform the first set among themselves. Unless S' replaces each of the symbols $x_{11}, x_{12}, \ldots, x_{1r}$ by the same multiple of itself, G, so far as it affects this set, would be reducible, which is contrary to the supposition made. Hence either

$$x_{2i} = e x_{1i}, \quad (i = 1, 2, \ldots, r),$$

where e is a constant multiplier, or the second set is independent of the first. Further, since S' is permutable with every operation of G. every operation of G effects exactly the same transformation on

$$x_{21},\ x_{22},\ \ldots,\ x_{2r},$$

as it does on

$$x_{11},\ x_{12},\ \ldots,\ x_{1r}.$$

If some other transformation S'' of G' changes the first set into

$$x_{31},\ x_{32},\ \ldots,\ x_{3r},$$

precisely similar reasoning shows that either these are linearly independent of the first two sets, or else

$$x_{3i} = ex_{1i} + fx_{2i}, \quad (i = 1, 2, \ldots, r),$$

where e and f are constant multipliers. Moreover, the third set are transformed among themselves by G in exactly the same way as the first and second sets. By considering further substitutions of G' this process can be continued till no new independent set can be formed. Suppose that just s sets thus arise

$$\begin{array}{l} x_{11},\ x_{12},\ \ldots,\ x_{1r}, \\ x_{21},\ x_{22},\ \ldots,\ x_{2r}, \\ \ldots\quad\ldots\quad\ldots \\ x_{s1},\ x_{s2},\ \ldots,\ x_{sr}. \end{array}$$

The symbols in each line are transformed among themselves by each substitution of G in exactly the same way, and the most general substitution of G' changes x_{1i} into

$$e_1x_{1i} + e_2x_{2i} + \ldots + e_sx_{si}, \quad (i = 1, 2, \ldots, r).$$

Further, the substitutions S'^{-1} and $S'^{-1}S''$ of G' change

$$x_{21},\ x_{22},\ \ldots,\ x_{2r},$$

into

$$x_{11},\ x_{12},\ \ldots,\ x_{1r},$$

and

$$x_{31},\ x_{32},\ \ldots,\ x_{3r},$$

respectively. Hence the most general substitution of G' changes x_{2i} into

$$f_1x_{1i} + f_2x_{2i} + \ldots + f_sx_{si}, \quad (i = 1, 2, \ldots, r).$$

It follows thus that the symbols in each column of the preceding scheme are transformed among themselves by every operation of G', and that any operation of G' gives the same transformation for the set of symbols in each column. The groups of transformations

effected by G and G' on the sr symbols in the scheme must each be transitive, since G and G' are transitive; and, since G and G' are reciprocal, the two groups must each be simply transitive. Hence, if ρ is the number of independent parameters in the group of transformations of

$$x_{11},\ x_{12},\ \ldots,\ x_{1r},$$

and σ the number in that of

$$x_{11},\ x_{21},\ \ldots,\ x_{s1},$$

$$\rho = \sigma = rs.$$

But, in any case, $\rho \leqslant r^2,\ \sigma \leqslant s^2.$

Hence $r = s$

and $\rho = \sigma = r^2.$

Whatever the simply transitive group G in n variables may be, it must therefore be possible to choose r^2 $(r \geqslant 1)$ independent linear functions of the variables and an equal number of independent linear functions of the parameters, so that the first r^2 of the finite equations of the group take the form

$$x_i' = \sum_{k=1}^{k=r} e_{jk} x_{ik} \quad (i, j = 1, 2, \ldots, r);$$

while those of the reciprocal group are

$$x_{ij}' = \sum_{k=1}^{k=r} e_{ik}' x_k \quad (i, j = 1, 2, \ldots, r).$$

So far as it affects this set of variables G is simply isomorphic with the general linear homogeneous group in r variables.

2. If, when G and G' have been transformed so that the first r^2 equations of each have the form arrived at in the preceding section, $x_{11}, x_{12}, \ldots, x_{rr}, x_{11}', x_{12}', \ldots, x_{rr}'$ are all made zero, the first r^2 equations of each are satisfied identically, and the remaining $n - r^2$ equations of each are the equations of linear homogeneous groups in $n - r^2$ variables. These groups must be transitive, or G and G' themselves would be intransitive; and, since G and G' are reciprocal, they must be simply transitive. Hence the process of the preceding section may be applied to them. It is therefore possible to find r_1^2 $(r_1 \geqslant 1)$ linear functions of the original variables, y_{ij} $(i, j = 1, 2, \ldots, r_1)$, linearly independent of each other and of the preceding r^2 functions

such that when the x_{ij}'s are made zero the y's undergo the substitution

$$y'_{ij} = \sum_{k=1}^{k=r_1} f_{jk} y_{ik} \quad (i, j = 1, 2, \ldots, r_1),$$

corresponding to the general substitution of G, and

$$y'_{ij} = \sum_{k=1}^{k=r_1} f'_{ik} y_{kj} \quad (i, j = 1, 2, \ldots, r_1),$$

corresponding to that of G'. The r_1^2 symbols f_{ij} represent linearly independent functions of the original parameters; but whether the f_{ij}'s are independent of the e_{ij}'s requires further discussion. The general equations of the group, so far as these symbols are concerned, will contain for each y'_{ij} a linear function $f_{ij}(x)$ of the x_{ij}'s with linear functions of the original parameters as coefficients.

The process already used may be repeated for the simply transitive group that arises on making all the x_{ij}'s and y_{ij}'s zero, leading to r_2^2 new variables independent of each other and of the x's and the y's, whose transformations are given by

$$z'_{ij} = g_{ij}(x) + G_{ij}(y) + \sum_{k=1}^{k=r_2} g_{jk} z_{ik} \quad (i, j = 1, 2, \ldots, r_2),$$

where $g_{ij}(x)$ and $G_{ij}(y)$ represent linear functions of the x_{ij}'s and y_{ij}'s with coefficients which are linear functions of the original parameters; and the process may be continued till the whole of the original variables are accounted for.

3. To discuss the question of the independence of the parameters $e_{ij}, f_{ij}, \ldots$, consider first the two sets of equations

$$x'_{1j} = \sum_{k=1}^{k=r} e_{jk} x_{1k} \quad (j = 1, 2, \ldots, r),$$

$$y'_{1j} = \sum_{k=1}^{k=r_1} f_{jk} y_{1k} \quad (j = 1, 2, \ldots, r_1).$$

These constitute a group. Suppose first that r and r_1 are not equal, and let r_1 be greater than r. The totality of the substitutions which replace each x by the same multiple of itself form a self-conjugate sub-group. For this sub-group

$$e_{11} = e_{22} = \ldots = e_{rr},$$

$$e_{rs} = 0 \quad (r \neq s).$$

The r_1^2 f's are linearly independent among themselves, and hence, even when there are as many linear relations as possible between them and the e's, at least $r_1^2-r^2+1$ of them remain arbitrary when the e's are subject to the above r^2-1 relations. The group in the y's would therefore contain a self-conjugate sub-group of order $r_1^2-r^2+1$ at least. Now the general linear group has no self-conjugate sub-group except that which replaces every symbol by the same multiple of itself. Hence, if r and r_1 are unequal, the e's and the f's are linearly independent. If $r_1 = r$, and if the e's and the f's are not independent, suppose that there are s linear relations connecting them. Then when the e's satisfy the preceding relations at least r^2-s+1 of the f's remain arbitrary. Hence the condition as regards the self-conjugate sub-group can only be satisfied if $s = r^2$, *i.e.*, if the f's are linearly independent functions of the e's. But when this is the case to every operation of the group in the x's will correspond a definite operation of the group in the y's. Hence the group of the y's can be so transformed that each f is equal to the corresponding e.

Suppose next that $r = r_1 = r_2$. If each f is equal to the corresponding e, then from the previous case either the g's must be independent of the e's or the z's can be so chosen that each g is equal to the corresponding e. If the e's and the f's are distinct, the possibility arises that the g's may be functions of the e's and f's together. If this is so, consider the sub-group given by

$$e_{11} = e_{22} = \ldots = e_{rr}, \quad f_{11} = f_{22} = \ldots = f_{rr},$$

$$e_{rs} = f_{rs} = 0 \quad (r \neq s).$$

Of the g's then just two would be independent, and the group of the z's would have a self-conjugate sub-group of order 2, which is not the case. Hence the g's must be independent, or each g must be equal to the corresponding e, or each g must be equal to the corresponding f. The same method may be used to deal with any number of equalities among the r's, each fresh set of parameters being shown to be either independent of the preceding ones or identical with one of the preceding sets.

4. The linear functions denoted in the second section by $f_{ij}(x)$, $g_{ij}(x)$, $G_{ij}(y)$, ..., remain now to be discussed. For this purpose suppose again that all the variables except those of two sets, say the y's and the z's (these are taken to preserve the notation, and not as

representing necessarily the second and the third sets), are made zero. Suppose also that the f's and the g's are independent. The equations of G for the y's and z's then become a set of $r_1^2+r_2^2$ equations constituting a linear group, as also do those of G'. Since the f's and the g's give $r_1^2+r_2^2$ parameters, these groups are transitive, and, since G and G' are reciprocal, they must be simply transitive.

The equations

$$y'_{ij} = \sum_1^{r_1} f_{jk} y_{ik} \quad (i, j = 1, 2, \ldots, r),$$

$$z'_{ij} = G_{ij}(y) + \sum_1^{r_2} g_{jk} z_{ik} \quad (i, j = 1, 2, \ldots, r_2),$$

in which the f's and the g's are linearly independent, constituting a simply transitive group, the coefficients of the y's in $G_{ij}(y)$ must be linear functions of the f's and g's.

Now the values of the parameters

$$f_{11} = f_{22} = \ldots = f_{r_1 r_1} = \alpha, \quad f_{ij} = 0 \quad (i \neq j),$$

$$g_{11} = g_{22} = \ldots = g_{r_2 r_2} = \beta, \quad g_{ij} = 0,$$

constitute a sub-group of order 2, all of whose operations are self-conjugate. For these values the above equations become

$$y'_{ij} = \alpha y_{ij},$$

$$z'_{ij} = \alpha g_{ij}(y) + \beta h_{ij}(y) + \beta z_{ij},$$

where $g_{ij}(y)$ and $h_{ij}(y)$ represent certain linear functions of the y's; and, unless all the functions $g_{ij}(y)$ and $h_{ij}(y)$ are zero, these equations have not the group property. This is verified at once by combining the substitutions with α, β and α', β' as parameters. Hence the self-conjugate operations are

$$y'_{ij} = \alpha y_{ij}, \quad z'_{ij} = \beta z_{ij}.$$

Every operation of the group, being permutable with these operations, must interchange the y's and z's separately among themselves. Hence all the functions denoted by $G_{ij}(y)$ are identically zero. From this it follows that no two sets of the variables which occur in connection with distinct sets of parameters can appear in each other's equations.

To carry the discussion further than this would involve the discussion and discrimination between particular cases, which it is not

proposed here to enter on. If the set of parameters

$$\begin{matrix} e_{11}, & e_{12}, & \ldots, & e_{1r}, \\ e_{21}, & e_{22}, & \ldots, & e_{2r}, \\ \ldots & \ldots & \ldots & \\ e_{r1}, & e_{r2}, & \ldots, & e_{rr} \end{matrix}$$

occurs in connection with kr sets, of r variables each,

$$x_{ijt}, \quad (i, j = 1, 2, \ldots, r\,;\ t = 1, 2, \ldots, k),$$

the finite equations corresponding to these variables can, from the preceding discussion, be written in the form

$$x'_{ijt} = f_{ij}^{1,t}(x_1) + f_{ij}^{2,t}(x_2) + \ldots + f_{ij}^{t-1,t}(x_{t-1}) + \sum_{k=1}^{k=r} e_{jk} x_{ikt}$$

$$(i, j = 1, 2, \ldots, r\,;\ \ldots\,;\ t = 1, 2, \ldots, k),$$

where $f_{ij}^{s,t}(x_s)$ represents a linear function of the symbols

$$x_{ijs} \quad (i, j = 1, 2, \ldots, r)$$

with coefficients depending on the original parameters.

Corresponding to each distinct set of parameters, *i.e.*, to each irreducible factor of the determinant of the original equations, there will be a similar system of equations; and the nature of the reduction of the group is then determined, so far as it depends on the distinct factors of the determinant. How far the above equations, which correspond to a particular irreducible factor, may be further simplified or modified will depend on the value of k in each case, and will lead to a discussion similar to that I have undertaken, for the case $r = 1$ and the group Abelian, in a paper "On Linear Homogeneous Groups whose Operations are Permutable" (*Proc. Lond. Math. Soc.*, Vol. XXIX., p. 325).

5. The results obtained so far may be conveniently summarized here as follows:—

(i) The determinant D_ϵ of the finite equations of a simply transitive linear homogeneous group in n variables with respect to the

variables is always of the form

$$D_e = \prod_{t=1}^{t=s} \begin{vmatrix} e_{11t}, & e_{12t}, & \ldots, & e_{1r_t t} \\ e_{21t}, & e_{22t}, & \ldots, & e_{2r_t t} \\ \ldots & & \ldots & \ldots \\ e_{r_t 1t}, & e_{r_t 2t}, & \ldots, & e_{r_t r_t t} \end{vmatrix}^{k_t r_t} = \prod_{t=1}^{t=s} E_t^{k_t r_t},$$

where $$n = \sum_{t=1}^{t=s} k_t r_t^2,$$

and the r, $$= \sum_{t=1}^{t=s} r_t^2,$$

symbols e_{ijk} are linear and independent functions of the n parameters entering in the equations of the group.

(ii) The variables may be chosen so that both G and the reciprocal group G' interchange them among themselves in the same s sets, the number in the t-th set being $k_t r_t^2$. The finite equations for G as regards the t-th set are of the form given near the end of § 4; and their determinant with respect to the variables is $E_t^{k_t r_t}$, the power of a single irreducible factor.

Both G and G' interchange among themselves the symbols of the t-th set, while the determinant of the equations of G for this set is the power of a single irreducible function $E_t^{k_t r_t}$. Now the determinants of any two reciprocal simply transitive groups differ only, if at all, in the power to which the different irreducible factors enter into them. Further, the determinant of G with respect to the parameters is obtained by writing variables for parameters in the determinant of G' with respect to the variables. Hence—

(iii) D_e and D'_e, the determinants of the finite equations of G and G' with respect to the variables, are identical with each other; and D_x, the determinant of G with respect to the variables, is obtained from D_e by replacing each symbol e_{ijk} by the x_{ijk} defined in § 4.

To these three general results the following special ones are added on account of their importance :—

The determinant D_e is expressed as a function of ν independent linear functions of the original n parameters. If no linear relation connects the n differential coefficients of D_e with respect to the

original parameters, ν must be equal to n and then each of the k's must be unity. Hence—

(iv) If D_e cannot be expressed in terms of less than n linear functions of the original parameters, every irreducible factor of D_e enters into D_e to a power equal to its degree; and G is then the direct product of a number of groups each of which is simply isomorphic with a general linear group.

(v) If D_e cannot be expressed in terms of less than n linear functions of the parameters, and if D_e is a power of one irreducible factor, then n must be a square, and G is simply isomorphic with the parameter group of the general linear group in $\sqrt{n}$ variables.

II.

The preceding results may be applied to deal with the determinant of any transitive linear group, homogeneous both in the variables and in the parameters, and with the reduction of such a group.

Let G be such a group in n variables and m ($n^2 > m > n$) parameters. The parameter group of G is a simply transitive linear homogeneous group in m variables. Denote its determinant with respect to the variables by D_e and that of G by Δ_e. Since $\Delta_e = 0$ determines the systems of values of the parameters for which the finite equations of G cannot be solved, while $D_e = 0$ gives all inadmissible systems of values of the parameters, the irreducible factors of Δ_e must be some or all of the irreducible factors of D_e. The form of Δ_e is thus determined from the previous sections as

$$\Delta_e = \Pi \begin{vmatrix} e_{11}, & e_{12}, & \ldots, & e_{1r} \\ e_{21}, & e_{22}, & \ldots, & e_{2r} \\ \ldots & \ldots & \ldots & \\ e_{r1}, & e_{r2}, & \ldots, & e_{rr} \end{vmatrix}^k,$$

where all the e's in the different factors of the determinant are linearly independent linear functions of the original parameters. Hence Δ_e must have more than one irreducible factor; for, being of the n-th degree, it would, if irreducible, be a determinant of n rows and columns all of whose elements were independent; and this is impossible, m being less than n^2. Choose now new parameters, so that the set $e_{11}, \ldots, e_{rr}$ occur among them. When we make

$$e_{11} = e_{12} = \ldots = e_{1r} = 0,$$

the finite equations of the group cannot be solved for the original variables. Hence these values of the parameters must either make one at least of the original variables disappear from the equations, or must involve at least one linear relation among the transformed variables. If together with the original group we consider the "transposed" group that arises on interchanging the rows and columns of coefficients in the finite equations, when of these two possibilities one holds for the original group, the other must hold for the "transposed." It is therefore sufficient to consider one of them. Suppose then that the assumed conditions for the e's lead to the relation

$$\Sigma a_i x_i' = 0$$

among the transformed variables. If new variables are taken, of which

$$\xi_1 = \Sigma a_i x_i$$

is the first, then ξ_1' vanishes when

$$e_{11} = e_{12} = \ldots = e_{1r} = 0,$$

whatever the other parameters may be. Moreover, since to the identical operation of the group there correspond the values

$$e_{11} = e_{22} = \ldots = e_{rr} = 1, \quad e_{rs} = 0 \quad (r \neq s),$$

of the e's, it follows that when

$$e_{11} = 1, \quad e_{12} = \ldots = e_{1r} = 0,$$

ξ_1' must be equal to ξ_1. Hence we may take

$$\xi_1' = e_{11}\xi_1 + e_{12}\xi_{12} + \ldots + e_{1r}\xi_r,$$

$\xi_2, \xi_3, \ldots, \xi_r$ being linearly independent functions of the original variables.

If now the transformations corresponding to accented and unaccented values of the e's are carried out in succession, the resulting substitution is that for which

$$\begin{aligned}
e_{11}'' &= e_{11}e_{11}' + e_{12}e_{21}' + \ldots + e_{1r}e_{r1}', \\
e_{12}'' &= e_{11}e_{12}' + e_{12}e_{22}' + \ldots + e_{1r}e_{r2}', \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
e_{1r}'' &= e_{11}e_{1r}' + e_{12}e_{2r}' + \ldots + e_{1r}e_{rr}'
\end{aligned}$$

are the first r parameters; this being given directly by the parameter group. But

$$e_{11}''\xi_1 + e_{12}''\xi_2 + \ldots + e_{1r}''\xi_r \equiv e_{11}\xi_1' + e_{12}\xi_2' + \ldots + e_{1r}\xi_r',$$

where $\xi_1', \xi_2', \ldots, \xi_r'$ are the transformed values of $\xi_1, \xi_2, \ldots, \xi_r$ corresponding to accented values of the e's. It follows that

$$\xi_1' = e_{11}'\xi_1 + e_{12}'\xi_2 + \ldots + e_{1r}'\xi_r,$$

$$\xi_2' = e_{12}'\xi_1 + e_{r1}'\xi_2 + \ldots + e_{2r}'\xi_r,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\xi_r' = e_{r1}'\xi_1 + e_{r2}'\xi_2 + \ldots + e_{rr}'\xi_r.$$

The first r variables are therefore transformed among themselves by every operation of G, and the group so far as it affects these variables is the general linear group in r symbols.

If, in the equations of G, $\xi_1, \xi_2, \ldots, \xi_r, \xi_1', \ldots, \xi_r'$ are made zero, the remaining $n-r$ equations give a transitive group in $n-r$ variables, and this may be dealt with as G has been. Moreover, as in the case of the simply transitive group, this process may be continued till all the variables are accounted for. Further, exactly as in § 4 of the first section, it may be shown that, if

$$\xi_1, \ \xi_2, \ \ldots, \ \xi_r,$$

$$\eta_1, \ \eta_2, \ \ldots, \ \eta_s$$

are two sets of variables thus arising in connection with two distinct sets of parameters (or, in other words, in connection with two distinct irreducible factors of Δ_e), neither set can occur in the equations giving the transformed values of the other. The results thus obtained may be summarized in the following form:—

(i) The determinant with respect to the variables, Δ_e, of a transitive linear homogeneous group in n variables, other than the general linear group, is never an irreducible function of the parameters. It can always be expressed in the form

$$\Delta_e = \prod_{t=1}^{t=s} \begin{vmatrix} e_{11t}, & e_{12t}, & \ldots, & e_{1r_t t} \\ e_{21t}, & e_{22t}, & \ldots, & e_{2r_t t} \\ \ldots & \ldots & & \ldots \\ e_{r_t 1t}, & e_{r_t 2t}, & \ldots, & e_{r_t r_t t} \end{vmatrix}^{k_t},$$

where the symbols e_{ijk} represent independent linear functions of the original parameters.

(ii) The variables may be so chosen that they are transformed

among themselves in s distinct sets. The number in the t-th set is $k_t r_t$; and, if they are denoted by

$$x_{ijt} \quad (i = 1, 2, \ldots, r_t;\ j = 1, 2, \ldots, k_t),$$

the general operation of G, so far as it affects the symbols of this set, is

$$x'_{ijt} = f_{j-1} + \sum_{k=1}^{k=r_t} e_{ikt} x_{kjt}$$

$$(i = 1, 2, \ldots, r_t; \quad = 1, 2, \ldots, k_t),$$

where f_{j-1} represents a linear function of the symbols

$$x_{ipt} \quad (i = 1, 2, \ldots, r_t;\ p = 1, 2, \ldots, j-1),$$

with coefficients linear in the parameters.

(iii) An irreducible homogeneous function of degree m in μ variables, between whose first different coefficients with respect to the variables no linear relation with constant coefficients exists cannot be a relative invariant for a transitive group, linear both in the variables and in the parameters, unless $\mu = m^2$; and the function must be of the form

$$\begin{vmatrix} x_{11}, & x_{12}, & \ldots, & x_{1m} \\ x_{21}, & x_{22}, & \ldots, & x_{2m} \\ \ldots & \ldots & & \ldots \\ x_{m1}, & x_{m2}, & \ldots, & x_{mm} \end{vmatrix},$$

where the elements of the determinant are linearly independent linear functions of the variables which are transformed by the group.

III.

I wish to draw attention to the great simplification which the results of the first section of the present paper permit in proving the main propositions of the second section of my paper "On the Continuous Group defined by any given Group of Finite Order" (*Proc. Lond. Math. Soc.*, Vol. XXIX., p. 526) already referred to.

If D_y is the determinant with respect to the variables of the group G,

$$x'_s = \sum_{i=1}^{i=n} y_{si-1} x_i \quad (i = 1, 2, \ldots, n),$$

there considered, and if

$$y_1,\ y_2,\ \ldots,\ y_n,$$

$$z_1,\ y_2,\ \ldots,\ z_n,$$

are the values of the parameters corresponding to an arbitrary operation T of G and to its inverse T^{-1}, it may be shown, almost exactly as on pp. 556, 557, that

$$z_{i-1} = \frac{1}{nD_y}\frac{\partial D_y}{\partial y_i}.$$

Now, if the y's have arbitrary values, so also have the z's; and therefore the n differential coefficients $\dfrac{\partial D_y}{\partial y_i}$ are linearly independent. Hence, by (iv) of § 5, every irreducible factor of D_y enters into D_y to a power equal to its degree; and, if there are s such irreducible factors, G is the direct product of s general linear groups in $m_1, m_2, \ldots, m_s$ variables, where

$$m_1^2 + m_2^2 + \ldots + m_s^2 = n.$$

This being the case, the self-conjugate operations of G constitute a sub-group of order s. But it had previously been proved that this sub-group had r for its order. Hence $s = r$, and the variables may be chosen so that every operation of both G and G' interchange them in the same r sets, the number in each set being a square. These are the main results on which the remainder of the discussion turns.

The Propagation of Light in a Uniaxal Crystal. By A. W. Conway. Received November 5th, 1902. Read November 13th, 1902.

Introduction.

The following paper is an adaptation of the analysis used by Prof. A. E. H. Love ("Integration of the Equations of Propagation of Electric Waves," *Phil. Trans.*, Series A, 1901) to the case of a uniaxal crystalline medium, together with some deductions from the

general equations and applications to physical optics. §§ 1, 2, and 3 are occupied in integrating the equations of motion in terms of certain boundary conditions. §§ 4 and 5 treat of the direction of the vibration and the flow of energy. §§ 6, 7, and 8 are applications of the general integral obtained in § 3 to Huygens' principle for a crystalline medium with a plane face, and to the passage of parallel and divergent beams of light through a thin crystalline plate. A note is added on Huygens' principle, which was suggested by some of the results in § 6.

1. *Particular Solutions.*

If the medium be considered to be magnetically isotropic, but to possess an axis of electric symmetry, and if the magnetic permeability be taken to be unity, the differential equations of wave propagation on the basis of the electromagnetic theory of light are

$$\left.\begin{aligned} a^{-2}\dot{X} &= \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \\ a^{-2}\dot{Y} &= \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} \\ c^{-2}\dot{Z} &= \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \\ -\dot{\alpha} &= \frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \\ -\dot{\beta} &= \frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} \\ -\dot{\gamma} &= \frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \end{aligned}\right\} \quad (1)$$

where (X, Y, Z) and (α, β, γ) denote the electric and magnetic forces respectively in electromagnetic units.

From these equations we can deduce

$$a^{-2}\frac{\partial X}{\partial x} + a^{-2}\frac{\partial Y}{\partial y} + c^{-2}\frac{\partial Z}{\partial z} = 0,$$

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} = 0,$$

wherever these functions have no singularities. For light waves it

is known that the constants a and c vary with the wave-length. In what follows they will be taken to be constants, and the results obtained will then be applicable to the case of monochromatic light.

If we put $a = c$ in equations (1), we get the case of an isotropic medium. In this case there are two classes of solutions which are infinite only at the origin. The simplest of these solutions, from which all the others can be derived, are (a) the case of a Hertzian oscillator or vibrating electric doublet, and (b) the case of a vibrating magnetic doublet.

If $F(t)$ and $f(t)$ are arbitrary functions which are uniform, finite, and continuous for all real values of t, and if we denote by χ the function

$$\int_0^{\sqrt{x^2+y^2}} \frac{a\,d\lambda}{\lambda} \{F(a^{-1}\sqrt{\lambda^2+z^2}+t) - F(\sqrt{c^{-2}\lambda^2+a^{-2}z^2}+t) + f(a^{-1}\sqrt{\lambda^2+z^2}-t) - f(\sqrt{c^{-2}\lambda^2+a^{-2}z^2}-t)\},$$

we can show that χ has no infinity for real values of $\sqrt{x^2+y^2}$ and z, and that near the z-axis it can be expanded in positive integral powers of x^2+y^2, and hence that expressions of the form $\dfrac{\partial^{l+m+n}}{\partial x^l \partial y^m \partial z^n}\chi$ have no singularity on the z-axis. We have also the relations

$$\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2} = \frac{\partial}{\partial t}\left\{\frac{F(a^{-1}\sqrt{x^2+y^2+z^2}+t)}{\sqrt{x^2+y^2+z^2}} - \frac{a}{c^2}\frac{F[\sqrt{c^{-2}(x^2+y^2)+a^{-2}z^2}+t]}{\sqrt{c^{-2}(x^2+y^2)+a^{-2}z^2}}\right.$$

$$\left. - \frac{f(a^{-1}\sqrt{x^2+y^2+z^2}-t)}{\sqrt{x^2+y^2+z^2}} + \frac{a}{c^2}\frac{F[\sqrt{c^{-2}(x^2+y^2)+a^{-2}z^2}-t]}{\sqrt{c^{-2}(x^2+y^2)+a^{-2}z^2}}\right\} \quad (2)$$

$$= \phi - \psi,$$

where ϕ denotes the part involving $\sqrt{x^2+y^2+z^2}$ and ψ denotes the part involving $\sqrt{c^{-2}(x^2+y^2)+a^{-2}z^2}$,

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = a^{-2}\frac{\partial^2\phi}{\partial t^2}, \quad (3)$$

$$a^{-2}\frac{\partial^2\psi}{\partial x^2}+a^{-2}\frac{\partial^2\psi}{\partial y^2}+c^{-2}\frac{\partial^2\psi}{\partial z^2} = a^{-2}c^{-2}\frac{\partial^2\psi}{\partial t^2}, \quad (4)$$

$$\frac{\partial^2\chi}{\partial z^2} = a^{-2}\frac{\partial^2\chi}{\partial t^2} - \phi + \frac{c^2}{a^2}\psi. \quad (5)$$

Using the above notation, let us consider the following values for the components of magnetic force :—

$$\left.\begin{aligned} \alpha_1 &= a^{-2}\frac{\partial^3\dot{\chi}}{\partial x\partial y\partial z} \\ \beta_1 &= a^{-2}\frac{\partial\dot{\phi}}{\partial z} - a^{-2}\frac{\partial^2\dot{\chi}}{\partial x^2\partial z} \\ \gamma_1 &= -a^{-2}\frac{\partial\dot{\phi}}{\partial y} \end{aligned}\right\}. \tag{6}$$

Using the equations (1), we find for the electric force

$$\left.\begin{aligned} X_1 &= -\left(\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) + \frac{\partial^4\chi}{\partial x^2\partial z^2} \\ Y_1 &= \frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^4\chi}{\partial x\partial y\partial z^2} \\ Z_1 &= \frac{c^2}{a^2}\frac{\partial^2\phi}{\partial x\partial z} - \frac{c^2}{a^2}\left(\frac{\partial^4\dot{\chi}}{\partial x^3\partial z} + \frac{\partial^4\chi}{\partial x\partial y^2\partial z}\right) \end{aligned}\right\}, \tag{7}$$

which take the simpler forms, on making use of equations (2), (3), and (5),

$$\left.\begin{aligned} X_1 &= -a^{-2}\ddot{\phi} + \frac{c^2}{a^2}\frac{\partial^2\psi}{\partial x^2} + a^{-2}\frac{\partial^2\ddot{\chi}}{\partial x^2} \\ Y_1 &= \frac{c^2}{a^2}\frac{\partial^2\psi}{\partial x\partial y} + a^{-2}\frac{\partial^2\ddot{\chi}}{\partial x\partial y} \\ Z_1 &= \frac{c^2}{a^2}\frac{\partial^2\psi}{\partial x\partial z} \end{aligned}\right\}. \tag{8}$$

On taking the curl of (X_1, Y_1, Z_1) in the above forms we find that it is identical with $-(\dot{\alpha}_1, \dot{\beta}_1, \dot{\gamma}_1)$. Hence these expressions satisfy equations (1). On examination of the terms it will be seen that they are of two kinds. The first kind involves arbitrary functions of $a^{-1}\sqrt{x^2+y^2+z^2} \pm t$, which refer evidently to a radiation travelling with a velocity a. The second kind involves arbitrary functions of $\sqrt{c^{-2}(x^2+y^2)+a^{-2}z^2} \pm t$, which refer to a radiation travelling with a velocity $[c^{-2}(l^2+m^2)+a^{-2}n^2]^{-\frac{1}{2}}$ in the direction l, m, n. Hence the wave surface corresponding to this disturbance consists of two sheets.

one being a sphere and the other a spheroid. To see the physical meaning of the expressions (6) and (7), we may let F and f be simply periodic functions with a long period, and we see that the disturbance is due to an electric doublet whose axis is along the axis of x. On letting $a = c$ the functions ϕ and ψ become identical and χ vanishes, and we obtain the well known solution of Hertz.

In a similar manner the electric and magnetic vectors for a doublet along the axis of y are given by the equations

$$\left.\begin{aligned} X_2 &= \frac{c^2}{a^2}\frac{\partial^2\psi}{\partial x\partial y} + a^{-2}\frac{\partial^2\ddot{\chi}}{\partial x\partial y} \\ Y_2 &= -a^{-2}\ddot{\phi} + \frac{c^2}{a^2}\frac{\partial^2\psi}{\partial y^2} + a^{-2}\frac{\partial^2\ddot{\chi}}{\partial y^2} \\ Z_2 &= \frac{c^2}{a^2}\frac{\partial^2\psi}{\partial y\partial z} \end{aligned}\right\}, \tag{9}$$

$$\left.\begin{aligned} \alpha_2 &= -a^{-2}\frac{\partial\dot{\phi}}{\partial z} + a^{-2}\frac{\partial^3\dot{\chi}}{\partial y^2\partial z} \\ \beta_2 &= -a^{-2}\frac{\partial^3\dot{\chi}}{\partial x\partial y\partial z} \\ \gamma_2 &= a^{-2}\frac{\partial\dot{\phi}}{\partial x} \end{aligned}\right\}. \tag{10}$$

For an electric doublet along the axis of z the components take the simpler forms

$$\left.\begin{aligned} X_3 &= \frac{c^2}{a^2}\frac{\partial^2\psi}{\partial x\partial z} \\ Y_3 &= \frac{c^2}{a^2}\frac{\partial^2\psi}{\partial y\partial z} \\ Z_3 &= -\frac{c^4}{a^4}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) \end{aligned}\right\}, \tag{11}$$

$$\left.\begin{aligned} \alpha_3 &= \frac{c^2}{a^4}\frac{\partial\dot{\psi}}{\partial y} \\ \beta_2 &= -\frac{c^2}{a^4}\frac{\partial\dot{\psi}}{\partial x} \\ \gamma_3 &= 0 \end{aligned}\right\}. \tag{12}$$

It may be noticed that an electric doublet along the axis of z gives rise only to a single wave surface—a spheroid—and it will be seen from the forms below (15) that a magnetic doublet along the same axis gives rise only to a spherical wave surface.

Besides electric doublets we can have magnetic doublets, which can also be regarded as currents flowing around small circuits. If the moment or the current varies with the time in any manner, a disturbance is propagated. We give below in (13), (14), and (15) the electric forces due to doublets of this kind having their axes along the axes of x, y, and z respectively,

$$\left.\begin{aligned} X_1' &= a^{-2}\frac{\partial^3\chi}{\partial x\partial y\partial z} \\ Y_1' &= -a^{-2}\frac{\partial\phi}{\partial z}+a^{-2}\frac{\partial^3\chi}{\partial y^2\partial z} \\ Z_1' &= \frac{c^2}{a^4}\frac{\partial\psi}{\partial y} \end{aligned}\right\}, \tag{13}$$

$$\left.\begin{aligned} X_2' &= a^{-2}\frac{\partial\phi}{\partial z}-a^{-2}\frac{\partial^3\chi}{\partial x^2\partial z} \\ Y_2' &= -a^{-2}\frac{\partial^3\chi}{\partial x\partial y\partial z} \\ Z_2' &= -\frac{c^2}{a^4}\frac{\partial\psi}{\partial x} \end{aligned}\right\}, \tag{14}$$

$$\left.\begin{aligned} X_3' &= -a^{-2}\frac{\partial\phi}{\partial y} \\ Y_3' &= a^{-2}\frac{\partial\phi}{\partial x} \\ Z_3' &= 0 \end{aligned}\right\}. \tag{15}$$

Introducing a vector—the magnetic displacement—which is connected with the magnetic force, thus

$$(\dot{u}_1\dot{v}_1\dot{w}_1) = (\alpha_1\beta_1\gamma_1),$$
$$(\dot{u}_2\dot{v}_2\dot{w}_2) = (\alpha_2\beta_2\gamma_2),$$
$$(\dot{u}_3\dot{v}_3\dot{w}_3) = (\alpha_3\beta_3\gamma_3),$$

we see that the expressions X_1, Y_1, Z_1, u_1, v_1, w_1 are identical with

X_1, X_2, X_3, X_1', X_2', X_3', respectively; and that X_2, Y_2, Z_2, u_2, v_2, w_2 are identical with Y_1, Y_2, Y_3, Y_1', Y_2', Y_3', respectively; and that X_3, Y_3, Z_3, u_3, v_3, w_3 are identical with Z_1, Z_2, Z_3, Z_1', Z_2', Z_3', respectively.

2. *The Reciprocal Theorem.*

If U and V satisfy the relations

$$\nabla^2 U = 0, \quad \nabla^2 V = 0,$$

it was shown by Green that

$$\iint \left\{ U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right\} dS = 0,$$

provided that the surface integrals are taken over the boundary of a region not containing singularities of U and V. Similar theorems have been proved by Poisson and Kirchhoff for solutions of

$$a^2 \nabla^2 U = \frac{\partial^2 U}{\partial t^2}.$$

For the equations (1), if (X, Y, Z), (X', Y', Z') be two possible solutions of the same period, we can verify the theorem

$$\begin{aligned} \iint \Big\{ & X \frac{\partial X'}{\partial n} - X' \frac{\partial X}{\partial n} + Y \frac{\partial Y'}{\partial n} - Y' \frac{\partial Y}{\partial n} + Z \frac{\partial Z'}{\partial n} - Z' \frac{\partial Z}{\partial n} \\ & - (lX + mY + nZ) \left(\frac{\partial X'}{\partial x} + \frac{\partial Y'}{\partial y} + \frac{\partial Z'}{\partial z} \right) \\ & + (lX' + mY' + nZ') \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) \Big\} dS = 0. \quad (16) \end{aligned}$$

By taking $V = \{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{-\frac{1}{2}}$,

Green has shown that

$$\iint \left\{ U \frac{\partial V}{\partial n} - V \frac{\partial U}{\partial n} \right\} dS = 4\pi U',$$

where U' is the value of U at a point x', y', z' outside the boundary which encloses the singularities of U. This equation can be interpreted in the theory of electrostatics and of hydrodynamics, just as the theorems of Kirchhoff and Poisson have interpretations in the theory of sound. In like manner, by properly choosing X', Y', Z', we might, by the aid of (16), obtain the value of the electric force

at an external point in terms of a surface integral. Such a theorem would be open to the objections urged by Prof. A. E. H. Love.* We shall therefore make use of the reciprocal theorem given by him.

Let (X, Y, Z), (u, v, w) and (X', Y', Z'), (u', v', w') be two solutions of equation (1). Then, if the surface integration extends over the boundary of a region which is free from the singularities of these functions,

$$\int_{t_0}^{t_1} dt \iint \{ l(vZ' - v'Z - wY' + w'Y) + m(wX' - w'X - uZ' + u'Z) + n(uY' - u'Y - vX' + v'X) \} dS$$

$$= \int_{t_0}^{t_1} dt \iiint d\tau \left\{ X' \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) - X \left(\frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} \right) + u' \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \right) - \dots \right\}. \tag{17}$$

But

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = a^{-2} X,$$

$$\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} = -\dot{\alpha} = -\ddot{u};$$

therefore the expression becomes, on changing the order of integration,

$$= \iiint d\tau \left[\dot{u}u' - u\dot{u}' + \dot{v}v' - v\dot{v}' + \dot{w}w' - w\dot{w}' \right]_{t_0}^{t_1} = 0$$

if we take u', v', w' to be insensible at the times t_0, t_1.

3. *The Integration of the Equations and Huygens' Principle.*

Let us take for (X', Y', Z'), (u', v', w') the forms (X_1, Y_1, Z_1), (u_1, v_1, w_1) given in (6) and (7). We have at our disposal two arbitrary functions F and f. Let $f = 0$, and let

$$\frac{\partial F(x)}{\partial x} = \zeta(x).$$

Then we shall give $\zeta(x)$ the following two properties. $\zeta(x)$ is very nearly zero except in the interval from $-\eta_0$ to η_1, where η_0 and η_1 are

* "The Integration of the Equations of Propagation of Electric Waves," *Phil. Trans.*, Series A, 1901, from which this section and the next are adapted.

very small positive quantities, and

$$\int_{-\eta_0}^{\eta_1} \zeta(x)\, dx = 1.$$

If ϵ is a small positive number, and x, y, z refer to any point on the small spheroid

$$\frac{x^2}{c^2} + \frac{y^2}{c^2} + \frac{z^2}{a^2} = \epsilon^2,$$

we can find t_1 such that

$$a^{-1}\sqrt{x^2+y^2+z^2}+t_1 \quad \text{and} \quad \sqrt{c^{-2}(x^2+y^2)+a^{-2}z^2}+t_1 > \eta_1,$$

and t_0 such that

$$a^{-1}\sqrt{x^2+y^2+z^2}+t_0 \quad \text{and} \quad \sqrt{c^{-2}(x^2+y^2)+a^{-2}z^2}+t_0 > -\eta_0.$$

Then
$$\int_{t_0}^{t_1} \zeta(a^{-1}\sqrt{x^2+y^2+z^2}+t)\, dt = \int_{-\eta_0}^{\eta_1} \zeta(x)\, dx = 1,$$

$$\int_{t_0}^{t_1} \sqrt{x^2+y^2+z^2}\ \zeta'(a^{-1}\sqrt{x^2+y^2+z^2}+t)\, dt$$
$$= \sqrt{x^2+y^2+z^2}\int_{t_0}^{t_1} \zeta'(a^{-1}\sqrt{x^2+y^2+z^2}+t)\, dt = 0,$$

if $\sqrt{x^2+y^2+z^2}$ is sufficiently small, and similar relations hold for $\zeta\{\sqrt{c^{-1}(x^2+y^2)+a^{-2}z^2}+t\}$. Making use of this function $\zeta(\ )$, it is clear that (17) is satisfied. An example of a function fulfilling these conditions is

$$\zeta(x) = \frac{\mu}{\sqrt{\pi}}\, e^{-\mu^2 x^2},$$

provided that μ is very large and ϵ is small of order μ^{-6}. The surface of integration in (17) is a surface (S_1) surrounding the singularities of (X, Y, Z), (u, v, w), and a surface (S_2) surrounding the origin which is the only singularity of (X_1, X_1, Z_1), (u_1, v_1, w_1). This latter surface we shall take to be the small spheroid

$$\frac{x^2}{c^2} + \frac{y^2}{c^2} + \frac{z^2}{a^2} = \epsilon^2.$$

We shall proceed now to find the value of the integral on the left-hand side of (17) when taken over S_2. Referring to (7) and (8), consider first the terms arising from differentiations of the function χ.

They are

$$\int_{t_0}^{t_1} dt \iint dS \left\{ a^{-2} \frac{\partial^2 \ddot{\chi}}{\partial x^2} (mw - nv) + a^{-2} \frac{\partial^2 \ddot{\chi}}{\partial x \partial y} (nu - lw) \right.$$
$$\left. + a^{-2} \frac{\partial^3 \chi}{\partial x \partial y \partial z} (mZ - nY) - a^{-2} \frac{\partial^3 \chi}{\partial x^2 \partial z} (nX - lZ) \right\}.$$

On integrating with respect to t and letting ϵ become very small, we find that the only term which does not vanish is

$$-\int_{t_0}^{t_1} dt \iint dS\, a^{-2} n X \frac{\partial^3 \chi}{\partial x^2 \partial z},$$

which, from symmetry,

$$= -\frac{a^{-2}}{2} \int_{t_0}^{t_1} dt \iint dS\, n X \frac{\partial}{\partial z} \left(\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} \right)$$

$$= -\frac{a^{-2}}{2} \int_{t_0}^{t_1} dt \iint dS\, n X \left(\frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial z} \right)$$

$$= -\tfrac{1}{2} \iint dS\, z n X \left(-\frac{1}{a^2 r^3} + \frac{1}{a^3 c^2 \rho^3} \right),$$

where $$r^2 = x^2 + y^2 + z^2$$

and $$\rho^2 = c^{-2}(x^2 + y^2) + a^{-2} z^2.$$

The contribution from the other terms of X_1, Y_1, Z_1, u_1, v_1, w_1 will, in like manner, be found to be

$$\iint dS \left(-\frac{1}{a c^2 \rho^3} (mw - nv) - \frac{X}{a^2 r^3} (nz + my) \right).$$

On expanding v and w near the origin in the form

$$v = v_0 + x \left(\frac{\partial v}{\partial x} \right)_0 + \ldots,$$

we find $$-\iint dS \frac{(mw - nv)}{a c^2 \rho^3} = -\frac{4\pi}{3} \left\{ \left(\frac{\partial w}{\partial y} \right)_0 - \left(\frac{\partial v}{\partial z} \right)_0 \right\}$$

$$= -\frac{4\pi}{3} a^{-2} X_0.$$

We have also

$$-\iint \left\{ \frac{X}{a^2r^3}(nz+my) - \tfrac{1}{2}\frac{X}{a^2r^3}nz \right\} dS = -2\pi a^{-2}X_0$$

and

$$-\tfrac{1}{2}\iint dS\,\frac{znX}{a^3c^2\rho^3} = -\frac{2\pi a^{-2}}{3}X_0.$$

Hence the integration over S_2 gives altogether $-4\pi a^{-2}X_0$, when $t=0$.

We have next to consider the integration over the surface S_1. By making use of the following theorems we can integrate at once with respect to t. Let U be any function of t which does not become infinite for any real value of t.

$$\int_{t_0}^{t_1} U\zeta\left(\frac{r}{a}+t\right)dt = \int_{-\eta_0}^{\eta_1} (U)_{t=-r/a}\,\zeta(t)\,dt = (U)_{t=-r/a}.$$

In a similar manner,

$$\int_{t_0}^{t_1} U\frac{\partial\zeta\left(\frac{r}{a}+t\right)}{\partial r}dt = -\frac{1}{a}\left(\frac{\partial U}{\partial t}\right)_{t=-r/a},$$

$$\int U\frac{\partial^2\zeta\left(\frac{r}{a}+t\right)}{\partial r^2}dt = \frac{1}{a^2}\left(\frac{\partial^2 U}{\partial t^2}\right)_{t=-r/a},$$

$$\int U\frac{\partial\zeta(\rho+t)}{\partial\rho}dt = -\left(\frac{\partial U}{\partial t}\right)_{t=-\rho}.$$

Thus of the terms in the surface integral, one set is formed at the time $t=-\frac{r}{a}$, whilst the other set is formed at the time $t=-\rho$. If the origin be outside the closed surface S_1, it will be necessary to add a third boundary, an infinite sphere having its centre at the origin. Its contribution will vanish if we take the initial disturbance to be confined to the region S_1.

Taking a new origin so that the old origin becomes x', y', z', and reckoning t from a different time, we can now write down the values of X, Y, Z at the point x', y', z' and time t in terms of a surface integral. The notation $[\ ]_{t-r/a}$, $[\ ]_{t-\rho}$ will be used to denote that the quantities inside the brackets are to be formed at the times $t-\frac{r}{a}$ and $t-\rho$ respectively; we shall also use the symbol

$$\varpi^2 = (x-x')^2+(y-y')^2.$$

$$
\begin{aligned}
4\pi X = \iint dS \Big\{ & (x-x')(y-y')(z-z') \left(\frac{2}{a\rho\varpi^4} + \frac{1}{ac^2\rho^3\varpi^2} + \frac{1}{ac^2\varpi^2\rho^2} \frac{\partial}{\partial t} \right) [mZ-nY]_{t-\rho} \\
& - (x-x')(y-y')(z-z') \left(\frac{2}{r\varpi^4} + \frac{1}{r^3\varpi^2} + \frac{1}{r^2\varpi^2 a} \frac{\partial}{\partial t} \right) [mZ-nY]_{t-r/a} \\
& + (z-z') \left[-\frac{1}{r^3} - \frac{1}{r^2 a} \frac{\partial}{\partial t} - \frac{1}{\varpi^2 r} + (x-x')^2 \left(\frac{2}{r\varpi^4} + \frac{1}{r^3\varpi^2} + \frac{1}{r^2\varpi^2 a} \frac{\partial}{\partial t} \right) \right] [nX-lZ]_{t-r/a} \\
& + (z-z') \left[-\frac{1}{\varpi^2 a\rho} + (x-x')^2 \left(\frac{2}{a\rho\varpi^4} + \frac{1}{ac^2\rho^3\varpi^2} + \frac{1}{ac^2\varpi^2\rho^2} \frac{\partial}{\partial t} \right) \right] [nX-lZ]_{t-\rho} \\
& + (y-y') \left(\frac{1}{r^3} + \frac{1}{r^2 a} \frac{\partial}{\partial t} \right) [lY-mX]_{t-r/a} \\
& - \left[\frac{a}{r} \left(\frac{1}{\varpi^2} - \frac{2(x-x')^2}{\varpi^4} \right) \frac{\partial}{\partial t} + \frac{(y-y')^2}{\varpi^2 r} \frac{\partial^2}{\partial t^2} \right] [mw-nv]_{t-r/a} \\
& + \left[-\frac{a}{c^2} \left(\frac{1}{\rho^3} + \frac{1}{\rho^2} \frac{\partial}{\partial t} \right) + \frac{a(x-x')^2}{c^4} \left(\frac{3}{\rho^5} + \frac{3}{\rho^4} \frac{\partial}{\partial t} + \frac{1}{\rho^3} \frac{\partial^2}{\partial t^2} \right) + \frac{a^2}{c^2} \left(\frac{1}{\varpi^2} - \frac{2(x-x')^2}{\varpi^4} \right) \frac{\partial}{\partial t} - \frac{a}{c^2} \frac{x^2}{\varpi^2\rho} \frac{\partial^2}{\partial t^2} \right] \\
& \qquad \times [mw-nv]_{t-\rho} \\
& + (x-x')(y-y') \left[\frac{a}{c^4} \left(\frac{3}{\rho^5} + \frac{3}{\rho^4} \frac{\partial}{\partial t} + \frac{1}{\rho^3} \frac{\partial^2}{\partial t^2} \right) - \frac{2a^2}{c^2\varpi^4} \frac{\partial}{\partial t} - \frac{a}{c^2\varpi^2\rho} \frac{\partial^2}{\partial t^2} \right] [nu-lw]_{t-\rho} \\
& + (x-x')(y-y') \left[-\frac{2a}{r\varpi^4} \frac{\partial}{\partial t} + \frac{1}{\varpi^2 r} \right] [nu-lw]_{t-r/a} \\
& + \frac{(x-x')(z-z')}{c^2 a} \left(\frac{3}{\rho^5} + \frac{3}{\rho^4} \frac{\partial}{\partial t} + \frac{1}{\rho^3} \frac{\partial^2}{\partial t^2} \right) [lv-mu]_{t-\rho} \Big\}, \qquad (18)
\end{aligned}
$$

$$4\pi Y = \iint dS \Big\{ (z-z') \Big[\frac{1}{r^3} + \frac{1}{r^2 a}\frac{\partial}{\partial t} + \frac{1}{\varpi^2 r} - (y-y')^2 \Big(\frac{2}{r\varpi^4} + \frac{1}{r^3\varpi^2} + \frac{1}{r^2\varpi^2 a}\frac{\partial}{\partial t} \Big) \Big] [mZ-nY]_{t-r/a}$$

$$-(z-z') \Big[\frac{1}{\varpi^2 a\rho} - (y-y')^2 \Big(\frac{2}{a\rho\varpi^4} + \frac{1}{ac^2\rho^3\varpi^2} + \frac{1}{ac^2\varpi^2\rho^2}\frac{\partial}{\partial t} \Big) \Big] [mZ-nY]_{t-\rho}$$

$$+(x-x')(y-y')(z-z') \Big(\frac{2}{r\varpi^4} + \frac{1}{r^3\varpi^2} + \frac{1}{r^2\varpi^2 a}\frac{\partial}{\partial t} \Big) [nX-lZ]_{t-r/a}$$

$$-(x-x')(y-y')(z-z') \Big(\frac{2}{a\rho\varpi^4} + \frac{1}{ac^2\rho^3\varpi^2} + \frac{1}{ac^2\varpi^2\rho^2}\frac{\partial}{\partial t} \Big) [nX-lZ]_{t-\rho}$$

$$-(x-x') \Big(\frac{1}{r^3} + \frac{1}{r^2 a}\frac{\partial}{\partial t} \Big) (lY-mX)_{t-r/a}$$

$$+(x-x')(y-y') \Big[\frac{a}{c^4} \Big(\frac{3}{\rho^5} + \frac{3}{\rho^4}\frac{\partial}{\partial t} + \frac{1}{\rho^3}\frac{\partial^2}{\partial t^2} \Big) - \frac{2a^2}{c^2\varpi^4}\frac{\partial}{\partial t} - \frac{a}{c^2\varpi^2\rho^2}\frac{\partial^2}{\partial t^2} \Big] [mw-nv]_{t-\rho}$$

$$+(x-x')(y-y') \Big[-\frac{2a}{r\varpi^4}\frac{\partial}{\partial t} + \frac{1}{\varpi^2 r} \Big] [mw-nv]_{t-r/a}$$

$$+ \Big[-\frac{a}{c^2} \Big(\frac{1}{\rho^3} + \frac{1}{\rho^2}\frac{\partial}{\partial t} \Big) + \frac{a(y-y')^2}{c^4} \Big(\frac{3}{\rho^5} + \frac{3}{\rho^4}\frac{\partial}{\partial t} + \frac{1}{\rho^2}\frac{\partial^2}{\partial t^2} \Big) + \frac{a^2}{c^2} \Big(\frac{1}{\varpi^2} - \frac{2(y-y')^2}{\varpi^4} \Big) \frac{\partial}{\partial t} - \frac{a}{c^2}\frac{x^2}{\varpi^2\rho}\frac{\partial^2}{\partial t^2} \Big]$$
$$\times [nu-lw]_{t-\rho}$$

$$-\Big[\frac{a}{r} \Big(\frac{1}{\varpi^2} - \frac{2(y-y')^2}{\varpi^4} \Big) \frac{\partial}{\partial t} + \frac{(x-x')^2}{\varpi^2 r}\frac{\partial^2}{\partial t^2} \Big] [nu-lw]_{t-r/a}$$

$$+\frac{(y-y')(z-z')}{c^2 a} \Big(\frac{3}{\rho^5} + \frac{3}{\rho^4}\frac{\partial}{\partial t} + \frac{1}{\rho^3}\frac{\partial^2}{\partial t^2} \Big) [lv-mu]_{t-\rho} \Big\}, \qquad (19)$$

$$4\pi Z = \iint dS \left\{ -\frac{(y-y')}{ac^2}\left(\frac{1}{\rho^3}+\frac{1}{\rho^2}\frac{\partial}{\partial t}\right)[mZ-nY]_{t-\rho}\right.$$

$$+\frac{(x-x')}{ac^2}\left(\frac{1}{\rho^3}+\frac{1}{\rho^2}\frac{\partial}{\partial t}\right)[nX-lZ]_{t-\rho}$$

$$+\frac{(x-x')(z-z')}{ac^2}\left(\frac{3}{\rho^5}+\frac{3}{\rho^4}\frac{\partial}{\partial t}+\frac{1}{\rho^3}\frac{\partial^2}{\partial t^2}\right)[mw-nv]_{t-\rho}$$

$$+\frac{(y-y')(z-z')}{ac^2}\left(\frac{3}{\rho^5}+\frac{3}{\rho^4}\frac{\partial}{\partial t}+\frac{1}{\rho^3}\frac{\partial^2}{\partial t^2}\right)[nu-lw]_{t-\rho}$$

$$+\left[+\frac{2}{a}\left(\frac{1}{\rho^3}+\frac{1}{\rho^2}\frac{\partial}{\partial t}\right)\right.$$

$$\left.\left.-\frac{(x-x')^2+(y-y')}{ac^2}\left(\frac{3}{\rho^5}+\frac{3}{\rho^4}\frac{\partial}{\partial t}+\frac{1}{\rho^3}\frac{\partial^2}{\partial t^2}\right)\right][lv-mu]_{t-\rho}\right\}. \qquad (20)$$

Thus we have expressed the value of X, Y, Z at a point x', y', z' in terms of a surface integral. This is the analytical expression of the principle of Huygens. The disturbance at any point at a time t is expressed in terms of the values of the electric force and magnetic displacement on any given surface at times $t-\frac{r}{a}$ and $t-\rho$. In other words, each element of the surface can be regarded as a source of disturbance sending out two waves, one with velocity a and the other with velocity $\{(l^2+m^2)c^{-2}+n^2a^{-2}\}^{-\frac{1}{2}}$. Let us consider, for example, the parts of (18), (19), and (20) which involve $(mZ-nY)$ under the sign of integration. On examination they will be ound to differ only from the expression X_1, Y_1, Z_1, (7), by having $[mZ-nY]_{t-r/a}$ and $[mZ-nY]_{t-\rho}$ in place of $\frac{\partial}{\partial t}F\left(\frac{r}{a}+t\right)$ and $\frac{\partial}{\partial t}F(\rho+t)$; so that these portions of the integral can be interpreted as the electric force due to a magnetic doublet of moment proportional to $mZ-nY$ and having its axis along the axis of x. Taking the other terms in like manner, we shall find that the electric force at any point can be regarded as due to a double system of sources situated on the surface, one system being electric doublets of moments proportional at each point to the vector $mw-nv$, $nu-lw$, $lv-mu$, and the other system being magnetic doublets of moments proportional to $mZ-nY$, $nX-lZ$, $lY-mX$.

4. *The Direction of Vibration.*

If the components of electric force at any point due to a system of *plane* waves be denoted by $X_0 f(lx+my+nz-kt)$, $Y_0 f(lx+my+nz-kt)$, $Z_0 f(lx+my+nz-kt)$ where $l^2+m^2+n^2=1$ and X_0, Y_0, Z_0 are independent of x, y, z, and t, then on substituting in equations (1) we find either

$$\text{(I.)}\quad k^2 = c^2(l^2+m^2)+a^2n^2$$

or

$$\text{(II.)}\quad k^2 = a^2,$$

showing that the plane waves at any time t touch

$$\text{(I.)}\quad \frac{x^2}{c^2}+\frac{y^2}{c^2}+\frac{z^2}{a^2}=t^2$$

or

$$\text{(II.)}\quad x^2+y^2+z^2=a^2t^2.$$

These are the extraordinary and ordinary wave surfaces respectively. It will also be found that the direction of electric vibration (or electric displacement) is the trace of the radius vector on the tangent to the wave surface at the point where the radius vector meets it. This is the well known construction of Huygens and Fresnel for plane wave trains. In questions dealing with a divergent disturbance it is usually assumed that this construction holds also for such cases. But we shall now examine if any solution of the equations (1) exists infinite only at one point and having the electric displacement obeying this law of Huygens and Fresnel. We shall consider first an extraordinary wave diverging from a point. Putting

$$\varpi^2 = x^2+y^2,$$

we have to determine U such that $\left\{\frac{xz}{\varpi^2}U, \frac{yz}{\varpi^2}U, -U\right\}$ are possible values for the components of electric force, and that they are infinite only at the origin. From these values, since

$$a^{-2}\frac{\partial X}{\partial x}+a^{-2}\frac{\partial Y}{\partial y}+c^{-2}\frac{\partial Z}{\partial z}=0,$$

we have

$$\frac{a^{-2}}{\varpi}\frac{\partial U}{\partial \varpi}=\frac{c^{-2}}{z}\frac{\partial U}{\partial z};$$

therefore

$$U = f(c^{-2}\varpi^2+a^{-2}z^2) = f(\rho), \text{ say.}$$

But
$$\frac{\partial^2 Z}{\partial x^2}+\frac{\partial^2 Z}{\partial y^2}+\frac{\partial^2 Z}{\partial z^2}-c^{-2}\frac{\partial^2 Z}{\partial t^2}=\frac{\partial}{\partial z}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right)$$
$$=\left(1-\frac{a^2}{c^2}\right)\frac{\partial^2 Z}{\partial z^2};$$

therefore
$$c^2\frac{\partial^2 Z}{\partial x^2}+c^2\frac{\partial^2 Z}{\partial y^2}+a^2\frac{\partial^2 Z}{\partial z^2}=\frac{\partial^2 Z}{\partial t^2}$$

or
$$\frac{1}{\rho}\frac{\partial^2}{\partial \rho^2}(\rho Z)=\frac{\partial^2 Z}{\partial t^2};$$

therefore
$$Z=\frac{f_1(\rho-t)+f_2(\rho+t)}{\rho},$$

$$X=-\frac{xz}{\varpi^2\rho}\{f_1(\rho-t)+f_2(\rho+t)\},$$

$$Y=-\frac{yz}{\varpi^2\rho}\{f_1(\rho-t)+f_2(\rho+t)\}.$$

Hence the only solutions we can find obeying Huygens' construction have X and Y infinite along the optic axis. These solutions, it may be seen, represent waves diverging from a source and running along a perfectly conducting wire extending along the axis of z in both directions or else the disturbance due to a continuous stream of electrons being projected from the origin along the axis of z with velocity a.* We can find solutions for the ordinary wave, and they will also be infinite along the axis of z. It does not therefore seem legitimate to make any statement about the direction of vibration in a divergent beam close up to the source. It may be seen, however, from the forms (8), (9), and (10) given for simple singular solutions that at a very great distance from the origin the vibration will satisfy the same laws as for plane waves. In § 7 an attempt is made to treat a problem in convergent light without any assumption as to the direction of vibration.

5. *The Ray Direction.*

It is usually defined that the ray direction is the direction in which the energy of radiation, as defined by the Poynting flux, travels. We shall now consider if any solution exists which has a singularity only at the origin and in which the Poynting flux is radial in the

* *Cf.* Heaviside, *Electromagnetic Theory*, pp. 53 *et seq.*

immediate neighbourhood of the origin. Considering the case of frequency p, the most general solution is of the form

$$X = (X_0 + pX_1 + p^2X_2 + \ldots)\, e^{ip(t-r/a)} + (X_0' + pX_1' + p^2X_2' + \ldots)\, e^{ip(t-\rho)},$$

$$\alpha = (\alpha_0 + p\alpha_1 + p^2\alpha_2 + \ldots)\, e^{ip(t-r/a)} + (\alpha_0' + p\alpha_1' + \ldots)\, e^{ip(t-\rho)},$$

with similar forms for Y, Z, β, γ; $X_0, X_0', \ldots, \alpha_0, \alpha_0', \ldots$ are functions of x, y, and z. We can assume that the twelve terms $X_0, X_0', Y_0, Y_0', Z_0, Z_0', \alpha_0, \alpha_0', \beta_0, \beta_0', \gamma_0, \gamma_0'$ are not simultaneously zero. If they were, we could, by integrating with respect to t, restore the above forms.

Since the Poynting flux is assumed to be radial,

$$xX + yY + zZ = 0,$$

$$x\alpha + y\beta + z\gamma = 0.$$

We have also

$$a^{-2}\frac{\partial X}{\partial x} + a^{-2}\frac{\partial Y}{\partial y} + c^{-2}\frac{\partial Z}{\partial z} = 0,$$

$$\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} = 0.$$

With respect to these relations the portions multiplying $e^{ip(t-r/a)}$ and $e^{ip(t-\rho)}$ must separately vanish. For, since they must hold at any point for every value of t on putting

$$t = \frac{r}{a} + \frac{\pi}{2} \quad \text{or} \quad = \rho + \frac{\pi}{2},$$

we can deduce the above result, and the results must also hold on putting $p = 0$, *i.e.*, making the period infinitely long.

Considering first $\alpha_0, \beta_0, \gamma_0$, we have

$$\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} = \left(\frac{\partial \gamma_0}{\partial y} - \frac{\partial \beta_0}{\partial z}\right) e^{ip(t-r/a)} + \ldots = a^{-2}\dot{X}$$

$$= \text{a quantity multiplied by } p;$$

therefore

$$\frac{\partial \gamma_0}{\partial y} = \frac{\partial \beta_0}{\partial z}.$$

Hence we can put

$$\alpha_0 = \frac{\partial U}{\partial x}, \quad \beta_0 = \frac{\partial U}{\partial y}, \quad \gamma_0 = \frac{\partial U}{\partial z}$$

and we find

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} + z\frac{\partial U}{\partial z} = 0,$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0.$$

U therefore is independent of r, and depends only on the direction angles θ, ϕ $\left(\cos\theta = \frac{z}{r},\ \sin\theta\sin\phi = \frac{y}{r},\ \ldots\right)$. Putting

$$\tan\tfrac{1}{2}\theta\cos\phi = \xi,$$

$$\tan\tfrac{1}{2}\theta\sin\phi = \eta,$$

we have

$$\frac{\partial^2 U}{\partial\xi^2} + \frac{\partial^2 U}{\partial\eta^2} = 0,$$

and U is finite for all real values of ξ and η. From this it follows that U is a constant. Hence $\alpha_0 = 0, \beta_0 = 0, \gamma_0 = 0, \ldots$; in like manner $\alpha_0' = 0, \beta_0' = 0, \ldots$, contrary to our hypothesis. Hence no solution exists in which the Poynting flux is radial and which has a singularity only as the origin. (The type of solution found in § 4 can be shown to satisfy this condition.*)

6. *Application of Huygens' Principle to a Crystalline Medium with a Plane Face.*

Let us take the plane of xy to be the surface of a crystalline medium, the positive direction of the z axis coinciding with the axis of the crystal, and let us suppose the face of the crystal to be disturbed by plane waves which travel in the outside medium with velocity V and fall on the face in the direction $(0, \sin i, \cos i)$. If the values of electric force and magnetic displacement at the origin just inside the crystal be represented by the real parts of $(X_0 e^{ipt},\ Y_0 e^{ipt},\ Z_0 e^{ipt})$ and $(u_0 e^{ipt},\ v_0 e_{ipt},\ w_0 e^{ipt})$ respectively, then the values at any other part of the face of the crystal will be $X_0 e^{ip[t-(y\sin i)/V]}$, $Z_0 e^{ip[t-(y\sin i)/V]}$, $Y_0 e^{ip[t-(y\sin i)/V]}$, and $u_0 e^{ip[t-(y\sin i)/V]}$, $v_0 e^{ip[t-(y\sin i)/V]}$, $w_0 e^{ip[t-(y\sin i)/V]}$.

With these values let us evaluate the integrals on the right-hand sides of (18), (19), and (20), the region of integration being an infinite rectangle on the plane xy with the origin as centre. We have

$$l = 0, \quad m = 0, \quad n = 1, \quad dS = dx\,dy,$$

$$[mZ - nY]_{t-\rho} = -y_0 e^{ip[t-\rho-(y\sin i)/V]}, \ldots .$$

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* It has been pointed out to me by one of the referees that the results of §§ 4 and 5 amount to saying that the wave fronts (if defined as containing in the tangent plane both the electric displacement and magnetic force) near a source are not ellipsoidal.

If we write

$$\phi' = \frac{e^{-ip(r/a-t)}}{r},$$

$$\psi' = \frac{a}{c^2}\,\frac{e^{-ip(\rho-t)}}{\rho},$$

$$\chi' = \int_0^{\varpi} \frac{a d\lambda}{ip\lambda}\left\{e^{-ip\sqrt{c^{-2}\lambda^2+a^{-2}z^2}+ipt} - e^{-ip/a\sqrt{\lambda^2+z^2}+ipt}\right\},$$

it will be found that ϕ', ψ', χ' are connected by the same relations (2), (3), (4), and (5) as ϕ, ψ, and χ, and we can write the expressions for the electric force at any point x', y', z' in the crystal in the forms

$$X = \frac{1}{4\pi}\iint dx\,dy\,e^{-(ipy\sin i)/V}\left\{-\frac{\partial^3\chi'}{\partial x\partial y\partial z}Y_0 + \left(\frac{\partial\psi'}{\partial z} - \frac{\partial^3\chi'}{\partial x^2\partial z}\right)X_0 \right.$$
$$\left. -\left(-\ddot{\phi}+c^2\frac{\partial^2\psi'}{\partial x^2}+\frac{\partial^2\ddot{\chi}'}{\partial x^2}\right)v_0 + \left(c^2\frac{\partial^2\psi'}{\partial x\partial y}+\frac{\partial^2\ddot{\chi}'}{\partial x\partial y}\right)u_0\right\},$$

$$Y = \frac{1}{4\pi}\iint \partial x\,\partial y\,e^{-(ipy\sin i)/V}\left\{-\left(-\frac{\partial\phi'}{\partial z}+\frac{\partial^3\chi'}{\partial y^2\partial z}\right)Y_0 - \frac{\partial^3\chi'}{\partial x\partial y\partial z}X_0 \right.$$
$$\left. -\left(c^2\frac{\partial^2\psi'}{\partial x\partial y}+\frac{\partial^2\ddot{\chi}'}{\partial x\partial y}\right)v_0 + \left(-\ddot{\phi}+c^2\frac{\partial^2\psi'}{\partial y^2}+\frac{\partial^2\ddot{\chi}'}{\partial y^2}\right)u_0\right\},$$

$$Z = \frac{1}{4\pi}\iint dx\,dy\,e^{-(ipy\sin i)/V}\left\{-\frac{c^2}{a^2}\frac{\partial\psi'}{\partial y}Y_0 + \frac{c^2}{a^2}\frac{\partial\psi'}{\partial x}X_0 - c^2\frac{\partial^2\psi'}{\partial x\partial z}v_0\right.$$
$$\left. + c^2\frac{\partial^2\psi}{\partial y\partial z}u_0\right\}.$$

We shall now show that the integrations depend on the two integrals

$$\iint \phi e^{-(ipy\sin i)/V}\,dx\,dy = P \text{ (say)},$$

$$\iint \psi e^{-(ipy\sin i)/V}\,dx\,dy = Q.$$

We have

$$\iint \frac{\partial^3\chi'}{\partial x\partial y\partial z}e^{-(ipy\sin i)/V}\,dx\,dy = \int\left[\frac{\partial^2\chi'}{\partial y\partial z}\right]_{-\infty}^{\infty} e^{-(ipy\sin i)/V}\,dy = 0,$$

$$\iint \frac{\partial^3\chi'}{\partial x^2\partial z}e^{-(ipy\sin i)/V}\,dx\,dy = 0,$$

$$\iint \frac{\partial^2\ddot{\chi}}{\partial x\partial y}e^{-(ipy\sin i)/V}\,dx\,dy = 0,$$

$$\iint \frac{\partial \phi'}{\partial z} e^{-(ipy \sin i)/V} dx\,dy = -\frac{\partial P}{\partial z'},$$

$$\iint \ddot{\phi} e^{-(ipy \sin i)/V} dx\,dy = \frac{\partial^2 P}{\partial t^2};$$

$$\iint \left(-\frac{\partial \phi'}{\partial z} + \frac{\partial^3 \chi'}{\partial y^2 \partial z}\right) e^{-(ipy \sin i)/V} dx\,dy$$

$$= \iint \left(-\frac{\partial \psi'}{\partial z} - \frac{\partial^3 \chi'}{\partial x^2 \partial z}\right) e^{-(ipy \sin i)/V} dx\,dy = \frac{\partial Q}{\partial z'}, \ldots;$$

so that the integrals become

$$4\pi X = -X_0 \frac{\partial P}{\partial z'} + v_0 \frac{\partial^2 P}{\partial t^2},$$

$$4\pi Y = -Y_0 \frac{\partial Q}{\partial z'} + u_0 \left(-\frac{\partial^2 Q}{\partial t^2} + c^2 \frac{\partial^2 Q}{\partial y'^2}\right),$$

$$4\pi Z = +Y_0 \frac{c^2}{a^2} \frac{\partial Q}{\partial y'} + u_0 c^2 \frac{\partial^2 Q}{\partial y' \partial z'},$$

$$P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-ip\left(\frac{r}{a} + \frac{y \sin i}{V} - t\right)}}{r} dx\,dy.$$

Transform the variable x, thus

$$x - x' = \frac{a}{2}\left(\xi - \frac{\varpi^2}{a^2 \xi}\right),$$

so that $$r^2 = (x-x')^2 + \varpi^2 = \frac{a^2}{4}\left(\xi + \frac{\varpi^2}{a^2 \xi}\right)^2,$$

where $$\varpi^2 = (y-y')^2 + z'^2.$$

Then $$P = \int_{-\infty}^{\infty} dy \int_0^{\infty} \frac{d\xi}{\xi} e^{-ip\left(\frac{\xi}{2} + \frac{\varpi^2}{2a^2 \xi} + \frac{y \sin i}{V} - t\right)}.$$

Making use of the theorem

$$\int_{-\infty}^{\infty} e^{-i\alpha y^2} dy = e^{-\frac{1}{4}i\pi} \sqrt{\frac{\pi}{\alpha}}$$

(where α is real and positive), and putting $\xi = \eta^2$, we get

$$P = 2a \sqrt{\frac{2\pi}{p}} e^{-\frac{ipy \sin i}{V} - \frac{i\pi}{4}} \int_0^{\infty} e^{-ip\left[\frac{\eta^2}{2}\left(1 - \frac{a^2 \sin^2 i}{V^2}\right) + \frac{z'^2}{2a^2 \eta^2}\right]} d\eta.$$

But $$\int_0^{\infty} e^{-i(\alpha x^2 + \beta/x^2)} dx = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-2i\alpha\beta - \pi},$$

if α and β are real and positive. We get, finally,

$$P = -\frac{2i\pi}{p} \frac{a}{\sqrt{1-\frac{a^2\sin^2 i}{V^2}}} e^{ip\left(t-\frac{y'\sin i}{V}-\frac{z'}{a}\sqrt{1-\frac{a^2\sin^2 i}{V^2}}\right)}.$$

We may notice that, if we integrate first with respect to x, we get

$$P = \int_{-\infty}^{\infty} K_0\left(\frac{ip\varpi}{a}\right) e^{-ip\left(\frac{y\sin i}{V}-t\right)} dy$$

in the usual notation of Bessel functions. In fact we are replacing the point-sources by line-sources. In like manner, we find

$$Q = -\frac{2i\pi}{p} \frac{a}{\sqrt{1-\frac{c^2\sin^2 i}{V^2}}} e^{ip\left(t-\frac{y'\sin i}{V}-\frac{z'}{a}\sqrt{1-\frac{c^2\sin^2 i}{V^2}}\right)}.$$

Using these values, we get

$$X = \left(\tfrac{1}{2}X_0+\tfrac{1}{2}v_0 \frac{ipa}{\sqrt{1-\frac{a^2\sin^2 i}{V^2}}}\right) e^{ip\left(t-\frac{y'\sin i}{V}-\frac{z'}{a}\sqrt{1-\frac{a^2\sin^2 i}{V^2}}\right)},$$

$$Y = \left(\tfrac{1}{2}Y_0-\tfrac{1}{2}u_0 ipa\sqrt{1-\frac{c^2\sin^2 i}{V^2}}\right) e^{ip\left(t-\frac{y'\sin i}{V}-\frac{z'}{a}\sqrt{1-\frac{c^2\sin^2 i}{V^2}}\right)},$$

$$Z = \frac{c^2}{a^2}\left(-\tfrac{1}{2}Y_0 \frac{a\sin i}{V\sqrt{1-\frac{c^2\sin^2 i}{V^2}}} + \tfrac{1}{2}u_0 \frac{ipa^2\sin i}{V}\right) e^{ip\left(t-\frac{y'\sin i}{V}-\frac{z'}{a}\sqrt{1-\frac{c^2\sin^2}{V^2}}\right.}$$

The quantities X_0, Y_0, u_0, v_0, caused by two trains of waves, one incident and the other reflected, are not independent. We can in this case find the relation between them by noticing that the above expressions for X and Y must be equal to X_0 and Y_0 at the origin. This leads to the following simplified expressions:—

$$X = X_0 e^{ip\left(t-\frac{y'\sin i}{V}-\frac{z'}{a}\sqrt{1-\frac{a^2\sin^2 i}{V^2}}\right)},$$

$$Y = Y_0 e^{ip\left(t-\frac{y'\sin i}{V}-\frac{z'}{a}\sqrt{1-\frac{c^2\sin^2 i}{V^2}}\right)},$$

$$Z = -\frac{c^2}{a^2} Y_0 \frac{a\sin i}{V\sqrt{1-\frac{c^2\sin^2 i}{V^2}}} e^{ip\left(t-\frac{y'\sin i}{V}-\frac{z'}{a}\sqrt{1-\frac{c^2\sin^2 i}{V^2}}\right)}$$

$$= Z_0 e^{ip\left(t-\frac{y'\sin i}{V}-\frac{z'}{a}\sqrt{1-\frac{c^2\sin^2 i}{V^2}}\right)}.$$

The disturbance in the crystal thus consists of two parts, one refracted at an angle r given by

$$\sin r = \frac{a}{V} \sin i,$$

the vibration (*i.e.*, electric displacement) being perpendicular to the plane of incidence and travelling with a velocity a, the other refracted at an angle r' given by

$$\sin r' = \frac{a}{V} \sin i \left\{ 1 + \frac{a^2 - c^2}{V^2} \sin^2 i \right\}^{-\frac{1}{2}}$$

with velocity $a \left\{ 1 + \frac{a^2 - c^2}{V^2} \sin^2 i \right\}^{-\frac{1}{2}}$, the vibration being in the plane of incidence. It is to be noticed that the electric doublets and the magnetic doublets which we imagine to be on the surface each contribute half the total amount of the disturbance.

7. *The Passage of Plane Waves through a Thin Crystalline Plate.*

If we consider any set of plane waves which fall on a plate of crystal with parallel sides, the axis of the crystal being normal to both faces, on reaching the crystal a reflected system is set up, whilst the remainder is divided into two plane-polarized beams, each of which gives rise to reflected beams on reaching the other face of the crystal, whilst we get finally an emergent beam travelling in the same direction as the incident beam. It is unnecessary to consider the effect of multiple reflections if the plate we are considering is taken to be much thicker than those which give rise to the phenomena of interference analogous to Newton's rings. The emergent beam consists of two parts in different phases. In fact, we may easily see from the equations above that the difference in phase is

$$\delta = \frac{pd}{a} \left\{ \sqrt{1 - \frac{c^2 \sin^2 i}{V}} - \sqrt{1 - \frac{a^2 \sin^2 i}{V}} \right\},$$

where d is the thickness of the crystal; the amplitudes of the electric displacement in those two beams may be shown to be nearly proportional to the parts of the incident vibration resolved in and perpendicular to the plane of incidence (Preston's *Light*, chap. xiii.). Making use of these considerations, we find that a train of plane waves travelling in the direction l, m, n and having the trace of the

vibration on the plane xy parallel to the axis of x will, after traversing the crystalline plate, yield components of electric force given by

$$X = -n\frac{l^2+m^2e^{i\delta}}{l^2+m^2}Ge^{ip[t-(lx+my+nz)/V]},$$

$$Y = -n\frac{lm(1-e^{i\delta})}{l^2+m^2}Ge^{ip[t-(lx+my+nz)/V]},$$

$$Z = lGe^{ip[t-(lx+my+nz)/V]},$$

where G is independent of x, y, z, and t, and

$$\delta = \frac{pd}{a}\left\{\sqrt{1-\frac{c^2\sin^2 i}{V}}-\sqrt{1-\frac{a^2\sin^2 i}{V}}\right\},$$

$$\sin i = \sqrt{1-n^2}.$$

8. *The Passage of a Divergent Beam of Plane-polarized Light through a Thin Crystalline Plate.*

It is an easy deduction from the fundamental electro-magnetic equations in free æther that no solution exists infinite at a point and having the electric force in a given direction. E. T. Whittaker (*Brit. Assoc. Rep.*, 1902) has shown that any solution of these equations can be made up of a series of plane waves. We shall consider a plane-polarized divergent beam to consist of an infinite series of plane waves having their vibrations when resolved along the face of the crystal parallel to a fixed direction, which we may take to be the axis of x. Taking the origin on the side from which the light is coming, and the axis of z parallel to the optic axis which is parallel to both faces, we get for the emergent beam

$$X = \iint -n\frac{l^2+m^2e^{i\delta}}{l^2+m^2}Ge^{ip[t-(lx+my+nz)/V]}\,d\omega,$$

with similar values of Y and Z where, to secure generality, we take $G = f(l, m, n)$ to be an arbitrary function of l, m, n, and $d\omega$ is the element of solid angle. The region of integration is determined by the aperture of the incident beam, and does not depend on x, y, or z. Here

$$-n\frac{l^2+m^2e^{i\delta}}{l^2+m^2} = -n(l^2+m^2e^{i\delta})(1+n^2+n^4+\ldots),$$

whilst δ can be expanded in positive integral powers of n. If we replace l, m, n by $-\frac{V}{ip}\frac{\partial}{\partial x}$, $-\frac{V}{ip}\frac{\partial}{\partial y}$, $-\frac{V}{ip}\frac{\partial}{\partial z}$, and suppose δ' to be

what δ becomes when we put $-\frac{V}{ip}\frac{\partial}{\partial z}$ for n, we get

$$X = f\left(-\frac{V}{ip}\frac{\partial}{\partial x}, -\frac{V}{ip}\frac{\partial}{\partial y}, -\frac{V}{ip}\frac{\partial}{\partial z}\right)\frac{V}{ip}\frac{\partial}{\partial z}\left(\frac{V^2}{p^2}\frac{\partial^2}{\partial x^2}+\frac{V^2}{p^2}\frac{\partial^2}{\partial y^2}e^{i\delta'}\right)$$
$$\times\left(1-\frac{V^2}{p^2}\frac{\partial^2}{\partial z^2}+\frac{V^4}{p^4}\frac{\partial^4}{\partial z^4}-\ldots\right) I,$$

where $$I = \iint e^{ip[t-(lx+my+nz)/V]}\, d\omega.$$

To find I, let $$lx+my+nz = r\cos\theta,$$

and let φ be the angle made by any plane through the direction $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ a fixed plane, so that

$$d\omega = \sin\theta\, d\theta\, d\phi.$$

Then $$I = \iint e^{ip[t-(r\cos\theta)/V]}\sin\theta\, d\theta\, d\phi$$
$$= -\frac{2\pi V}{ip}\frac{e^{ip(t-r/V)}}{r} + \text{a term independent of } x, y, \text{ and } z.$$

If we now consider points at a considerable distance from the origin, we have

$$-\frac{V}{ip}\frac{\partial I}{\partial x} = \frac{x}{r} I,$$

$$-\frac{V}{ip}\frac{\partial I}{\partial y} = \frac{y}{r} I,$$

$$-\frac{V}{ip}\frac{\partial I}{\partial z} = \frac{z}{r} I,$$

$$-\frac{V^2}{p^2}\frac{\partial^2}{\partial x^2} I = \frac{x^2}{r^2} I, \ldots.$$

Now, putting λ, μ, ν for $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, we get finally (on writing ν for n in the values of δ)

$$X = -\nu\frac{\lambda^2+\mu^2 e^{i\delta}}{\lambda^2+\mu^2} f(\lambda, \mu, \nu)\, I,$$

$$Y = -\nu\frac{\lambda\mu\,(1-e^{i\delta})}{\lambda^2+\mu^2} f(\lambda, \mu, \nu)\, I,$$

$$Z = \lambda f(\lambda, \mu, \nu)\, I.$$

These are the expressions for the electric force due to a beam of divergent plane-polarized light which has passed through a thin plate. For directions near the axis of Z we can neglect Z, and, if we suppose an analyzer to be an instrument which allows to pass only vibrations in a certain plane, we can find an expression for the resultant analyzed vibration which will coincide with the usual expression given in text-books of physical optics. For instance, if the principal planes of the analyzer and polarizer are at right angles, the resulting vibration is given by

$$Y = -\nu \frac{\lambda\mu(1-e^{i\delta})}{\lambda^2+\mu^2} f(\lambda, \mu, \nu) I$$

and $f(\lambda, \mu, \nu)$ never vanishes, as the plate is taken of a certain thickness. We have then a black cross given by $\lambda = 0$, $\mu = 0$, and a series of rings given by $(1-e^{i\delta})I = 0$.

Note on Huygens' Principle.

Prof. Love, in his paper already quoted, has discussed the question of the intensity at any point of the wave front of a secondary wavelet as used in Huygens' principle. The question is indeterminate, and in this note I propose to point out the relationship between the various solutions.

In the case of an isotropic medium, putting $a = c = V$ and taking the incident radiation to be given by

$$X = X_0 e^{ip(t-z/V)}, \quad Y = 0, \quad Z = 0,$$

$$u = 0, \quad v = -\frac{iX_0}{pV} e^{ip(t-z/V)}, \quad w = \frac{iX_0}{pV} e^{ip(t-z/V)}.$$

Putting

$$\Phi = \frac{e^{ipr/V}}{r},$$

then the values of X, Y, Z at a point x', y', z' on the *positive* side of the plane of xy are given by

$$4\pi X = \iint dx\,dy\, X_0 e^{ip(t-z/V)} \left\{ \frac{\partial\Phi}{\partial z} + \frac{i}{pV}\left(-\ddot{\Phi} + V^2 \frac{\partial^2\Phi}{\partial x^2} \right) \right\},$$

$$4\pi Y = \iint dx\,dy\, X_0 e^{ip(t-z/V)} \frac{iV}{p} \frac{\partial^2\Phi}{\partial x\,\partial y},$$

$$4\pi Z = \iint dx\,dy\, X_0 e^{ip(t-z/V)} \left[\frac{\partial\Phi}{\partial x} + \frac{iV}{p} \frac{\partial^2\Phi}{\partial x\,\partial z} \right].$$

The terms $\frac{\partial\Phi}{\partial z}$ and $\frac{\partial\Phi}{\partial x}$ are due to magnetic doublets, whilst the others are due to electric doublets. All these integrals vanish if x', y', z' be on the negative side of (x, y) and $\frac{\partial\Phi}{\partial z}$ and $\frac{\partial^2\Phi}{\partial x\partial z}$ change sign on crossing the plane xy; hence the electric and magnetic doublets each contribute half the total amount of the magnetic force. Putting

$$\lambda = \frac{x'-x}{r}, \quad \mu = \frac{y'-y}{r}, \quad \nu = \frac{z'}{r},$$

we find that at a great distance the electric force contributed by a magnetic doublet is proportional to $(-\nu, 0, \lambda)$, whilst the electric force contributed by an electric doublet is proportional to $[-(\mu^2+\nu^2), \lambda\mu, \lambda\nu]$. If we take the sources as they stand in the integrals above, we find that the electric force contributed by an element of the surface varies as $1+\nu$. This is Prof. Love's expression. If we took only the magnetic doublets, the force would vary as $\sqrt{\nu^2+\lambda^2}$. If we took only the electric doublets, we should get Lord Rayleigh's form $\sqrt{\mu^2+\nu^2}$. Since X, Y, Z are independent of x, y, z, then

$$\frac{\partial X}{\partial z} = 0, \quad \frac{\partial Y}{\partial z} = 0, \quad \frac{\partial Z}{\partial z} = 0.$$

This means that a certain system of electric quadruplets distributed on the surface have a null effect. Combining these sources with the electric doublets, we find Sir G. Stokes's expression $\sqrt{\mu^2+\nu^2}(1+\nu)$. The region of integration is supposed to be an infinite rectangle.

Sets of Intervals on the Straight Line. *By* W. H. Young.

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The consideration of the theory of linear sets of points leads, in a natural manner, to that of sets of intervals on a straight line. Indeed, in some respects it is more natural to begin with the latter than with the former. For example, every set of non-overlapping

intervals defines a set of points, (namely, those points which are not interior to the intervals); whereas the converse is not necessarily true unless the set of points be closed.

A branch of the theory where this latter order seems the more natural, as well as the simpler, is that of the theory of content. We propose in the present paper to investigate certain fundamental theorems on sets of intervals, with the object of subsequently applying our results to the general theory of closed sets of points.

The sequence of thought here presented runs, to some extent, on parallel lines to that of Borel in his *Leçons;* * but the mode of presentation is different, and some of the results have, we believe, never been formally stated. The introduction of the explicit distinction between external and semi-external points and the avoidance of the Heine-Borel theorem,† (which is the key-stone of Borel's account), may, perhaps, be said to characterize the present introductory account. The proof of this theorem given by Borel in the *Leçons*, (Borel's second proof), is very elegant in conception; but it can scarcely be said to give the reader an insight into the *raison d'être* of the theorem.

§ 1. We define the *content I_δ of a finite number of intervals* to be the sum of their lengths. With this definition we see at once that

(1) the content I_δ is positive and less than or equal to the length l of the segment (A, B), (supposed finite), of the straight line in which the set lies.

(2) If the content I_δ be less than l, then there exists a complementary set of intervals, whose content I_p is equal to the excess of l over the content of the given set, *i.e.*,

$$I_\delta + I_p = l.$$

(3) If the content I_δ be equal to l, then

(i.) There are no complementary intervals;

(ii.) There are no points of (A, B) exterior to all the intervals;

(iii.) There are no end points (except, of course, A and B), which do not belong to two intervals.

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* Emile Borel, *Leçons sur la Théorie des Fonctions*, 1898.

† § 11, footnote.

§ 2. Next consider an infinite number or set of non-overlapping intervals in the segment (A, B). What are analogous theorems to (1), (2), and (3) of the previous article? Cantor has proved * that in this case the set of intervals is countable; they will therefore have a definite sum * less than or equal to l. *This we define as the content* I_δ. It then follows at once that (1) holds as it stands for any set of non-overlapping intervals.

When $I_\delta = l$ it follows from the meaning of this equation * that no complementary interval can exist. (2), however, falls entirely to the ground; for not only is it not necessary for the complementary points to fill up a set of intervals, but it may even happen that there is no single complementary interval, *i.e.*, no interval whatever of (A, B) free from *interior* points of the intervals δ. Indeed, given any quantity ϵ, however small, it is possible to construct a set of non-overlapping intervals whose content I_δ is less than ϵ, yet such that *no complementary interval* exists at all.

Incredible as this at first sight appears, this is not the only paradoxical circumstance connected with such a set of intervals.

In spite of the absence of even one complementary interval, the relation $I_\delta < l$ will be found sufficient to necessitate the existence of a more than countable set of points *exterior* to every interval δ†; indeed, the potency ‡ of the *exterior* points is that of the linear continuum, provided only $I_\delta < l$, (whether *or no* there are complementary intervals).

It is, perhaps, even more surprising that, whereas this *must* be true if $I_\delta < l$, it *may* be true when $I_\delta = l$. In this case, as already stated, no complementary interval exists; so that (3, i.) holds as it stands; but (3, ii.) falls to the ground, since not only may exterior points exist, but they may even be more than countable in number, (of potency c). This shows that the familiar relations between the interior, end, and exterior points of a finite set of non-overlapping intervals cannot be assumed in dealing with an infinite set. We proceed to investigate these relations in the most general case.

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* See §§ 3 and 4. † § 22 and § 29. ‡ *Mächtigkeit.*

§ 3. *Cantor's Theorem.**

Every set of intervals on a straight line is countable, provided no two overlap.

For let $e_1, e_2, e_3, \ldots$ be any sequence of numbers having zero as limit, and let us consider only the case when the intervals all lie in a finite segment (A, B) of length l. There is in this way no loss of generality, since we can bring the whole infinite straight line into (1, 1)-correspondence with (A, B). The number of intervals of the given set whose magnitude lies between e_r and e_{r+1} must be finite, since the intervals do not overlap. Let these, arranged in any order, be denoted by G_r. Then G_r is finite and the whole set can be arranged in the order $G_1, G_2, \ldots$, and "counted" as they stand; which proves the theorem.

Cor.—From the above it is evident that the intervals can be arranged in order of magnitude $\delta_1, \delta_2, \ldots$, and that, if (A, B) be a finite segment, given any positive quantity ϵ, we can assign an integer m so that, for all values of $n \geq m$, $\delta_n < \epsilon$.

§ 4. Now the sum of any number of the intervals cannot be greater than l. There must therefore be an upper limit I_δ, less than or equal to l, such that the sum of any finite number of the intervals is always less than I_δ, but can be made as near as we please to I_δ by taking sufficient of the intervals when arranged in order of magnitude. That is to say, given any ϵ, we can find an integer m such that, for all values of $n \geq m$, $I_\delta - \epsilon < \sum_1^n \delta_r < I_\delta$.

In the usual manner we express this fact in other words by saying that the series

$$\sum_1^\infty \delta \equiv \delta_1 + \delta_2 + \ldots \text{ ad inf.}$$

is convergent, and has I_δ for its sum. *I_δ we call the content of the set of intervals.* This evidently agrees with the definition of the content in the case when the set consists of a finite number of intervals only.

It now follows from the corollary of § 3 that, *given any small positive quantity σ, we can assign a small quantity ϵ such that the sum of all the*

* Stated and proved in precisely this way by Cantor, "Ueber unendliche lineare Punktmannigfaltigkeiten," *Ann.*, Vol. xx., p. 117, 1882.

intervals of the set which are less than ϵ *is less than* σ. Denoting this sum by $R(\epsilon)$, we have $R(\epsilon) < \sigma$.

§ 5. Having so defined the content, we can at once prove the following theorem :—

The content I_δ *of a set of intervals in a finite segment* (A, B) *is equal to the sum of the contents in any set of segments inside* (A, B) *such that each one of the given intervals is interior to some one of the segments.*

For let any small quantity σ be assigned ; then we determine ϵ so that $R(\epsilon) < \sigma$. Let the segments be denoted by $D_1, D_2, \ldots$ in order of magnitude, and let us determine m so that, for all values of $n > m$, $D_n < \epsilon$. Then every interval of the given set which is not less than ϵ, lies in $D_1, D_2, \ldots, D_m$, and the sum of these intervals is less than the content I_δ by less than σ. *A fortiori* the sum of the contents of *all* the intervals in $D_1, D_2, \ldots, D_m$ is less than I_δ by less than σ ; which is what we wanted to prove.*

§ 6. We proceed to discuss first the case when $I_\delta = l$. As already stated, (3, i.) holds universally. An example will best prove that (3, ii.) falls to the ground, and it will be seen that the possibility of the presence of external points is due to the fact that it is in the neighbourhood of certain *points* that the intervals δ become smaller than any assignable quantity.

Ex. 1.—In the interval $(0, 1)$ consider the intervals $(0, \frac{1}{4})$, $(\frac{1}{4}, \frac{3}{8})$,

$\left(\frac{2^2-1}{2^3}, \frac{2^3-1}{2^4}\right)$, ..., and all the intervals got by reflecting these in the point $\frac{1}{2}$. Then we shall have a (countably) infinite set of intervals in the segment $(0, 1)$, and the sum of them is

$$2\left(\frac{1}{2^2} + \frac{1}{2^3} + \ldots\right) = 1,$$

and yet the point $\frac{1}{2}$ is *exterior* to every interval.

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* The most general form of this theorem, which can be immediately deduced from the results of this paper, is the following :—

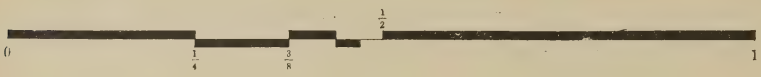
I_δ in (A, B) is equal to the sum of the contents in any set of segments contained in (A, B), provided the sum of those segments and parts of segments which lie inside each interval δ is equal to the content of that interval δ.

In particular, by Theorem 3, this will be the case if there is no external point of the segments which is not exterior to the intervals.

In fact, if we consider the given set as the "limiting set"* G of the finite sets G_1, G_2, ..., where G_1 consists of the two intervals $(0, \frac{1}{4})$, $(\frac{3}{4}, 1)$; G_2, ... of the four intervals

$$\left(0, \frac{1}{2^2}\right), \left(\frac{1}{2^2}, \frac{2^2-1}{2^3}\right), \left(\frac{2^2+1}{2^3}, \frac{2+1}{2^2}\right), \left(\frac{2+1}{2^2}, 1\right);$$

and so on, the complementary set of G_n always consists of one interval, of length successively $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{2^3}$, ...; so that in the limit the complementary interval evanesces, leaving us, however, with the point $\frac{1}{2}$, which is *interior* to the complementary interval of G_n for all values of n.



§ 7. *Ex.* 2.—Instead of reflecting all the intervals to the left of the point $\frac{1}{2}$ in that point, we might take as a new interval $(\frac{1}{2}, 1)$, the intervals to the left of $\frac{1}{2}$ being the same as before. In this case I_δ is still equal to 1; but the point $\frac{1}{2}$ is no longer external to every interval. It is, in fact, the left-hand end point of one interval. On the left of it, however, there is no interval of which it is an end point; so that, although (3, ii.) is not violated, (3, iii.) is so.

We shall find it convenient to use a new term to denote that a point is an end point of one interval only of the set.

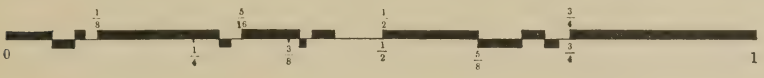
Definition.—*A point, other than A or B, is said to be semi-external to a set of non-overlapping intervals in a segment (A, B) when it is an end point of one interval only of the set.* A or B is, however, regarded as only semi-external when it is an end point of *no* interval; otherwise A or B is an ordinary end point. This agrees with the definition if we consider the whole straight line exterior to (A, B) as being, like the intervals, black.

§ 8. Since a semi-external point is an end point of one interval, it follows, by Cantor's theorem, that *the number of semi-external points of a set of non-overlapping intervals is at most countably infinite.*

* That is to say, any interval of G occurs as an interval of G_n for all values of n greater than a certain determinable integer m.

It is easy to construct examples where the number of semi-external points is actually countably infinite. For instance, as follows:—

Ex. 3.—Take the set of intervals of Ex. 2, viz., $\left(\frac{1}{2}, 1\right)$, $\left(0, \frac{1}{2^2}\right)$, $\left(\frac{1}{2^2}, \frac{2^2-1}{2^3}\right)$, ..., having the point $\frac{1}{2}$ as semi-external point, and divide each of these intervals similarly to the division in Ex. 2 of the segment (0, 1). Then in each of the above intervals the middle



point will be a semi-external point, and there will be no external points. The semi-external points will be $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{8}$, $\frac{5}{16}$, $\frac{21}{32}$, ..., and will be countably infinite in number.

§ 9. Similarly we might generalize Ex. 1, and obtain a set of content l having a countably infinite set of *external* points. We content ourselves here with giving these simple examples illustrating the truth of the statement that when the content $I_\delta = l$ there may be an infinite number of external points, violating (3, ii.), and there may be a countably infinite number of semi-external points, violating (3, iii.). We defer the introduction of examples of a more complicated nature illustrating the other statements till after the enunciation and proof of some general theorems.

§ 10. Theorem 1.—"*Of External and Semi-External Points.*"

If we have a (countably) infinite set of non-overlapping intervals $\delta_1, \delta_2, \ldots$, *lying in the segment* (A, B) *of length* l, *then, (even when* $I_\delta = l$*), there must be at least one point external or semi-external to the set.*

If there be any complementary interval, the theorem is obvious. If not, let the δ's be arranged as in § 3 in order of descending magnitude. Mark the interval δ_1 black. There remain over two segments or one; in any case there remains over at least one segment such that in it there lie intervals δ_k with index k higher than any assignable integer. Let (A_1, B_1) be this segment if determinate, or the left-hand segment if there are two possessing this property.

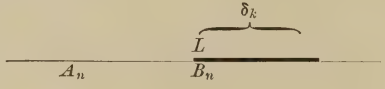
We treat (A_1, B_1) precisely as we did (A, B), marking black the

interval of lowest index in it, and passing on to that one of the segments left in (A_1, B_1) which contains intervals δ_k with index k higher than any assignable integer, and taking, if both segments have this property, the left-hand one.

Proceeding thus, we get an infinite series of segments (A, B), (A_1, B_1), (A_2, B_2), ..., each contained within the preceding and having with it one, and only one, end point common. These segments decrease without limit in length, since there is no interval of (A, B) free of internal points of δ's. They define, therefore, a definite limiting point L, which is either *internal* to *all* the segments (A_r, B_r), or, from and after a fixed integer m, is an end point of every segment (A_r, B_r), $r \geq m$.

This point L cannot be *interior* to any interval δ_k; for, if it were, denoting by h the distance of L from the nearer end point of δ_k (or $h = \frac{\delta_k}{2}$, if L lies in the middle), h will be finite, and we can determine an integer m such that for all values of $n \geq m$ the length of (A_n, B_n) is less than h. The interval (A_n, B_n), of which L is an interior or end point, will then lie entirely within δ_k, which, since the δ's do not overlap, is contrary to the hypothesis that (A_n, B_n) contained δ's with indices *higher* than k.

L may, however, be an *end* point of a determinate δ_k, while the δ's with indices as high as we please crowd themselves on the other side of L from δ_k. L will then be an end point of every (A_n, B_n) for all values of n from and after a determinable one m. For instance, L might coincide with B_n, $n \geq m$, and be the left-hand end point of δ_k, as in the figure.



The point L is therefore either *exterior* to all the black intervals δ or an end point of only one of them, *i.e.*, *semi-exterior*; which proves the theorem.

§ 11. The above proof shows that the point L is such that, at least on one side of L, the intervals δ with indices higher than any assignable quantity crowd themselves together. The lengths of these intervals become, as we saw (§ 3, Cor. 1), indefinitely small; so that the point L is a limiting point of ends of black intervals δ, or, if we

please, of the right-hand end points of δ's by themselves, or of the left-hand end points by themselves.

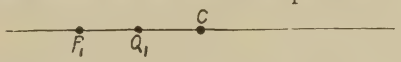
If L is a limiting point on *both* sides, it will certainly be *external* to all the black intervals. The two following theorems will show under what restrictions every point external to all the δ's is a limiting point on both sides of the end points. In the meanwhile we may enunciate the preceding theorem in the following more precise form :— *

An infinite (countable) set of non-overlapping intervals $\delta_1, \delta_2, \ldots$ *has always at least one limiting point of end points which is either external or semi-external to every black interval. If the limiting point be a limit on both sides, it is certainly external to every black interval; otherwise it is semi-external.*

§ 12. Theorem 2.—"*Of External Points.*"

Given a set of non-overlapping intervals, such that no complementary intervals exist, in a segment (A, B), *then, if there be any point* C *of* (A, B) *exterior to the set of intervals,* C *is a limiting point on both sides of intervals.*

For let e_1 be any small positive quantity, and construct on the left of C a segment of length e_1. Then, since no complementary interval exists, there must be at least one point within this segment



which is *not exterior* to every interval. Let P_1 be such a point. Then, since C is exterior to the interval δ containing P_1, there must be in the segment (P_1, C) the right-hand end point of the interval δ to which P_1 is not exterior. Let this point be Q_1, where Q_1 is either identical with P_1 or lies between P_1 and C; then $CQ_1 < e_1$.

* This theorem takes the place of the so-called Heine-Borel theorem, the proof being of the type of Borel's second proof, (*Leçons sur la Théorie des Fonctions*, p. 42), only direct instead of indirect. The enunciation of the Heine-Borel theorem is as follows :—

Given a countable set of intervals, (of course overlapping), such that each point of the closed segment (A, B) *is an internal point of at least one interval, then it is possible to choose out a finite number* N *of these intervals having the same property.*

This theorem may be deduced easily from our theorem. For, if we arrange the Heine-Borel intervals in countable order, then omitting any interval or part of an interval which was contained in any of the preceding intervals, and omitting any parts exterior to (A, B), we get a set of intervals such that *every* internal point of (A, B) is an interior point of one interval, or an end point of two intervals, while A and B are end points each of one interval. By our theorem the number of these intervals cannot be infinite. But each of these is the whole or a part of a definite interval of the given set, and these have the required property.

Now let e_2 be smaller than the length of (C, Q_1). We determine similarly a right-hand end point Q_2 such that $CQ_2 < e_2$. Proceeding thus, we determine a sequence $Q_1, Q_2, \ldots$ of right-hand end points of intervals δ, lying on the left of C and having C as limiting point.

Similarly we construct a sequence of left-hand end points of intervals δ, lying on the right of C and having C as limiting point. Thus C is a limiting point on both sides of end points of intervals.

Q. E. D.

Cor.—If $I_\delta = l$, *then, if there be any exterior point, it is a limiting point on both sides of intervals.* For in this case, (§ 4), no complementary interval can exist.

§ 13. In the case, then, when the complementary set of *intervals* does not exist, the whole continuum (A, B) is made up of the given set of non-overlapping intervals of content $I_\delta \leqslant L$ and the limiting points, (if any), on *both* sides, (exterior points). As to these exterior points, since they are not, like the semi-exterior points, each bound to some particular interval, we cannot assert that they are not more than countably infinite.

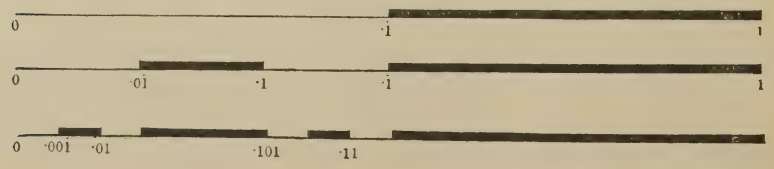
In fact, though there may be none at all, or only a finite or countably infinite number of them, they may be more than countably infinite, as we will now show by examples.

The first of the following examples where, although $I_\delta = l$, (so that certainly no complementary interval exists), the exterior points have the potency of the linear continuum is of classic interest.

§ 14. Consider the segment (0, 1), and let us take as right-hand end points of our intervals the point 1, and all those points whose numbers expressed as fractions in the ternary scale involve a finite number of 0's and 1's, and do not involve the figure 2.

The number corresponding to the left-hand end point of any interval is got from the number corresponding to the right-hand end point by changing the final one into 0$\dot{1}$.

Thus, in order of magnitude, the first few intervals are $(\cdot\dot{1}, 1)$, $(\cdot 0\dot{1}, \cdot 1)$, $(\cdot 00\dot{1}, \cdot 01)$, and $(\cdot 10\dot{1}, \cdot 11)$; or, which is the same thing,



using the everyday notation, $(\frac{1}{2}, 1)$, $(\frac{1}{6}, \frac{1}{3})$, $(\frac{1}{18}, \frac{1}{9})$, and $(\frac{1}{3}+\frac{1}{18}, \frac{1}{3}+\frac{1}{9})$. It is evident from the numbers that these intervals do not overlap nor abut anywhere, and that there are no complementary intervals, so that *every end point is a semi-external point.* These semi-external points correspond, therefore, by construction, to all the terminating ternary fractions, involving only 0's and 1's, (right-hand end points), and all the simple recurrers, involving only 0's and 1's and ending in $\dot{1}$, (left-hand end points).

Also it is easy to show that *all the interior points of black intervals correspond to ternary fractions involving at least one* 2, *and vice versa.* For, if N denote any combination of n figures, 0's and 1's only, and M any combination of 0's, 1's, and 2's, the ternary fraction $\cdot N02M$, as well as, (for all integral values of p), $N01^p2M$, lies, whatever M may be, between $\cdot N0\dot{1}$ and $\cdot N1$). These numbers correspond therefore to internal points of a determinate black interval, $(\cdot N0\dot{1}, N1)$. *Vice versa*, any number lying in this interval is expressible in one of the two given forms.

The remaining (*exterior*) *points consist therefore of all the non-terminating ternary fractions involving only* 0's *and* 1's, *other than the simple recurrers ending in* $\dot{1}$. Every such point is, as follows, easily shown to be a limiting point on both sides of the semi-external points.

Given any such non-terminating fraction, we can form a sequence of semi-external points, (right-hand end points), having that fraction as limiting point, by stopping in succession at each 1 inclusive. A similar sequence on the other side of the point in question is determined by stopping short of each 0 and appending a 1.

It is easy to see that *these external points are more than countably infinite* in number; indeed, the potency is actually c, (that of the linear continuum). For we only have to interpret these ternary fractions in the binary scale and we have set up a (1, 1)-correspondence between these points and the whole continuum from 0 to 1, with the exception of a countable number of binary points, which do not, of course, affect the potency.

We can, moreover, show that *the sum of the black intervals is actually* 1. For, since N consists of n figures, 0's and 1's, the number of semi-external points $\cdot N1$, (for the same n), is 2^n. Each of these is the right-hand end point of an interval of length $\frac{1}{2}\frac{1}{3^{n+1}}$.

Hence the sum of all the intervals is

$$\tfrac{1}{2}\left(1+\frac{1}{3}+\frac{2}{3^2}+\frac{2^3}{3^3}+\ldots\right)=\tfrac{1}{2}\left(1+\frac{1}{3}\frac{1}{1-\frac{2}{3}}\right)=1.$$

It has therefore been proved that *in the segment* (0, 1) *the points corresponding to all the ternary fractions involving the figure* 2 *fill up a set of intervals of content* 1,* *which do not overlap nor abut anywhere, so that the semi-external points are countably infinite in number. The external points are more than countably infinite, and yet are not dense in any interval, however small.*

§ 15. The mode in which we have treated this example was suggested by the context as throwing light on the preceding theorems and statements about external and semi-external points. Historically this example—the first of its kind—must be referred to H. J. S. Smith,† who, however, treated the subject purely geometrically and from a somewhat different point of view. H. J. S. Smith's set of intervals are not in *all* essentials identical with those given above,‡ since each of our black intervals is replaced in his treatment by an infinite set of abutting intervals exactly filling it up. Thus, a number of points which in our set were internal points become end points, and a number of semi-exterior points, (left-hand end points), become external points.

The law of formation of these abutting intervals in the interval $(\cdot N01, N1)$ is that the ends of the abutting intervals consist of the sequence $\cdot N02$, $\cdot N012$, $N0112$, ..., $N01^p2$, ..., (for all integral values of p), the limiting point of which is $N0\dot{1}$. The intervals are plotted down by H. J. S. Smith successively in order of magnitude.

On account of its historical interest, and the light the method throws on the general lie of the set in its relation to the continuum, and also because the method lends itself easily to a form of generalization which will be exceedingly valuable to us in the sequel, we shall now give a more detailed discussion of H. J. S. Smith's set of intervals from his point of view.

§ 16. *H. J. S. Smith's Ternary Set of Intervals of the First Kind.*

Divide first the segment (0, 1) into three equal parts, and blacken the right-hand part. At the second division divide each of the two unblackened parts into three equal parts and blacken the two right-hand parts. The two black intervals which abut, if amalgamated, would

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* So that certainly no complementary interval exists.

† *Proc. Lond. Math. Soc.*, Vol. VI., 1870.

‡ *Cf.* W. H. Young, "On the Density of a Linear Set of Points," *Proc. Lond. Math. Soc.*, Vol. XXXIV., p. 286, footnote.

form the germ of the first of the black intervals of the preceding section. At the third division divide each of the 2^2 unblackened segments, of length $\frac{1}{3^2}$, into three equal parts, and blacken the 2^2 right-hand parts. Generally at the $(n+1)$-th division divide each of the 2^n unblackened segments, of length $\frac{1}{3^n}$, into three equal parts and blacken the 2^n right-hand parts. (See G_1 in Fig.)

That this set of intervals is dense everywhere is almost immediately evident from the mode of construction, as are also the facts that there are a countably infinite set of simple end points, (left-hand),* and another countably infinite set of semi-external points, (right hand). It is not, however, easy to see geometrically that the external points are more than countable.†

These facts did not interest H. J. S. Smith at the moment; the question with him was one of content pure and simple.

The sum of the intervals blackened in the first n processes of division is

$$1+\frac{2}{3}+\ldots+\frac{2^{n-1}}{3^n}=\frac{1}{3}\,\frac{1-(\frac{2}{3})^n}{1-\frac{2}{3}}=1-\left(\frac{2}{3}\right)^n.$$

If, therefore, we continue the division *ad infinitum*, we shall get a set of black intervals of content 1.

§ 17. This method of H. J. S. Smith is capable of a natural modification which does not suggest itself when we work numerically. We are furnished in this way with the following example, among others, given also by H. J. S. Smith, without use of number, *loc. cit.* This example serves to show the fallacy of the naïve assumption that when I_δ is less than l at least one complementary interval must exist.

* Whose numbers all terminate with a 2, and involve otherwise (a finite number of) 0's and 1's only.

† For instance, the reasoning in Borel, p. 44, is only valid when $I_\delta < l$.

§ 18. *H. J. S. Smith's Ternary Set of Intervals of the Second Kind.*

As in the former example, divide the segment $(0, 1)$ into three equal parts, and blacken the right-hand part $(\cdot 2, 1)$, omitting it from further division. The origin together with the point of division $\cdot 1$, which lies outside the omitted segment, we shall denote by G_1.

At the second division we shall divide each of the two segments not blackened, $(0, \cdot 1)$ and $(\cdot 1, \cdot 2)$, into 3^2 equal parts, and blacken the right-hand segment. (See Γ in Fig. of p. 257.)

Denoting by G_2 the set of points consisting of G_1 and all the fourteen points of division which lie outside the blackened parts, we see that the numbers of G_2 are completetely characterized as being all the numbers of the form $\cdot e_1 e_2 e_3$, where

$$A_2 \begin{cases} (1) & e_1 \text{ is not } 2, \\ (2) & e_2 \text{ and } e_3 \text{ are not } \textit{both } 2. \end{cases}$$

At the third division we divide each of the $(3-1)(3^2-1)$ segments not already blackened into 3^3 parts and blacken the right-hand segment in each.

Denoting by G_3 the set of points consisting of G_2 and all the $(3-1)(3^2-1)(3^3-2)$ points of division which lie outside the blackened parts, we see that the numbers G_3 are completely characterized as being all the numbers $\cdot e_1 e_2 e_3 e_4 e_5 e_6$, where

$$A_3 \begin{cases} (1) & e_1 \text{ is not } 2, \\ (2) & e_2 \text{ and } e_3 \text{ are not both } 2, \\ (3) & e_4, e_5, \text{ and } e_6 \text{ are not } \textit{all } 2; \end{cases}$$

that is, where, in addition to the conditions A_2, the condition $A_3(3)$ is satisfied; and these conditions together are denoted by A_3.

It is now obvious how in turn $G_4, G_5, \ldots$ will be constructed and will become in turn the right-hand end points of black intervals. At the n-th division we divide each of the $(3-1)(3^2-1) \ldots (3^{n-1}-1)$ segments not already blackened into 3^n equal parts, and omit in each the right-hand segment. G_n will then consist of G_{n-1} and all the $(3-1)(3^2-1)(3^3-1) \ldots (3^{n-1}-1)(3^n-2)$ points of division outside the blackened parts, and the numbers of G_n are therefore completely characterized as being all the numbers $\cdot e_1 e_2 \ldots e_{\frac{1}{2}[n(n+1)]}$, where, in addition to the conditions A_{n-1}, we have the condition $A_n(n)$, viz., $e_{\frac{1}{2}[(n-1)n]+1}, e_{\frac{1}{2}[n(n-1)]+2}, \ldots, e_{\frac{1}{2}[n(n+1)]}$ are not all to be 2. These conditions we denote by A_n. The limiting set * of G_1, G_2, we denote by G.

* That is, that set G such that any assigned point of G has been a point of every G_n, from and after an assignable index m.

§ 19. Each point of G is a right-hand end-point of a set of abutting black intervals. If we consider all abutting intervals as amalgamated, and continue these intervals up to their limiting points on the left, we obtain a set of intervals analogous to those of § 14. Those limiting points, which were external and have now become semi-external, can be assigned as follows :—

Suppose $\cdot P$ to be a point of G which belongs to G_n,* but not to G_{n-1}, and let $\cdot Q$ be the point of G_n immediately to the left of $\cdot P$.† The left-hand end point of the interval of which $\cdot P$ is the right-hand end point is obtained by appending to $\cdot Q$, if necessary, so many 0's as to make the number of figures $\frac{n(n+1)}{2}$, and then appending λ_n, where

$$\lambda_1 = 21221222122221\ldots$$

and

$$\lambda_n = 2^n 12^{n+1} 12^{n+2} 1\ldots$$

the number of 2's between consecutive 1's increasing each time by one.

Thus, for example, the black interval whose right-hand end point is $\cdot 0\ 12\ 001\ 02$ has for left-hand end point $\cdot 0\ 12\ 001\ 0122\ 2^4\ 12^5\ 12^6\ 1\ \ldots$. These numbers show us that the left-hand end points are, as was asserted, semi-external points, since, stopping at any figure in one of these numbers, we get a number obeying the conditions A_n for a certain definite integer n, and therefore a sequence of numbers of G, having the given left-hand end point as limit on the right. The right-hand end points are also semi-external, since we can add on at the end of any one of these numbers any number of 0's followed by a 1, and so get a sequence of points of G, having the desired right-hand end point as limit on the left.

Both the left- and the right-hand end points being semi-external, it is evident that *no complementary interval can exist.* This could be deduced from the geometrical construction.

The content of the set of intervals is, however, no longer 1. It is,

---

* Given $\cdot P$, it is quite easy practically to determine n : for instance,

$$\cdot P = \cdot 1\ 02\ 101\ 2121\ 00011\ 12$$

belongs to G_6, but not to G_5.

† $\cdot Q$ is known at once as soon as $\cdot P$ is given, since

$$\cdot P = \cdot Q - \frac{1}{3^{\frac{1}{2}[n(n+1)]}}.$$

in fact, evident from the geometrical construction that the content is

$$\frac{1}{3}+\frac{(3-1)}{3^{2+1}}+\frac{(3-1)\,(3^2-1)}{3^{3+2+1}}+\ldots+\frac{(3-1)\,(3^2-1)\ldots(3^{n-1}-1)}{3^{1+2+\ldots+n}}+\ldots,$$

which is

$$\frac{1}{3}+\frac{1}{3^2}\left(1-\frac{1}{3}\right)+\frac{1}{3^3}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{3^2}\right)+\ldots,$$

whence $\quad \frac{1}{3} < I_\delta < \frac{1}{2}.$

Any point which violates the condition $A_n(n)$ for any value of n is evidently an interior point of a black interval which we can at once assign. The right-hand end point of this black interval is obtained by stopping at the first $\frac{n(n-1)}{2}$ figures of the number in question: *e.g.*, the points whose numbers begin with the figures 1 02 001 2222 ... are interior to the black interval (1 02 000 $2^3$1 $2^4$1 ..., .1·02 001).

It was asserted that when $I_\delta < l$, (that is, here < 1), there must be a more than countable set of exterior points. The only ternary fractions at present unaccounted for are all the infinite ternary fractions which do not violate the conditions A_n for any value of n, (other than those ending in λ_n). These are the exterior points of the set. That they have the potency of the continuum is evident when we consider that all the non-terminating ternary fractions involving only 0's and 1's are among them (§ 14).

Numerically we have proved the following theorem:—

Let $\cdot e_1 e_2 e_3 \ldots e_{\frac{1}{2}[n(n-1)]+1}\, e_{\frac{1}{2}[n(n-1)]+2} \cdots e_{\frac{1}{2}[n(n+1)]}$ *denote any ternary fraction with* $\frac{n(n+1)}{2}$ *figures, and let A_n denote the conditions that for no positive integral value of $r \leqslant n$ all the figures* $e_{\frac{1}{2}[r(r-1)]+1}, \ldots, e_{\frac{1}{2}[r(r+1)]}$ *should be 2's. Then all the ternary fractions which violate the conditions A_n for some value of n fill up a set of intervals of content I_δ, where*

$$\frac{1}{3} < I_\delta < \frac{1}{2},$$

*which do not overlap nor abut anywhere, so that the semi-external points are countably infinite in number. The external points are more than countably infinite, (of potency c), and yet are not dense in any interval, however small.**

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* The numbers corresponding to the simple end points of H. J. S. Smith's ternary set of the second kind are evidently those terminating fractions which violate some condition A_n, but only in the final stage $A_n(n)$.

§ 20. *H. J. S. Smith's Sets with Base m.*

Instead of the base 3, we might take any base $m \geq 3$ and perform the construction of § 16, or that of § 18, *mutatis mutandis*. The former series of sets, in which at each division we divide the unblackened parts of the straight line into m equal parts, gives us nothing particular of interest beyond the set for $m = 3$. But the latter series of sets, in which at the n-th division we divide the unblackened parts into m^n parts, presents this new and highly interesting feature, that

$$\frac{1}{m} < I_\delta < \frac{1}{m-1},$$

the other properties of the set being the same as in § 18.

Hence we see how it is possible to construct a set of intervals whose content is less than any assigned magnitude, and having a more than countable set of exterior points, (of potency c), which, however, are not dense in any interval, however small.

§ 21. Similarly by using any base $m \geq 3$, we can, as in §§ 14 and 18, construct sets of *non-abutting* intervals, with contents ranging from L down to *less than any assignable quantity*, and yet dense everywhere.

The numerical statements of these facts are interesting.

(1) *In the segment* (0, 1), *the points corresponding to all the m-ary fractions involving the figure* $(m-1)$ *fill up a set of intervals of content* 1, *which do not overlap or abut anywhere; so that the semi-external points are countably infinite in number. The external points are more than countably infinite, (of potency c), and yet are not dense in any interval, however small.*

These semi-external points correspond to all the terminating fractions not involving the figure $(m-1)$ and all the non-terminating fractions ending in "$(m-2)$ circulating." The external points correspond to all the other non-terminating fractions not involving the figure $(m-1)$.

(2) *Let* $\cdot e_1 e_2 e_3 \ldots e_{\frac{1}{2}[n(n-1)]+1} \ldots e_{\frac{1}{2}n(n+1)}$ *denote any m-ary fraction, and let* A_n *denote the conditions that for no positive integral value of* $r \leq n$ *all the figures* $e_{\frac{1}{2}[r(r-1)]+1}, \ldots, e_{\frac{1}{2}r(r+1)}$ *should be* $(m-1)$*'s. Then all the m-ary fractions which violate the conditions* A_n *for some value of n fill up a set of intervals of content* I_δ, *where*

$$\frac{1}{m} < I_\delta < \frac{1}{m-1},$$

which do not overlap nor abut anywhere; so that the semi-external points are countably infinite in number. The external points are more than countably infinite, (of potency c), and yet are not dense in any interval, however small.

§ 22. The properties of the sets which we have denoted as H. J. S. Smith's sets of the second kind are of the most general character possible, as is indicated by the following theorem :—

THEOREM 3.—*If we have a set of non-overlapping intervals* $\delta_1, \delta_2, \ldots$, *whose content* I_δ *is less than* l, *lying in a segment* (A, B) *of length* l, *then the points of* (A, B) *which are* external *to all the intervals* δ *form an infinite non-countable set.** (It will appear from the sequel that this set has the potency c of the linear continuum.)

If any complementary interval exists, the theorem is obvious. If not, every external or semi-external point C will be a limiting point of intervals δ, and there will be at least one such point. The number of semi-external points is at most countably infinite.

This being the case, let us, if possible, arrange all the external and semi-external points in countable order $C_1, C_2, \ldots$. Round C_1 describe an interval d_1 of length $\frac{e}{4}$ with C_1 as middle point, where

$$e = l - I_\delta.$$

Let C_i be the next point of the series $C_1, C_2, \ldots$ which is exterior to d_1, and describe similarly an interval of length $\frac{e}{2^{i+1}}$ with C_i as middle point. If this overlap or abut with d_1, we amalgamate them into one interval; if not, we call it d_2.

Proceeding in order with the points C which are not already interior or end points of intervals d, we get a finite or countably infinite set of non-overlapping and non-abutting intervals d, whose content is not greater than

$$e \sum_{2}^{\infty} \frac{1}{2^r} = \frac{e}{2},$$

containing all the points C. Adding to these such intervals δ, or parts of intervals δ, as are external to the intervals d, and amalgamating them with the intervals d where they overlap with them, we get a set of non-overlapping intervals such that no point of (A, B) is exterior or semi-exterior to them. By Theorem 1, "Of External and Semi-

* Borel, *Lecons*, p. 41.

external Points," the number of these intervals can only be finite. *A fortiori* the number of intervals d is finite. Let them be denoted by $d_1, d_2, \ldots, d_n$.

Marking these black, we are left with a finite number ($\leqslant n-1$) of segments (A, B) whose content also lies between $l-\frac{e}{2}$ and l. In each such unblackened segment there is no external or semi-external point of the intervals δ. Hence, by Theorem 1, in each, and therefore in all, of these segments there is a *finite* number of intervals δ entirely filling them up, and whose content therefore also lies between $l-\frac{e}{2}$ and l. But this is inconsistent with the fact that $I = l-e$. Hence the assumption was false, and, by a *reductio ad absurdum*, we are obliged to deny the possibility of counting the points C.

Q .E. D.

§ 23. *The Typical Ternary Set of Intervals.*

We have seen that, given any set of intervals, we have a perfectly determinate countable set of semi-external points, and that any end point of an interval where it does not abut on another interval is either (1) a semi-external point or (2) not a limiting point at all. In the latter case the set of intervals is *not* dense everywhere.

The question of content is obviously unaffected if we amalgamate all abutting intervals, in which case the end points of the new intervals will consist entirely of the two classes of points above referred to, and, if the original set of intervals be dense everywhere, will consist exclusively of semi-external points.

This demonstrates the importance of considering in detail the properties of sets of intervals whose end points are *all* semi-external points, that is, *sets of non-abutting intervals which are dense everywhere.*

The sets considered in §§ 14 and 18 belong to this class; but we shall find it desirable to modify them and take another set* as typical of the class, its numerical equivalent being the most convenient, as giving us a system of indices for the δ's in the general case, by means of which any two sets of the class are arranged at once in (1, 1)-correspondence.

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* *Cf.* G. Cantor, *Math. Ann.*, Vol. xxi., p. 590, and Schoenflies's *Bericht*, p. 102.

§ 24. Take the segment (0, 1) of the x axis and divide it into three equal parts and blacken the middle one: this is (·1, ·2), or, as we prefer to write it, (·0$\dot{2}$, ·2). This we denote by δ_1. In each of the two unblackened segments repeat the process: we get two new black intervals,

$$\delta_{01} = (\cdot 00\dot{2},\ \cdot 02),$$

$$\delta_{11} = (\cdot 20\dot{2},\ \cdot 22).$$

In each of the unblackened segments repeat the process, and so on.

Denoting by N any combination of n figures, 0's and 1's, and by $(2N)$ the number got by multiplying N by 2, we see that any black interval obtained by our process may be denoted by δ_{N1}, and characterized by the symbolic equation

$$\delta_{N1} = [\cdot(2N)\,0\dot{2},\ \cdot(2N)\,2].$$

We notice that the order of the intervals is precisely that of the binary fractions $\cdot N1$. The internal points of the black intervals are characterized by the appearance of a proper 1, not $\dot{2}$, in their ternary factions.

Given any ternary fraction with a proper 1 in it, we can at once assign the black interval to which it belongs. For example, the number ·020022021 ..., (where the dots denote *any* subsequent figures), lies between ·0200220202$\dot{}$ and ·020022022, and therefore is interior to the black interval $\delta_{010011011}$.

The only numbers unaccounted for are the non-terminating ternary fractions other than the simple recurrers ending in $\dot{2}$ involving only the figures 0 and 2. These must then represent the exterior points of our set of black intervals. Dividing each such number by 2, and interpreting in the scale of 2, we evidently get the whole continuum, with the exception of a countable set; so that the potency of this set of numbers is c. Hence *the potency of the exterior points is c.*

§ 25. If we are given *any* set of non-overlapping and non-abutting intervals, dense everywhere in an interval (A', B'), then A' and B' themselves may be ordinary end points or may be semi-external points. In the former case there are black intervals (A, A'), (B', B) belonging to the set, and A' and B' are not semi-external points. Such intervals, if they exist, do not materially affect the character of the set; they may therefore be conveniently omitted from con-

sideration in the present section. We are working, then, in a segment (A, B) where A and B are semi-external points of our set of intervals. We now propose to set up a (1, 1) correspondence, *maintaining the order* between the given set and the typical set of intervals, § 24.

§ 26. Divide (A, B) into three equal parts at C and D. Then, since the set is dense everywhere, either (C, D) forms part of a determinate black interval or else there is a black interval inside (C, D), possibly coinciding with it. Let us then choose some particular black interval, for instance, the largest possible in (C, D), and denote it by δ_1, and, in the first case, let δ_1 be the black interval of which (C, D) forms a part.

Since A and B, being semi-external points, are not the end points of any black interval, there will be two unblackened segments left after we have blackened δ_1. The end points of each of these two segments are semi-external points; hence we may repeat the process in each of these two segments and choose out two new intervals of our set, one in each of these segments, and these we may call δ_{01} and δ_{11}. The order of the three intervals δ_{01}, δ_1, δ_{11} is evidently the same as in the typical case, (§ 24), and the same as that of the binary fractions ·01, ·1, ·11.

We are now left with four segments, whose end points are all semi-external to our set of black intervals; so that we can repeat our process in each of them, and choose in each a black interval of our set. These we denote by δ_{001}, δ_{011}, δ_{101}, δ_{111} in order from left to right; so that the order of the binary fractions is maintained.

Proceeding thus, we can evidently use the terminating binary fractions, (omitting the point), as a general system of indices, not merely proving the countableness of our set of intervals, but also indicating exactly their order in relation to the continuum. This sets up, *ipso facto*, a (1, 1)-correspondence, maintaining the order, between the general set of this class and the typical set of § 24, enabling us to solve many problems for the general set by means of the known properties of the typical set.

§ 27. One consequence of the mode adopted for determining the indices is that, *given any positive quantity* ϵ, *we can determine an integer* m *such that, for all values of* $n \geqslant m$, $\delta_{N1} < \epsilon$ (n being, as always, the number of figures in N). For, by the construction, the two segments left after the blackening of δ_1 are each less than $\frac{2}{3}$ of (A, B),

and at each stage a similar statement can be made as to the length of each unblackened segment. Thus we have only to determine m so that $(\frac{2}{3})^m (A, B) < \epsilon$, and this m will certainly satisfy our requirements.

§ 28. Since the order has been maintained, it follows from the above that any sequence of intervals of the given set, defining a single limiting point, will correspond to a sequence of intervals of the typical set, defining a single limiting point, and *vice versa*. We can most easily express this correspondence between the limiting points by denoting the left- and right-hand end points of any black interval of δ_{N1} by $P_{N0\dot{2}}$, and $P_{N\dot{2}}$, and any exterior point by P, with, as index, the ternary number denoting the limiting point of the corresponding intervals of the typical set.

We see that, the semi-external points being in (1, 1)-correspondence by themselves, the exterior points will be so also. *Hence it follows that the potency of the external points is c.*

§ 29. We have proved by our correspondence that *the external points of a set of non-abutting intervals have the potency of the linear continuum*, it being unnecessary to postulate that the set should be dense everywhere, since, when at least one complementary interval exists, the theorem is obvious. The following can now be deduced:—

THEOREM 3′.—*Of the Potency of the External Points.*

If we have a set of non-overlapping intervals $\delta_1, \delta_2, \ldots$, *whose content* $I_\delta < l$, *lying in a segment of length* l, *then the points which are* external *to all the intervals* δ *form an infinite non-countable set of potency c.*

If the set is not dense everywhere, the theorem is obvious. We assume, therefore, that the set is dense everywhere.

We shall, for reasons which will immediately be evident, denote as a *point of arrest* a point such that in any segment, however small, containing it as internal point, there is a more than countable set of external points.

Let (A', B') be the segment in which the set exists. If A' and B' are not points of arrest, we will show how to replace (A', B') by a segment (A, B), lying within it, so that the potency of the external points is unaltered, while A and B are points of arrest.

Let us bisect (A', B') at M. Then, since, by Theorem 3, there is a

more than countable set of external points in (A', B'), this must be the case in one or both of (A', M), (M, B'). We determine whether or no (A', M) has this property. Next we bisect again, and determine that segment which (i.) contains a more than countable set of external points, and (ii.) lies nearest to A'. Continuing this process, since there is nothing to prevent our doing so, *ad infinitum*, we have a series of segments, each contained within the preceding, and of half its length. This determines a limiting point, which will evidently coincide with A', if A' be a point of arrest, and will otherwise be the first point of arrest on the right of A', so that between it and A' there is at most a countable set of external points. This point of arrest we denote by A, and blacken the whole interval (A', A). In amalgamating all the intervals between A' and A in this way we have, *at most*, affected a countable set of external points.

We notice, then, that the content of the amalgamated intervals is, by Theorem 3, equal to that of (A', A). Similarly, working from right to left, we obtain B from B'.

Now, as in § 26, divide (A, B) into three equal parts and determine a black interval of the given set, which either coincides with the middle segment, or contains it, or is contained in it. This interval we subject to the already described process of amalgamation, blackening up to and including the first point of arrest on the right and left respectively.

During our process of amalgamation we have, as before, affected at most a countable set of exterior points, and the content of the amalgamated intervals is the same as that of the new interval we have constructed and blackened. This new interval does not abut with (A', A) nor (B', B) nor with the remaining intervals of the given set, and, if we denote it by (A_1, B_1), each of the segments (A, A_1) and (B_1, B) has its end points points of arrest, and the length of either of them is less than $\frac{2}{3}(A, B)$.

We now proceed separately with (A, A_1) and (B_1, B), as we did with (A, B). Subsequently we repeat the same process separately with the 2^2 segments left after the amalgamation and blackening have been carried out in (A, A_1) and (B_1, B) ; and so on.

After the n-th stage we shall have blackened

$$1+2+2^2+\ldots+2^{n-1}=2^n-1$$

distinct intervals, obtained by amalgamation, without affecting more than a countable set of external points. The 2^n segments left over

are such that the length of each is less than $(\frac{2}{3})^n(A, B)$, while the end points of each are points of arrest.

Thus, after a countably infinite series of steps, we determine a definite new set of black intervals. By our mode of construction these black intervals are dense everywhere in (A, B) and abut nowhere; while, with the possible exception of a finite or countably infinite set of points, our new set of black intervals has the same external points as the given set. By § 28 the potency of the external points is therefore c, as was asserted.

§ 30. We notice, further, that in each of the 2^n-1 new black intervals at the end of the n-th stage the content of the amalgamated intervals is equal to that of the new black interval in which they lie; while, since in the remaining segments there is no interval of the given set greater than $(\frac{2}{3})^n(A, B)$, (which may be made as small as we please by choosing n sufficiently large), the sum of the intervals of the given set in the remaining segments may be made as small as we please by choosing n sufficiently large; for the same reason the sum of all the intervals of the new set which lie in one of these segments can be made as small as we please. It follows, therefore, that *the new set of black intervals has the same content as the given set.** Thus we have incidentally proved the following important theorem:—

THEOREM 4.—*Of the Ultimate Set.*

Any set of non-overlapping intervals determines uniquely, (after a finite or countably infinite series of steps of the type described), an ultimate set of non-abutting *intervals having the same content and, (with the possible exception of a finite or countably infinite set of points), the same external points.*

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* *Cf.* § 5.

On Closed Sets of Points defined as the Limit of a Sequence of Closed Sets of Points. *By* W. H. Young. Received and Read February 13th, 1902. Received, in revised form, October 3rd, 1902.

§ 1. Let $G_1, G_2, \ldots, G_n, \ldots$ be a countably infinite sequence of sets of points, each of which is closed and nowhere dense and is contained in all the succeeding sets.

This sequence of sets defines a limiting set G, defined by the two properties: (1) Every point of G_n is a point of G for all values of n; (2) no point of G exists which does not belong to some definite G_n of the sequence. This limiting set G may or may not be itself closed. Moreover, if we express it as the sum of sets of points thus,

$$G = G_1 + (G_2 - G_1) + (G_3 - G_2) + \ldots + (G_n - G_{n-1}) + \ldots, \text{ ad inf.},$$

the terms of the series may or may not be closed sets.

As regards the latter property, we make no assumption. We assume, however, in what follows that G is closed and nowhere dense. Notice the advantage of regarding G as defined by a sequence instead of by a series. We can in this way confine our attention to closed sets.

§ 2. The theorem we propose to prove is the following:—

Given any small positive quantity σ, *we can determine an integer* m *and a small positive quantity* ϵ *so that, for all values of* $n \geqslant m$, *all the intervals* $\geqslant \epsilon$ *of* G_n *are identical with all the intervals* $\geqslant \epsilon$ *of* G; *and the sum of the remaining intervals of* $G_n < \sigma$.

This result throws light on the mode in which G_n approaches its limit G. Moreover, it gives us as an immediate corollary Osgood's content theorem *that the content of* G *is the limit when* n *is infinite of the content of* G_n.

§ 3. Since each set is closed and nowhere dense, the complementary points fill up a set of intervals, which, in the enunciation of the theorem, we have called the intervals of the set. We shall in what follows refer to them as the "black intervals" of the set, the points of the set being conceived as white on a black ground. These in-

tervals are "open," or, in other words, the end points of the intervals are points of the set.

§ 4. It is, perhaps, here the place to point out a confusion which is easily made in the case when G is open.

Suppose Γ to be the set got by closing G. Then *

$$\Gamma = \mathfrak{M}(G, G'),$$

where G' is the first derived of G. If Γ is perfect,

$$\Gamma = G'.$$

We have by definition

$$\mathrm{Lt}\, G_n = G;$$

but we cannot deduce that

$$\mathrm{Lt}\, G_n' = G' = \Gamma$$

when Γ is perfect. In fact, as G_n is closed, G_n' will be contained in G_n, and cannot, therefore, generate more than, at most, G. Also, since by hypothesis G is open, G' contains points which are not contained in G. Thus $\mathrm{Lt}\, G_n'$ cannot be G' when G is open; while G' and Γ are not identical unless Γ is perfect.

It is probably this confusion which has led to the incorrect presentation of the subject of this paper in Schoenflies's *Bericht über die Mengenlehre*, p. 91. The following is a literal translation of the passage referred to:—

"If Γ be a closed set, and G_n a component set of Γ, and if G_n, as n increases, becomes dense everywhere † in Γ, then the limit of the content of G_n is the content of Γ. For the content of G_n is equal to that of G_n', (the first derived of G_n). And, if G_n, as n increases, becomes dense everywhere in Γ, *then* G_n' *converges towards* Γ itself, whence the theorem follows."

The fallacy in the proof is in the words in italics. We may add, it will appear from the sequel that the *statement* of the theorem given by Schoenflies is incorrect.

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* That is, the set consisting of all the elements of G and G', common elements being counted once only, *Math. Ann.*, p. 355, Schoenflies, p. 6.

† It is not quite clear in what sense the expression "If G_n is dense everywhere in Γ" is used. On p. 80, *loc. cit.*, Schoenflies gives the definition: A component set U of a closed set Q is said to be dense everywhere in Q, if the first derived set U' of U coincides with Q. This definition is clearly untenable, however, for, in the passage referred to, Schoenflies especially includes the consideration of closed sets which are not perfect; now for such a set Q the definition could have no meaning, for no component set of it could possibly have Q for its derived set. In what follows we shall ourselves always say that the component set U of a closed set Q is dense everywhere in Q, if U coincides with Q when the remaining limiting points of U are added to U. In other words, when $Q = \mathfrak{M}(U, U')$.

§ 5. A simple example has been given by Osgood, in which G is unclosed. Let G_n stand for all the proper fractions with n as denominator. Then G consists of all the rational numbers between 0 and 1. Evidently the content of G_n is 0, however great n may be. Here Γ is the continuum from 0 to 1, and its content is therefore unity.

§ 6. In the above Γ is dense everywhere. In the following example Γ is nowhere dense. We give it at some length, as it illustrates the essential features of the matter in hand and throws light on the mode of proof adopted in the sequel.

It will be noted that (1) G' may contain a more than countable set of points which are not points of any G'_n; (2) the content of G' may be different from the limit of the content of G'_n when n is made infinite.

§ 7. Consider the following sequence of sets :—

G_1 is H. J. S. Smith's ternary closed set* of the first kind in the segment (0, 1).

By means of repetitions of the processes by which G_1 was constructed in the segment (0, 1), we propose to construct a series of closed sets whose limit G, when closed by the addition of those limiting points not already included in it, is identical with H. J. S. Smith's ternary closed set of the second kind. This we denote by Γ. For this purpose the following should be noticed :—

(1) If we divide the segment in which an H. J. S. Smith's set of the first kind is given into 3^n equal parts, certain of them will be entirely black for the set, and in each of the others there is an H. J. S. Smith's set of the first kind.

(2) If we divide the segment in which an H. J. S. Smith's set of the second kind is given into $3^{\frac{1}{2}n(n+1)}$ parts, certain of them will be entirely black for the set, and in each of the others the given set has precisely the same form, though this form is not an H. J. S. Smith's set of the second kind, because the largest black interval in each part is not $\frac{1}{3}$ of that part.

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* *Proc. Lond. Math. Soc.*, Vol. VI., p. 948 ; *cf. Proc. Lond. Math. Soc.*, Vol. XXXIV., p. 286, footnote, for the reason of the insertion of the term "closed," *i.e.*, the set in question is that got by adding to H. J. S. Smith's ternary set of the first kind its limiting points. See, for a full discussion of these sets, the preceding paper by the present author on "Sets of Intervals."

Having premised this, we proceed to the construction. (See Fig. 1.) The largest black interval of G_1 is the same as the largest

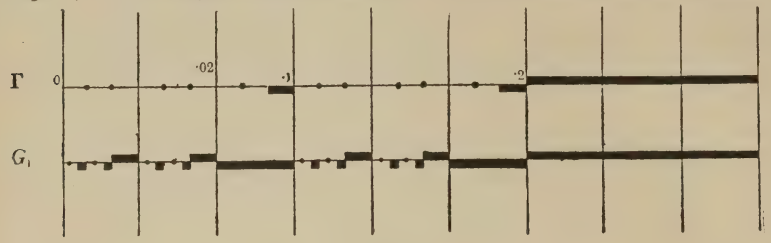

FIG. 1.

black interval of Γ, and in each of the two remaining segments (0, ·1) and (·1 to ·2) in the ternary notation, Γ has the same form, and G_1 consists of an H. J. S. Smith's set of the first kind. We need, therefore only consider what modification is necessary in the segment (0, ·1), the same modification being supposed made in the segment (·1, ·2), and the segment (·2, 1) being left unaltered. In the segment (0, ·1) Γ has its largest black interval of length $\frac{1}{3^{1+2}}$ on the extreme right; in all the other segments of the same length,

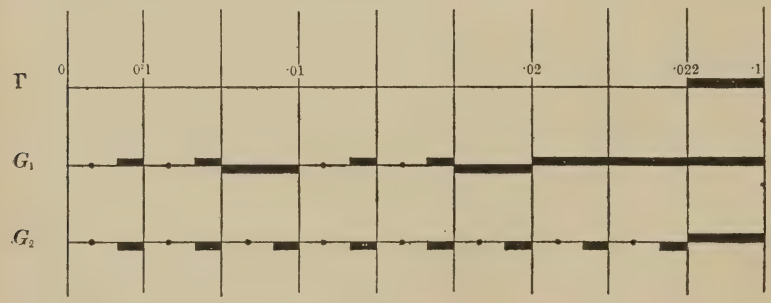

FIG. 2.

(0, ·0²1), &c., it has the same form, whereas G_1 has the form of an H. J. S. Smith's set of the first kind in only some of these segments, (·0²2, ·01), (·012, ·02), and (·02, ·1) being entirely black. If, however, in each of these three largest black intervals of G_1 we insert an H. J. S. Smith's set of the first kind, then the extreme right-hand segment (·022, ·1) will be entirely black for the new set G_2, and in each of the other eight segments G_2 will consist of an H. J. S. Smith's set of the first kind.

We can, therefore, as before, consider what modifications are necessary in the segment $(0, \cdot 0^2 1)$ only, the same modification being supposed made in the other seven segments, and the segment $(\cdot 022, \cdot 01)$ being left unaltered.

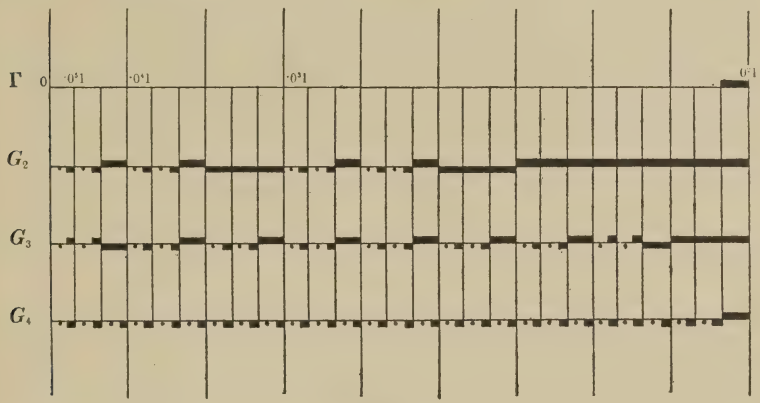

FIG. 3.

By a precisely similar argument as before, it is easily seen that, if we form G_3 by inserting in each of the three largest intervals of G_2 an H. J. S. Smith's set of the first kind, and G_4 by inserting in each of the 3^2 largest intervals of G_3 an H. J. S. Smith's set of the first kind, the extreme right-hand segment of length $\frac{1}{3^{1+2+3}}$, $(\cdot 0^3 222, \cdot 0^2 1)$, will be entirely black, as it is for Γ, and, in all the other segments of length $\frac{1}{3^{1+2+3}}$, G_4 will consist of an H. J. S. Smith's set of the first kind.

The general law is now obvious. We shall only have to consider the modifications necessary in $G_{1+1+2+3+\ldots+n}$ in the segment $(0, \cdot 0^{1+2+\ldots+n}1)$, in which it consists of an H. J. S. Smith's set of the first kind, the same modification being made in all the other segments of the same length except those which are by our construction already black for Γ.

The modification will consist in inserting H. J. S. Smith's sets of the first kind in the 3, 3^2, 3^3, ... 3^{n+1} largest intervals in turn to form $G_{\frac{1}{2}[n(n+1)]+2}$, $G_{\frac{1}{2}[n(n+1)]+3}$, up to $G_{\frac{1}{2}[(n+1)(n+2)]+1}$.

8. Comparing this series of sets with H. J. S. Smith's ternary closed set of the second kind, we see that, given any small quantity ϵ,

we can assign a stage m in the series such that for all values of $n \geqslant m$ all the black intervals of G_n which are $\geqslant \epsilon$ are identical with those $\geqslant \epsilon$ of H. J. S. Smith's ternary closed set of the second kind. But the content of G_n *is always zero ;* while that of H. J. S. Smith's ternary closed set of the second kind lies between $\frac{2}{3}$ and $\frac{1}{2}$.

The set G obviously consists of all the isolated points and limiting points on one side only of H. J. S. Smith's ternary set of the second kind, together with some of its limiting points on both sides. The set got by closing G *is* therefore H. J. S. Smith's ternary closed set of the second kind; that is, Γ.

We will now prove that G does not contain all the points of Γ, and is therefore unclosed. To do this it is sufficient to prove that the point

$$P = \cdot 1212212221\ldots$$

(where the right-hand side of the symbolic equation represents a ternary fraction, the number of 2's between consecutive 1's increasing each time by one), which is a limiting point on both sides of Γ (and a limiting point on one side only of Γ'), is an *internal* point of a definite black interval of G_n for *every* value n, and is therefore *not* a point of G. This interval is $(\cdot 12, \cdot 2)$ of G_1, $(\cdot 1212, \cdot 122)$ of G_2, $(\cdot 12122, \cdot 122)$ of G_3, $(\cdot 1212212, \cdot 121222)$ of G_4, $(\cdot 12122122, \cdot 12122)$ of G_5, $(\cdot 121221222, \cdot 121222)$ of G_6, $(\cdot 12122122212, \cdot 1212212222)$ of G_7, and so on. The general law is now evident, and hence the assertion is proved. Similarly it is evident that any ternary fraction of Γ which involves an infinite number of 2's * cannot belong to any G_n, since, apart from a finite number of figures at the beginning, the numbers of G_n involve only the figures 0 and 1. Each such point is a limiting point on both sides of Γ and is interior to a black interval of G_n whose length diminishes indefinitely as n increases indefinitely.

Hence the set G which we have constructed as the limiting set of a sequence of closed sets, nowhere dense, is open, and the content of the set obtained by closing G is different from that of the limit of G_n when n is made infinite.

9. In the following sections we proceed to prove the theorem enunciated in § 2. To do this we shall first state and prove a series of simple facts about the sets of the sequence.

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* Other than $\dot{2}$, of course.

10. *Any black interval* (P, Q) *of* G *or of* G_n "*leads up*" *to a definite black interval* (P', Q') *of* G_m $(m \leqslant n)$, *which is either identical with* (P, Q) *or of which* (P, Q) *forms a part.*

This follows at once from the fact that no point of G_m can lie between P and Q, since every point of G_m is a point of G_n and of G.

11. Hence *the sum of any number of black intervals* $\leqslant$ *the sum of the black intervals to which they lead up.*

12. *To each positive quantity* ϵ *there corresponds a perfectly definite finite integer* m *such that for all values of* $n \geqslant m$ *all the black intervals of* G_n *which are* $\geqslant \epsilon$ *are identical with all those* $\geqslant \epsilon$ *of* G.

For let (P, Q) be any black interval of G. Then, by the definition of G, there is a definite integer m_1 such that P and Q are points of G_{m_1}, but not both of G_{m_1-1}. The black interval (P, Q) will then belong to the set of black intervals of G_n for all values of $n \geqslant m_1$.

Now the number of black intervals of $G \geqslant \epsilon$ is finite, and hence it follows, by an immediate extension, that we can assign a definite finite integer m_2 such that all the black intervals $(P, Q) \geqslant \epsilon$ of G belong to the set of black intervals of G_n for all values of $n \geqslant m_2$. But G_{m_2} may have other black intervals $\geqslant \epsilon$.

Let (P', Q') be such an interval. Then P' and Q' are both points of G; but (P', Q') is not a black interval of G, nor does there lie between P' and Q' any interval $\geqslant \epsilon$ of G. Between P' and Q' we can mark a finite number of points of G, say $P_1, P_2, \ldots, P_k$, such that no one of the distances between successive points $P'P_1, P_1P_2, \ldots, P_kQ'$ is $\geqslant \epsilon$. Treating each such interval (P', Q') in the same way, we shall get a finite number of such points $P_1, P_2, \ldots$, since the number of *all—a fortiori, of some*—of the black intervals G_{m_2} which are $\geqslant \epsilon$ is finite. We can therefore determine a definite integer $m_3 \geqslant m_2$ such that G_{m_3} contains all these points, and G_{m_3} will therefore have no black interval $\geqslant \epsilon$ other than those belonging to G. G_{m_3} will be the first of our sequence of sets having this property, or else there will be a definite first set between it and G_{m_2} (this inclusive) having this property. Call this set G_m. Then m is evidently the integer sought.

13. To each positive quantity ϵ we have shown to correspond a definite integer m such that the black intervals of G_n $(n \geqslant m)$ and G differ only

in those intervals which are $< \epsilon$. It remains to show that we can choose ϵ so that the sum of the black intervals which are $< \epsilon$ is less than σ. Denote these sums by $R_n(\epsilon)$ and $R(\epsilon)$ respectively. We have only to refer to the example constructed in § 7 to convince ourselves that, without restrictions of some kind on the nature of G, this will not be the case, even when, without G being closed, the theorem of § 12 holds. Examination of that example suggests that the validity of this second part of the whole theorem enunciated in § 2 depends essentially on those points, which, without being end points of black intervals, are limiting points of G, and therefore limiting points *on both sides*. It is, in fact, evident that, provided G contains all its limiting points on *one side only*, (end points of black intervals), the first part of the theorem (§ 12) will hold, as it does in the example of § 7, but not necessarily the second part.

14. From the theorem of § 12 it follows immediately that, conversely, to each integer n there corresponds uniquely a positive quantity ϵ_n such that all the black intervals of G_n which are $\geqslant \epsilon_n$ are identical with all the black intervals of G which are $\geqslant \epsilon_n$, and ϵ_n is the *least* quantity of which this is true.

15. This being so, $\epsilon_m \geqslant \epsilon_n$, $(n > m)$, *and the quantities* ϵ_n *decrease without limit as* n *is increased indefinitely.* For, n being $> m$, any black interval of G_n which is $\geqslant \epsilon_m$ must lead up to a black interval of G_m which is $\geqslant \epsilon_m$; that is, to an interval which, being itself a black interval of G, contains no point of G except its end points; hence any black interval of G_n which is $\geqslant \epsilon_m$ must be identical with the black interval of G_m to which it leads up, and therefore is a black interval of G. On the other hand, any black interval of G which is $\geqslant \epsilon_m$ must lead up to a black interval $\geqslant \epsilon_m$ of G_n (§ 10), which is, therefore, identical with itself. Hence all the black intervals which are $\geqslant \epsilon_m$ of G and of G_n are identical, and therefore, by the definition of ϵ_n, $\epsilon_m \geqslant \epsilon_n$.

That the quantities ϵ_n form a sequence with zero as limit follows from the fact that, however small ϵ be taken, we can find a corresponding integer m, as in § 12, so that, by the definition of ϵ_m and by what has just been proved, $\epsilon_n \leqslant \epsilon_m \leqslant \epsilon$ $(n \geqslant m)$.

16. Now, since every black interval of $R_n(\epsilon_n)$ leads up to a black interval of $R_m(\epsilon_m)$, we have $R_n(\epsilon_n) \leqslant R_m(\epsilon_m)$; so *that the quantities*

$R_n(\epsilon_n)$ *form a sequence* whose limit is not greater than any one of them. If the limit of this sequence is zero, then it is evident that the theorem enunciated in § 2 is true, and we have a method of determining the integer m, while the quantity there denoted by ϵ might be taken as no other than ϵ_m, where we have determined m so that $R_m(\epsilon_m) < \sigma$.

17. Let us then, assuming that all that we have proved so far remains true, leave *out the assumption that G is closed,** and assume that the limit of the sequence of quantities $R_n(\epsilon_n)$ is other than zero. We know, by the example of § 7, that there is no contradiction in these assumptions.

We will now show that it is a consequence of the last assumption that G is unclosed. By § 16 this will complete the proof of the theorem of § 2.

18. Let the limit of the sequence $R_n(\epsilon_n)$ be the positive quantity λ. Then, assuming any small positive quantity $\tau < \frac{\lambda}{2}$ and any integer r, we can determine an integer m_r, such that, for all values of $n \geqslant m_r$,

$$R_n(\epsilon_n) - \lambda < \frac{\tau}{2^r}.$$

Now, taking, instead of the whole sequence $R_n(\epsilon_n)$, the partial sequence $R_{m_1}(\epsilon_{m_1})$, $R_{m_2}(\epsilon_{m_2})$, ..., and calling them for convenience $R_1(\epsilon_1)$, $R_2(\epsilon_2)$, ..., $R_n(\epsilon_n)$, ..., we have, for all values of n,

$$R_n(\epsilon_n) - \lambda < \frac{\tau}{2^n}. \tag{1}$$

Now take any small positive quantity $\sigma < \frac{\lambda}{2}$. We can determine a *finite* number, k_n, of the intervals $R_n(\epsilon_n)$, say, $d_{n,r}$, $(r = 1, 2, \ldots, k_n)$, such that the sum of the remaining intervals of $R_n(\epsilon_n)$ is less than $\frac{\sigma}{2^n}$. It follows that the sum of the intervals $d_{n,r}$, being greater

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* The assumptions are, stated at length, as follows:—G_1, G_2, ... are such that (1) each set is contained in all the preceding sets, (2) in any given segment, however small, it is possible to specify a part of it which is entirely black for every set G_n (so that G is nowhere dense), and (3), given any positive quantity ϵ, there corresponds a perfectly definite finite integer m, such that, for all values of $n \geqslant m$, all the black intervals of G_n which are $\geqslant \epsilon$, are identical with all those $\geqslant \epsilon$ of G.

than $R_n(\epsilon_n) - \frac{\sigma}{2^n}$, is greater than

$$\lambda - \frac{\sigma}{2^n}. \tag{2}$$

Now the intervals $d_{2,s}$, being intervals of $R_2(\epsilon_2)$, lead up to intervals of $R_1(\epsilon_1)$; and the sum of those intervals of $R_1(\epsilon_1)$ is, by (2), greater than $\lambda - \frac{\sigma}{2^2}$, and is therefore certainly greater than $\frac{\sigma}{2}$. Hence some of these intervals of $R_1(\epsilon_1)$ must be intervals $d_{1,r}$, because the sum of the remaining intervals of $R_1(\epsilon_1)$ was less than $\frac{\sigma}{2}$. For the same reason the sum of those intervals, (if any), $d_{2,s}$ which do not lead up to intervals $d_{1,r}$ is also less than $\frac{\sigma}{2}$. Hence, by (2), *the sum of those intervals* $d_{2,s}$ *which lead up to intervals* $d_{1,r}$ *is greater than* $\lambda - \frac{\sigma}{2^2} - \frac{\sigma}{2}$; *and the same is true of those intervals* $d_{1,r}$ *led up to by intervals* $d_{2,s}$. (3)

Now let us proceed a step further. The sum of the intervals $d_{3,t}$ is, as before, greater than $\lambda - \frac{\sigma}{2^3}$. The sum of those intervals of $R_2(\epsilon_2)$ which are not intervals $d_{2,s}$, or which, being intervals $d_{2,s}$, do not lead up to intervals $d_{1,r}$, is, by (3), less than

$$R_2(\epsilon_2) - \lambda + \frac{\sigma}{2} + \frac{\sigma}{2^2};$$

and therefore, by (1), is less than

$$\frac{\tau}{2^2} + \frac{\sigma}{2} + \frac{\sigma}{2^2}.$$

Hence the sum of those intervals $d_{3,t}$ which lead up to such intervals of $R_2(\epsilon_2)$ is also less than

$$\frac{\tau}{2^2} + \frac{\sigma}{2} + \frac{\sigma}{2^2}.$$

Therefore the sum of those intervals $d_{3,t}$ *which lead up to intervals* $d_{2,s}$ *contained in intervals* $d_{1,r}$ *is greater than*

$$\lambda - \frac{\sigma}{2^3} - \frac{\tau}{2^2} - \frac{\sigma}{2} - \frac{\sigma}{2^2},$$

which is

$$\lambda - \left\{ \frac{\sigma}{2} + \frac{\sigma}{2^2} + \frac{\sigma}{2^3} \right\} - \frac{\tau}{2^2},$$

while the same is true of the intervals $d_{2,s}$, *or the intervals* $d_{1,r}$ *to which they lead up.*

Similarly, going a step further, the sum of the remaining intervals of $R_3(\epsilon_3)$, being less than

$$R_3(\epsilon_3) - \lambda + \left(\frac{\sigma}{2} + \frac{\sigma}{2^2} + \frac{\sigma}{2^3}\right) + \frac{\tau}{2^2},$$

is, by (1), less than

$$\frac{\tau}{2^2} + \frac{\tau}{2^3} + \left(\frac{\sigma}{2} + \frac{\sigma}{2^2} + \frac{\sigma}{2^3}\right).$$

It follows that *the sum of those intervals* $d_{4,u}$ *which lead up to intervals* $d_{3,t}$, *which lead up to intervals* $d_{2,s}$, *contained in intervals* $d_{1,r}$, *is greater than*

$$\lambda - \frac{\sigma}{2^4} - \frac{\tau}{2^2} - \frac{\tau}{2^3} - \left\{\frac{\sigma}{2} + \frac{\sigma}{2^2} + \frac{\sigma}{2^3}\right\}$$

or

$$\lambda - \left\{\frac{\sigma}{2} + \frac{\sigma}{2^2} + \frac{\sigma}{2^3} + \frac{\sigma}{2^4}\right\} - \left\{\frac{\tau}{2^2} + \frac{\tau}{2^3}\right\},$$

while the same is true of the intervals $d_{3,t}$, *or* $d_{2,s}$, *or* $d_{1,r}$, *to which they lead up.*

In precisely the same manner it now follows by induction that there are a finite number of the intervals $d_{n,s}$ which lead up in turn to intervals $d_{m,r}$ for all values of $m < n$, and that the sum of these intervals $d_{n,s}$ is greater than

$$\lambda - \left\{\frac{\sigma}{2} + \frac{\sigma}{2^2} + \ldots + \frac{\sigma}{2^n}\right\} - \left\{\frac{\tau}{2^2} + \frac{\tau}{2^3} + \ldots + \frac{\tau}{2^{n-1}}\right\},$$

while the same is true of the intervals $d_{m,r}$ (for any fixed $m < n$) to which the intervals $d_{n,r}$ lead up. *A fortiori*, any one of these sums is greater than $\lambda - \sigma - \tau$, where σ and τ are as small as we please.

19. During this process we have been weeding out in turn (1) those intervals $d_{1,r}$ which do not contain intervals $d_{2,s}$; (2) those which, containing intervals $d_{2,s}$, do not contain intervals $d_{3,t}$ inside those intervals $d_{2,s}$; (3) those which, containing intervals $d_{2,s}$ with intervals $d_{3,t}$ inside them, do not contain intervals $d_{4,u}$ inside those intervals $d_{3,t}$; and so on. The sum of the intervals $d_{1,r}$ which are not weeded out, however, is, and remains always, greater than $\lambda - \sigma - \tau$. Hence the weeding process cannot, as far as it affects the intervals $d_{1,r}$, go on *ad infinitum;* but, after a finite number of stages, it comes to an

end. Let those intervals $d_{1,r}$ which are then left be denoted by

$$D_{1,r}, \quad (r = 1, 2, \ldots, \kappa_1;\ \kappa_1 \leqslant k_1).$$

Simultaneously we have been weeding out the intervals $d_{2,s}$. We have, firstly, weeded out intervals $d_{2,s}$ which are not contained in intervals $D_{1,r}$. But it may be that some of the intervals $d_{2,s}$ which lead up to intervals $D_{1,r}$ do not lead down to intervals $d_{n,t}$ for *all* values of $n > 2$. We have, however, weeded out from the intervals $d_{2,s}$ those which have not this property, just as we did the intervals $d_{1,r}$, and, since the sum of the intervals left is always $> \lambda - \sigma - \tau$, it follows, as before, that the weeding process, as far as it affects the intervals $d_{2,s}$, must also come to an end after a finite number of stages. Let us denote the intervals $d_{2,r}$ left at the end of the weeding process by

$$D_{2,r}, \quad (r = 1, 2, \ldots, \kappa_2;\ \kappa_2 \leqslant k_2).$$

Similarly we determine sets

$$D_{n,r}, \quad (r = 1, 2, \ldots, \kappa_n;\ \kappa_n \leqslant k_n)$$

for each successive value of n. The characteristics of these sets $D_{n,r}$ are (1) each interval $D_{n,r}$ leads up to intervals $D_{m,r}$ for all values of $m < n$; (2) each interval $D_{m,r}$ is led up to by intervals $D_{n,r}$ for all values of $n > m$; (3) the sum of all the intervals $D_{m,r}$ for given m is always $> \lambda - \sigma - \tau$; (4) the upper limit of the length of the intervals $D_{n,r}$, for given n, being $\leqslant \epsilon_n$, decreases without limit as n is indefinitely increased. Hence, if we choose one of these intervals for each successive value of n so that each of the chosen intervals lies within the preceding, they will define a limiting *point* P which will *not be exterior* to any one of the defining intervals. The end points of these defining intervals being points of G, P will certainly be a limiting point of G, (that is, a point of Γ), so that, unless P is an *end point* of each of the defining intervals after a definite stage, P, being *interior* to a black interval of every G_n, will belong to no G_n, and therefore not to G, so that G will be unclosed.

Since, however, the sum of the intervals $D_{n,r}$ for any assigned n is greater than $\lambda - \sigma - \tau$, it is easy to show that there must be limiting points which are *not* end points of the defining intervals. This is a consequence of a theorem which, being of general application, is stated and proved in the following section.

20. Theorem.—*If we have a finite number of non-overlapping intervals* $D_{n,r}, (r = 1, 2, \ldots, k_n)$, *for every integral value of* n *such that all the*

intervals $D_{n,r}$, for given n, lie inside the intervals $D_{m,r}$, for every value of $m < n$, and contain intervals $D_{p,r}$, for all values of $p > n$, then, if the sum S_n of all the intervals $D_{n,r}$, for every n, is greater than some assignable positive quantity λ, there are points such that each is interior to an interval $D_{n,r}$ for every value of n.

Let μ be any assigned small positive quantity. From each end of each interval $D_{n,r}$ let us cut off a fraction $\frac{1}{2^{n+r+1}} \frac{\mu}{S_n}$ of its length. The sum of the pieces cut off is less than $\frac{\mu}{2^n}$, and that of the curtailed intervals is greater than $S_n - \frac{\mu}{2^n}$.

Now inside the curtailed intervals $D_{m,r}$, $\left(\text{whose sum is} < S_m - \frac{\mu}{2^m}\right)$, there will fall parts of some, but not necessarily all, of the uncurtailed intervals $D_{m+1,r}$. But, since the sum of the parts cut off from the $D_{m,r}$ was less than $\frac{\mu}{2^m}$, it follows that the sum of the uncurtailed intervals and parts of intervals $D_{m+1,s}$ inside the curtailed intervals $D_{m,r}$ is greater than $S_{m+1} - \frac{\mu}{2^m}$. Hence the sum of the curtailed intervals and parts of intervals $D_{m+1,s}$ in the same is greater than

$$S_{m+1} - \frac{\mu}{2^m} - \frac{\mu}{2^{m+1}}.$$

The sum of the remaining intervals and part of intervals $D_{m+1,r}$ (uncurtailed) is less than $\frac{\mu}{2^m} + \frac{\mu}{2^{m+1}}$. Hence the sum of the uncurtailed intervals and parts of intervals $D_{m+2,t}$ which fall inside those curtailed intervals $D_{m+1,s}$ which lie in curtailed intervals $D_{m,r}$ is greater than $S_{m+2} - \frac{\mu}{2^m} - \frac{\mu}{2^{m+1}}$; and the sum of the *curtailed* intervals and parts of intervals $D_{m+2,t}$ in the same is greater than $S_{m+2} - \frac{\mu}{2^m} - \frac{\mu}{2^{m+1}} - \frac{\mu}{2^{m+2}}$.

In this way it appears that the sum of the curtailed intervals $D_{n,r}$ which fall inside curtailed intervals $D_{n-1,r}$, which themselves fall inside curtailed intervals $D_{n-2,r}$, and so on back to $D_{1,r}$, is greater than $S_n - \frac{\mu}{2} - \frac{\mu}{2^2} - \ldots - \frac{\mu}{2^n}$ and is therefore certainly greater than $S_n - \mu$; *a fortiori*, than $\lambda - \mu$.

Considering then the new sets of curtailed intervals, and leaving

out all such as do not lie in the preceding and contain all the following sets, we get a new series of sets, which we may denote by $D'_{n,r}$, and which have precisely the same attributes as before, except that $\lambda-\mu$ is substituted for λ.

If we now choose one of these intervals for each successive value of n, so that each lies within the preceding, they will determine a limiting point P or a limiting interval δ. In either case they determine one or more points which, not being *exterior* to any one of the defining intervals $D'_{n,r}$, is certainly *interior* to the corresponding uncurtailed intervals $D_{n,r}$. Q. E. D.

21. Applying this theorem to the results of the preceding article, we see that G will be unclosed; that is to say, *the hypothesis that the limit of the sequence of quantities* $R_n(\epsilon_n)$ *is other than zero is only compatible with our other conditions if* G *is unclosed, which proves the theorem of* § 2.

22. It should be noticed that the discussion of § 19 permits us to remove the restriction as to the finiteness of the number of intervals $D_{n,r}$. The more general theorem is as follows :—

If we have a set of non-overlapping intervals Δ_n *for every integral value of* n *such that all the intervals of* Δ_n *for given* n *lie inside the intervals of* Δ_m *for every value of* $m < n$, *and contain all the intervals of* Δ_p *for every value of* $p > n$, *then, if the sum* S_n *of all the intervals* Δ_n *for every* n *is greater than some assignable positive quantity* λ, *there are points such that each is interior to an interval of* Δ_n *for every value of* n.

23. It remains to deduce Osgood's theorem.* Denote by I and I_n the contents of G and G_n respectively, and by L the length of the finite segment over which they extend; then the contents of the corresponding sets of black intervals are $(L-I)$ and $(L-I_n)$.

By the theorem of § 2,

$$L-I_n-(L-I) = R_n(\epsilon)-R(\epsilon) < R_n(\epsilon) < \sigma \quad (n \geqslant m),$$

which proves the theorem.

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A Note on Unclosed Sets of Points defined as the Limit of a Sequence of Closed Sets of Points. *By* W. H. Young.

Received October 4th, 1902. Read November 13th, 1902.

1. The method pursued in the foregoing paper, "On Closed Sets of Points," enables us, without difficulty, to determine the most general condition that the limit of the content of such a sequence of sets G_n should be the content of the closed set Γ in which they become dense everywhere.

2. Let $G_1, G_2, \ldots$ be a countably infinite sequence of sets of points each of which is closed and nowhere dense and is contained in all the succeeding sets, and such that, given any segment, it is possible to specify a part of it which is entirely black for every set G_n, and let G be the limiting set and Γ the set obtained by adding to G those points, if any, which, without being points of G, are limiting points of G. *G and Γ will then be dense nowhere.* The following general theorem, of which the theorem of § 12 of the foregoing paper is a special case, viz., when G and Γ are identical, is the key to the whole matter.

3. *Given any two small positive quantities ϵ and σ, we can assign a definite integer m such that for all integers $n \geqslant m$ the difference between the sums of all the black intervals of Γ and of G_n which are $\geqslant \epsilon$ is less than σ.*

The proof is precisely on the lines of that of the corresponding theorem of § 12, *loc. cit.* We have, however, to notice that, if (P, Q) be a black interval of Γ which is $\geqslant \epsilon$, and we lengthen (P, Q) at each end by a length $\frac{\sigma}{2k}$ (where k is the number of black intervals of Γ which are $\geqslant \epsilon$), there must be a point P_1 of G in the little piece added on at P, though P is not necessarily a point of G, and similarly there must be a point Q_1 of G in the little piece added on at Q.

If we determine m_1 so that both P_1 and Q_1 are points of G_{m_1}, then, since (P, Q) is certainly entirely black for G_{m_1}, either (P_1, Q_1)

is a black interval of G_{m_1} or else there is a black interval of G_{m_1} whose end points lie in (P, P_1) and (Q, Q_1) respectively.

Doing this with each black interval $\geqslant \epsilon$ of Γ, we determine an integer m_2 such that the sum of those black intervals of G_{m_2} which are led up to by all the black intervals of Γ which are $\geqslant \epsilon$ differs from the sum of the latter black intervals by less than σ.

As in § 12, *loc. cit.*, however, G_{m_2} may have other black intervals $\geqslant \epsilon$; but these can be disposed of and the proof completed precisely as was done there for the closed G.

4. Thus, if I and I_n be the contents of Γ and G_n respectively, and $R(\epsilon)$ and $R_n(\epsilon)$ be the sums of those black intervals of Γ and G_n respectively which are $< \epsilon$, we have

$$I - I_n - R_n(\epsilon) + R(\epsilon) < \sigma.$$

Now we can choose ϵ so as to make $R(\epsilon)$ as small as we please, so that *it is evidently necessary and sufficient for the equality of* I *and* $\lim I_n$ $(n = \infty)$ *that we may be able to choose* ϵ *so that, for all integers* n *greater than a certain integer,* $R_n(\epsilon)$ *may be less than any assigned small quantity.*

Summation of a certain Series. *By* A. C. Dixon.

Received October 10th, 1902. Read November 13th, 1902.*

The object of the present note is to find the sum of the infinite series

$$1 + \frac{\alpha\beta\gamma}{\delta\epsilon} + \frac{\alpha(\alpha+1)\,\beta(\beta+1)\,\gamma(\gamma+1)}{2!\,\delta(\delta+1)\,\epsilon(\epsilon+1)} + \ldots$$

in the case when $\beta + \delta = \gamma + \epsilon = \alpha + 1.$

The condition for convergency will be supposed satisfied; that is, the real part of $\delta + \epsilon - \alpha - \beta - \gamma$ will be taken to be positive.

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* The original MS. was lost in transit. The paper has been rewritten by the author, January, 1903.

Let $F(\alpha, \beta, \gamma, x)$ and $F(\alpha, \beta, \gamma, \delta, \epsilon, x)$ stand for the two series

$$1+\frac{\alpha\beta}{\gamma}\,x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)}\,x^2+\ldots,$$

$$1+\frac{\alpha\beta\gamma}{\delta\epsilon}\,x+\frac{\alpha(\alpha+1)\beta(\beta+1)\gamma(\gamma+1)}{2!\,\delta(\delta+1)\epsilon(\epsilon+1)}\,x^2+\ldots.$$

Let $\int_{(a,b)}$ denote integration round a closed path consisting of four loops, each of which begins and ends at a certain point c on the straight line between a and b, and which pass respectively round b positively, a positively, b negatively, a negatively; the subject of integration will contain powers of such factors as x, $1-x$ which are initially real and positive; its initial value will be fixed by taking the logarithms of these factors to have their real values.

Let $\int_{a\,(b)}$ denote integration along a loop starting and ending at a, and passing positively round b, the initial value of the subject being fixed as before. Thus we have, when r is positive,

$$\int_{(0,1)} x^{rm-1}(1-x^r)^{n-1}\,dx = -\int_{(1,0)} x^{rm-1}(1-x^r)^{n-1}\,dx$$

$$= \frac{1}{r}(1-e^{2rm\iota\pi})(1-e^{2n\iota\pi})\,B(m,n)$$

$$= -\frac{4}{r}\,e^{(rm+n)\iota\pi}\sin rm\pi\,\sin n\pi\,B(m,n),$$

and, if the real part of m is also positive,

$$\int_{0\,(1)} x^{rm-1}(1-x^r)^{n-1}\,dx = \frac{1}{r}(1-e^{2n\iota\pi})\,B(m,n)$$

$$= -\frac{2\iota}{r}\,e^{n\iota\pi}\sin n\pi\,B(m,n).$$

Now $F(\alpha,\beta,\gamma,\delta,\epsilon,t)\int_{(0,1)} z^{\gamma-1}(1-z)^{\epsilon-\gamma-1}\,dz$

$$= \int_{(0,1)} F(\alpha,\beta,\delta,tz)\,z^{\gamma-1}(1-z)^{\epsilon-\gamma-1}\,dz$$

and $F(\alpha,\alpha-\delta+1,\delta,x) = (1+x)^{-\alpha}F\left(\frac{\alpha}{2},\ \frac{\alpha+1}{2},\ \delta,\ \frac{4x}{(1+x)^2}\right)$.

Hence

$$F(a, a-\delta+1, a-\epsilon+1, \delta, \epsilon, t)\int_{(0,1)} z^{a-\epsilon}(1-z)^{2\epsilon-a-2}\,dz$$

$$= \int_{(0,1)} F\left\{\frac{a}{2},\ \frac{a+1}{2},\ \delta,\ \frac{4tz}{(1+tz)^2}\right\}(1+tz)^{-a}\,z^{a-\epsilon}(1-z)^{2\epsilon-a-2}\,dz$$

$$= (1+t)^{2\epsilon-a-2}\,2^{1-a}\,t^{1-\epsilon}\int_{[(1-t)/(1+t),\ 1]} F\left(\frac{a}{2},\ \frac{a+1}{2},\ \delta,\ 1-u^2\right)$$
$$\times (1-u^2)^{a-\epsilon}\left(u-\frac{1-t}{1+t}\right)^{2\epsilon-a-2}du,$$

by the substitution $u = (1-tz)/(1+tz)$.

Now

$$F\left(\frac{a}{2},\ \frac{a+1}{2},\ \delta,\ 1-u^2\right)$$

$$= \frac{\Gamma\delta\,\Gamma(\delta-a-\frac{1}{2})}{\Gamma\left(\delta-\frac{a}{2}\right)\Gamma\left(\delta-\frac{a+1}{2}\right)}\,F\left(\frac{a}{2},\ \frac{a+1}{2},\ a-\delta+\tfrac{3}{2},\ u^2\right)$$

$$+\frac{\Gamma\delta\,\Gamma(a+\frac{1}{2}-\delta)}{\Gamma\frac{a}{2}\,\Gamma\frac{a+1}{2}}\,u^{2\delta-2a-1}\,F\left(\delta-\frac{a}{2},\ \delta-\frac{a+1}{2},\ \delta-a+\tfrac{1}{2},\ u^2\right).$$

Let this substitution be made in the integral, and consider the two parts separately. In the first the subject of integration has no singularity at 0, and thus the loop containing $\dfrac{1-t}{1+t}$ may be taken to contain 0 as well, and we may put $t=1$ at once. The expression to be calculated is therefore

$$\int_{(0,1)} F\left(\frac{a}{2},\ \frac{a+1}{2},\ a-\delta+\tfrac{3}{2},\ u^2\right)(1-u^2)^{a-\epsilon}\,u^{2\epsilon-a-2}\,du$$

$$\text{or}\quad \int_{(0,1)}\int_{(0,1)} x^{\frac{1}{2}a-1}(1-x)^{\frac{1}{2}a-\delta+\frac{1}{2}}(1-xu^2)^{-\frac{1}{2}(a+1)}(1-u^2)^{a-\epsilon}u^{2\epsilon-a-2}\,dx\,du$$
$$\div\int_{(0,1)} x^{\frac{1}{2}a-1}(1-x)^{\frac{1}{2}a-\delta+\frac{1}{2}}\,dx.$$

In the double integral change the order of integration and use the substitution

$$u^2 = y^2/(1-x+xy^2).$$

The expression thus becomes

$$\int_{(0,1)}\int_{(0,1)} (1-y^2)^{\alpha-\epsilon}\, y^{2\epsilon-\alpha-2}\,(1-x)^{\alpha-\delta-\epsilon+1}\, x^{\frac{1}{2}\alpha-1}\, dy\, dx$$

$$\div \int_{(0,1)} x^{\frac{1}{2}\alpha-1}\,(1-x)^{\frac{1}{2}(\alpha+1)-\delta}\, dx,$$

that is, $-2e^{\epsilon\iota\pi}\sin(\alpha-\epsilon)\,\pi\sin(2\epsilon-\alpha)\,\pi\, B\left(\alpha-\epsilon+1,\ \epsilon-\frac{\alpha+1}{2}\right)$

$$\times -4e^{(\frac{3}{2}\alpha-\delta-\epsilon)\iota\pi}\sin(\alpha-\delta-\epsilon)\,\pi\sin\frac{\alpha\pi}{2}\, B\left(\frac{\alpha}{2},\ \alpha-\delta-\epsilon+2\right)$$

$$\div -4e^{(\alpha-\delta-\frac{1}{2})\iota\pi}\sin\left(\frac{\alpha+3}{2}-\delta\right)\pi\sin\frac{\alpha\pi}{2}\, B\left(\frac{\alpha}{2},\ \frac{\alpha+3}{2}-\delta\right).$$

The coefficient of this in $F(\alpha, \alpha-\delta+1, \alpha-\epsilon+1, \delta, \epsilon, 1)$ is

$$2^{2\epsilon-2\alpha-1}\,\frac{\Gamma\delta\,\Gamma(\delta-\alpha-\frac{1}{2})}{\Gamma\left(\delta-\frac{\alpha}{2}\right)\Gamma\left(\delta-\frac{\alpha+1}{2}\right)}$$

$$\div\ -4e^{\epsilon\iota\pi}\sin(\alpha-\epsilon)\,\pi\sin(2\epsilon-\alpha)\,\pi\, B(\alpha-\epsilon+1,\ 2\epsilon-\alpha-1).$$

Hence the value of the term as a whole is, after reduction,

$$-2^{2\epsilon-2\alpha-2}\, e^{[\frac{1}{2}(\alpha+1)-\epsilon]\iota\pi}\,\Gamma\delta\,\Gamma\epsilon\,\Gamma(\delta-\alpha+\tfrac{1}{2})\,\Gamma(\alpha-\delta+\tfrac{1}{2})\,\Gamma\left(\epsilon-\frac{\alpha+1}{2}\right)$$

$$\div\,\Gamma(\delta+\epsilon-\alpha-1)\,\Gamma\left(\delta-\frac{\alpha}{2}\right)\Gamma\frac{\alpha+1}{2}\,\Gamma(\tfrac{3}{2}\alpha-\delta-\epsilon+2)\,\Gamma(2\epsilon-\alpha-1).$$

To find the other term we must evaluate

$$\lim_{\kappa=0}\int_{(\kappa,1)} F\left(\delta-\frac{\alpha}{2},\ \delta-\frac{\alpha+1}{2},\ \delta-\alpha+\tfrac{1}{2},\ u^2\right)$$

$$\times u^{2\delta-2\alpha-1}\,(1-u^2)^{\alpha-\epsilon}\,(u-\kappa)^{2\epsilon-\alpha-2}\, du,$$

where κ has been written for $(1-t)/(1+t)$.

Take κ to be small and positive and let the loops start from the point $\sqrt{\kappa}$. Then on the loops round 1 we have $|u| \geq \sqrt{\kappa}$, and thus $(u-\kappa)/u = 1$ in the limit, so that we may put $\kappa = 0$ at once in the subject for this part of the path. The loop round κ may be taken to be a circle having the points $\sqrt{\kappa}$ and $\frac{1}{2}\kappa$ as the ends of a diameter. On this circle we have $\left|\frac{u-\kappa}{u}\right| < 1$ and $> 1-\sqrt{\kappa}$; the integral is then a quantity of the same order of magnitude as $\int |u^{2\delta+2\epsilon-3\alpha-3}\, du|$ taken

round the same circle; that is, it vanishes in the limit if the real part of $2\delta+2\epsilon-3\alpha-2$ is positive. Hence, under this condition, which we have already supposed fulfilled, we may replace the integral by

$$(1-e^{(2\epsilon-\alpha)2\iota\pi})\int_{0,(1)} F\left(\delta-\frac{\alpha}{2},\ \delta-\frac{\alpha+1}{2},\ \delta-\alpha+\tfrac{1}{2},\ u^2\right) \times (1-u^2)^{\alpha-\epsilon}u^{2\delta+2\epsilon-3\alpha-3}\,du,$$

since the subject of integration at the end of the second loop round 1 has its original value multiplied by $e^{(2\epsilon-\alpha)2\iota\pi}$.

The value of this integral may be found by means of the same substitutions as before, and the second term of the expression $F(\alpha,\ \alpha-\delta+1,\ \alpha-\epsilon+1,\ \delta,\ \epsilon,\ 1)$ is the product of the coefficient

$$2^{2\epsilon-2\alpha-1}\frac{\Gamma\delta\,\Gamma(\alpha+\frac{1}{2}-\delta)}{\Gamma\frac{\alpha}{2}\,\Gamma\frac{\alpha+1}{2}}$$

$$\div\ -4e^{\epsilon\iota\pi}\sin(\alpha-\epsilon)\,\pi\sin(2\epsilon-\alpha)\,\pi\,B(\alpha-\epsilon+1,\ 2\epsilon-\alpha-1),$$

by the value of the integral, namely,

$$(1-e^{(2\epsilon-\alpha)2\iota\pi})\times -\iota e^{(\alpha-\epsilon+1)\iota\pi}\sin(\alpha-\epsilon+1)\,\pi\,B\left(\alpha-\epsilon+1,\ \delta+\epsilon-\frac{3\alpha}{2}-1\right)$$

$$\times -4e^{(\frac{3}{2}\alpha-\epsilon-\frac{1}{2})\iota\pi}\sin\left(\delta-\frac{\alpha+1}{2}\right)\pi\sin(\alpha-\delta-\epsilon+2)\,\pi \times B\left(\delta-\frac{\alpha+1}{2},\ \alpha-\delta-\epsilon+2\right)$$

$$\div\ -4e^{(\delta-\alpha-\frac{3}{2})\iota\pi}\sin\left(\delta-\frac{\alpha+1}{2}\right)\pi\sin\left(1-\frac{\alpha}{2}\right)\pi \times B\left(\delta-\frac{\alpha+1}{2},\ 1-\frac{\alpha}{2}\right).$$

After reduction this product becomes

$$-2^{2\epsilon-2\alpha-2}e^{(\frac{3}{2}\alpha-\delta-\epsilon)\iota\pi}\,\Gamma\delta\Gamma\epsilon\Gamma(\delta-\alpha+\tfrac{1}{2})\,\Gamma(\alpha-\delta+\tfrac{1}{2})\,\Gamma\left(\delta+\epsilon-\frac{3\alpha}{2}-1\right)$$

$$\div\,\Gamma(\delta+\epsilon-\alpha-1)\,\Gamma\left(\delta-\frac{\alpha}{2}\right)\Gamma\frac{\alpha+1}{2}\,\Gamma\left(\frac{\alpha+3}{2}-\epsilon\right)\Gamma(2\epsilon-\alpha-1),$$

and, adding this to the former result, we have as the sum of the

original series

$$2^{2\epsilon-2\alpha-2}\frac{\Gamma\delta\Gamma\epsilon\Gamma\left(\delta+\epsilon-\frac{3\alpha}{2}-1\right)\Gamma\left(\epsilon-\frac{\alpha+1}{2}\right)}{\Gamma(\delta+\epsilon-\alpha-1)\Gamma\left(\delta-\frac{\alpha}{2}\right)\Gamma\frac{\alpha+1}{2}\Gamma(2\epsilon-\alpha-1)}\frac{\pi}{\sin(\delta-\alpha+\frac{1}{2})\pi}$$

$$\times\left[-\frac{e^{[\frac{1}{2}(\alpha+1)-\epsilon]\,i\pi}}{\Gamma\left(\delta+\epsilon-\frac{3\alpha}{2}-1\right)\Gamma\left(\frac{3\alpha}{2}-\delta-\epsilon+2\right)}\right.$$

$$\left.-\frac{e^{(\frac{3}{2}\alpha-\delta-\epsilon)\,i\pi}}{\Gamma\left(\epsilon-\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+3}{2}-\epsilon\right)}\right].$$

The expression in square brackets

$$=-\frac{1}{\pi}e^{[\frac{1}{2}(\alpha+1)-\epsilon]\,i\pi}\sin\left(\frac{3\alpha}{2}-\delta-\epsilon\right)\pi+\frac{1}{\pi}e^{(\frac{3}{2}\alpha-\delta-\epsilon)\,i\pi}\sin\left(\frac{\alpha+1}{2}-\epsilon\right)\pi$$

$$=\frac{1}{\pi}\sin(\delta-\alpha+\tfrac{1}{2})\pi.$$

We also have

$$\Gamma(2\epsilon-\alpha-1)=\Gamma\left(\epsilon-\frac{\alpha}{2}\right)\Gamma\left(\epsilon-\frac{\alpha+1}{2}\right)2^{2\epsilon-\alpha-2}\pi^{-\frac{1}{2}}.$$

Hence, finally,

$$F(\alpha,\ \alpha-\delta+1,\ \alpha-\epsilon+1,\ \delta,\ \epsilon,\ 1)$$

$$=2^{-\alpha}\pi^{\frac{1}{2}}\frac{\Gamma\delta\Gamma\epsilon\Gamma(\delta+\epsilon-\frac{3}{2}\alpha-1)}{\Gamma\left(\delta-\frac{\alpha}{2}\right)\Gamma\left(\epsilon-\frac{\alpha}{2}\right)\Gamma\frac{\alpha+1}{2}\Gamma(\delta+\epsilon-\alpha-1)},$$

if the real part of $\delta+\epsilon-\frac{3}{2}\alpha-1$ is positive.

If in this we write $\delta=\epsilon=1$, we arrive at Prof. Morley's result (*Proceedings*, Vol. XXXIV., p. 401). The present investigation was suggested by the reading of Prof. Morley's paper. I learn from Prof. Morley that he had hoped to sum the series of this paper by his method.

Thursday, December 11th, 1902.

Prof. H. LAMB, F.R.S., President, in the Chair.

Eighteen members present.

Mr. J. H. Grace was admitted into the Society.

The Auditor (Mr. J. H. Grace) having made his report, the President moved that the Treasurer's report (read at the November meeting) be adopted, and that the thanks of the Society be given to the Treasurer and the Auditor. The motion was seconded by Mr. Sheppard, and carried *nem. con.*

The following papers were communicated by their authors:—

Dr. H. F. Baker: (1) On the Calculation of the Finite Equations of a Continuous Group. (2) On the Integration of Linear Differential Equations. (3) On some cases of Matrices with Linear Invariant Factors.

Mr. G. H. Hardy: The expression of the Double Zeta and Gamma Functions in terms of Elliptic Functions.

Prof. M. J. M. Hill: The Continuation of the Power Series for arc sin x.

Mr. E. T. Whittaker: The Functions associated with the Parabolic Cylinder in Harmonic Analysis.

Lieut.-Col. Cunningham took the Chair while the President gave an account of his recent researches on "Wave Motion in Two Dimensions."

The following papers were communicated from the Chair:—

Prof. L. E. Dickson: (1) The Abstract Group Simply Isomorphic with the Group of Linear Fractional Transformations in a Galois Field. (2) Generational Relations of an Abstract Simple Group of Order 4080.

Mr. H. M. Macdonald: Some Applications of Fourier's Theorem.

Rev. F. H. Jackson: Series connected with the Enumeration of Partitions.

Mr. W. H. Young: Sets of Intervals, Part II., Overlapping Intervals.

Mr. J. H. Grace: Perpetuants.

The following presents were made to the Library :—

"Educational Times," Vol. LV., No. 500 ; 1902.

"Indian Engineering," Vol. XXXII., Nos. 17–20 ; 1902.

"L'Enseignement Mathématique," Année IV., No. 6 ; Paris, 1902.

"Société des Naturalistes de Varsovie, Comptes Rendus," 1901.

Martin, Emilie Norton.—"On the Imprimitive Substitution Groups of Degree 15," 4to ; Baltimore, 1901.

Mittag-Leffler, G.—"Sur la représentation analytique d'une branche uniforme d'une fonction monogène," 4to ; 1902.

The following exchanges were received :—

"Bulletin of the American Mathematical Society," Vol. IX., No. 3 ; New York, 1902.

"Proceedings of the American Philosophical Society," Vol. XLI., No. 170 ; Philadelphia, 1902.

"Jornal de Sciencias Mathematicas," Vol. XV., No. 1 ; Coimbra, 1902.

"Beiblätter zu den Annalen der Physik," Bd. XXVI., Heft 11 ; Leipzig, 1902.

"Periodico di Matematica," Anno XVIII., Fasc. 3 ; Livorno, 1902.

"Supplemento al Periodico di Matematica," Anno VI., Fasc. 1 ; Livorno, 1902.

"Proceedings of the Physical Society," Vol. XVIII., Pt. 3 ; London, 1902.

"Bulletin des Sciences Mathématiques," Tome XXVI., Oct., Nov., 1902 ; Paris.

"Reale Accademia dei Lincei—Rendiconti," Vol. XI., Sem. 2, Fasc. 9, 10 ; Roma, 1902.

"Prace Matematyczno-Fizyczne," Tome XIII. ; Warsaw, 1902.

The Abstract Group Simply Isomorphic with the Group of Linear Fractional Transformations in a Galois Field. *By* L. E. Dickson, Ph.D. Received November 13th, 1902. Read December 11th, 1902.

1. The object of this paper is the determination of a simple set of generational relations for the abstract group $G_{\frac{1}{2}p^n(p^{2n}-1)}$ simply isomorphic with the group of all linear fractional transformations on one variable with coefficients belonging to the $GF[p^n]$, $p>2$, and having determinant unity. The group is known to be simple, except for $p^n = 3$.

For $n = 1$, the group may be generated by two operators T and S, subject to the generational relations (§ 20)

$$T^2 = I, \quad S^p = I, \quad (ST)^3 = I, \quad (S^\tau T S^{2/\tau} T)^2 = I,$$

where $\tau = 3, 4, \ldots, \frac{1}{2}(p-1)$. However, only one of the positive residues of t, $2/t$, $-2/t$ need be taken as a value of τ. Thus, for $p = 3$, or for $p = 5$, τ takes no values, so that only the first three relations occur. For $p = 7$, $\tau = 3$; for $p = 11$, $\tau = 3, 4$; for $p = 13$, $\tau = 3, 4$; for $p = 17$, $\tau = 3, 4, 6, 7$; for $p = 19$, $\tau = 3, 4, 5, 6$; for $p = 23$, $\tau = 3, 4, 5, 6, 9$.

For general n, the group may be generated by the operators T and S_λ (where λ runs through the series of p^n marks of the field) subject to the generational relations

$$(1) \quad T^2 = I, \quad S_0 = I, \quad S_\lambda S_\mu = S_{\lambda+\mu} \quad (\lambda, \mu \text{ any marks}),$$

$$(2) \quad (S_1 T)^3 = I,$$

$$(3) \quad (S_\tau T S_{2/\tau} T)^2 = I \quad (\tau \text{ any mark} \neq 0).$$

Owing to redundancies in these relations, we may restrict λ, μ, τ to a part of the marks of the field (compare §§ 20, 21).

While the preceding statements are doubtless true for any prime $p > 2$ and any positive integer n, they are here proved only when $p^n < 49$; viz., for $p^n = 3, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 37, 41, 43, 47$. Hence they hold at least for the groups of orders < 58800. For the general case, the validity of the theorems is made

to depend upon a problem in the theory of numbers. While the latter is readily treated in any particular case, its general solution is not undertaken. Perhaps direct proofs of the above theorems may be given; for example, by showing that the order of the abstract group defined by (1), (2), and (3) is at most $\frac{1}{2}p^n(p^{2n}-1)$.

2. The group $G_{\frac{1}{2}p^n(p^{2n}-1)}$ may be generated by the operators T and S_λ subject to the relations (1) and

$$(4)\quad S_\lambda T S_\mu T S_{(\lambda-1)/(\lambda\mu-1)} T S_{-(\lambda\mu-1)} T S_{(\mu-1)/(\lambda\mu-1)} T = I,$$

where λ and μ are arbitrary marks such that $\lambda\mu \neq 1$.* Relation (4) will be designated briefly by either of the symbols

$$(4')\quad [\lambda, \mu], \quad \left[\lambda, \mu, \frac{\lambda-1}{\lambda\mu-1}, -(\lambda\mu-1), \frac{\mu-1}{\lambda\mu-1}\right].$$

3. For $\lambda = 0$ or 1, or for $\mu = 0$ or 1, relation (4) reduces to (2) in view of (1). One of these cases arises if any subscript in (4) equals 0 or 1.

If λ and μ each differs from both 0 and 1, the product of any two consecutive subscripts in (4) differs from 1 (the first subscript being regarded as consecutive with the last), so that the following symbols may be employed for (4):—

(5)

$$\left[\mu, \frac{\lambda-1}{\lambda\mu-1}\right], \left[\frac{\lambda-1}{\lambda\mu-1}, -\lambda\mu+1\right], \left[-\lambda\mu+1, \frac{\mu-1}{\lambda\mu-1}\right], \left[\frac{\mu-1}{\lambda\mu-1}, \lambda\right].$$

Each relation (5) *has the same five subscripts and in the same cyclic order as the five subscripts in* $[\lambda, \mu]$. Hence each follows from $[\lambda, \mu]$ by simple transformation, upon applying (1). To prove the first statement, we need only verify that the operation† C which replaces λ, μ by μ, $\frac{\lambda-1}{\lambda\mu-1}$, respectively, will replace the latter pair by

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* A more general theorem, valid for an arbitrary field, was established by Prof. E. H. Moore, and communicated (without proof) to me in 1899. I then constructed the proof for finite fields given in my *Linear Groups*, § 278, Corollary.

† Note that we have a birational transformation of period 5,

$$C\begin{cases}\lambda' = \mu,\\ \mu' = \dfrac{\lambda-1}{\lambda\mu-1},\end{cases} \qquad C^4 = C^{-1}\begin{cases}\lambda' = \dfrac{\mu-1}{\lambda\mu-1},\\ \mu' = \lambda.\end{cases}$$

Then C, C^2, C^3, C^4 replace λ by the remaining four subscripts in (4).

$\frac{\lambda-1}{\lambda\mu-1}$, $-\lambda\mu+1$; the latter in turn by $-\lambda\mu+1$, $\frac{\mu-1}{\lambda\mu-1}$; the latter by $\frac{\mu-1}{\lambda\mu-1}$, λ; and the latter by λ, μ. Hence C produces a cyclic permutation of the subscripts in (4). Hence the relations (5) are derived from $[\lambda, \mu]$ by applying C, C^2, C^3, C^4, respectively.

4. For $\mu = -1$, so that $\lambda \neq -1$, relation (4) becomes

$$(S_{\lambda+1} T S_{2/(\lambda+1)} T)^2 = I,$$

upon replacing $TS_{-1}T$ by $S_1 TS_1$, as may be done by (2). Giving to λ all possible values, we obtain relations (3).

When any other subscript in (4) equals -1, the relation follows, as in § 3, from that in which the second subscript is -1, upon applying (1). Such a case arises only for $\lambda = -1$ or for

$$(6) \quad \mu = \frac{2}{\lambda} - 1, \quad \mu = \frac{2}{\lambda}, \quad \text{or} \quad \mu = \frac{2}{\lambda+1}.$$

5. The chief aim of the paper is to show that, for $p \neq 2$,* relations (4) all follow from relations (1), (2), (3). This statement has been proved in §§ 3, 4 for relations (4) in which any subscript equals 0, $+1$, or -1. In particular, it is true for relations $[\lambda, \mu]$ in which one of the relations (6) holds.

Henceforth, we say that a relation *follows* when it can be derived from (1), (2), and (3); that a relation *follows from another* when it can be derived from the latter and (1), (2), (3).

6. Theorem.—*Either of the relations* $[\lambda, \mu]$ *and* $\left[\mu, \frac{2}{\mu} - \lambda\right]$ *follows from the other.*†

Transforming both members of (4) by $S_{\lambda-2/\mu}$ and replacing $S_{2/\mu} TS_\mu T$, the initial product in the resulting relation, by its inverse $TS_{-\mu} TS_{-2/\mu}$, following from (3), we get

$$TS_{-\mu} TS_{(2-\mu-\lambda\mu)/[\mu(\lambda\mu-1)]} TS_{-\lambda\mu+1} TS_{(\mu-1)/(\lambda\mu-1)} TS_{\lambda-2/\mu} = I.$$

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* For $p = 2$, relation (3) follows from (1).

† Note that here also we have a birational transformation

$$O \begin{cases} \lambda' = \mu, \\ \mu' = \frac{2}{\mu} - \lambda, \end{cases} \qquad O^{-1} \begin{cases} \lambda' = \frac{2}{\lambda} - \mu, \\ \mu' = \lambda. \end{cases}$$

Taking the inverse of each member, we obtain relation $\left[\mu, \frac{2}{\mu} - \lambda\right]$. To denote briefly that the latter follows from $[\lambda, \mu]$, we write

$$(7) \quad [\lambda, \mu] \sim \left[\mu, \frac{2}{\mu} - \lambda\right].$$

7. Certain results, not used below, may be derived as corollaries to the preceding theorem. From (1), (2), (3) can be derived the following relations (λ being an arbitrary mark) :—

$$\left[\lambda, \frac{2}{\lambda}\right], \quad \left[\lambda, \frac{2}{\lambda} \pm 1\right], \quad \left[\lambda, \frac{2}{\lambda \pm 1}\right], \quad \left[\lambda, \frac{3-\lambda}{\lambda+1}\right],$$

$$\left[\lambda, \frac{\pm 2}{\lambda^2 \pm \lambda}\right], \quad \left[\frac{\pm 2}{\lambda^2 \pm \lambda}, \lambda\right], \quad \left[\lambda, \frac{\lambda^2 - \lambda + 2}{\lambda^2 + \lambda}\right].$$

Three of these, given by (6), were established directly in §§ 4, 5.

8. Before entering upon the technical discussion by means of which the computations are greatly simplified, I will show in a few examples how the two theorems of §§ 3 and 6 suffice to prove without device that relations (4) follow from (1), (2), (3).

For $p^n = 5$, the relations (4), with $\lambda \neq 0, 1$; $\mu \neq 0, 1$, are

$$[22222], \quad [24334], \quad [33424], \quad [34243], \quad [42433], \quad [43342].$$

Now $[2, 2]$ follows from $[2, \frac{2}{2} - 2] \equiv [2, -1]$ by § 6. The remaining five relations all have a subscript $4 \equiv -1 \pmod 5$. Hence all six follow from (1), (2), and (3).

For $p^n = 7$, the relations (4), with $\lambda \neq 0, 1$; $\mu \neq 0, 1$, are

$$[22545], \quad [33262], \quad [44363], \quad [55646],$$

together with those given by permuting the subscripts cyclically. Now $[2, 2]$ follows from $[2, -1]$; while $6 \equiv -1 \pmod 7$. Hence all follow from (1), (2), (3).

For $p^n = 9$, the field may be defined by

$$j^2 \equiv -1 \pmod 3.$$

The relations (4), with $\lambda \neq 0, 1$; $\mu \neq 0, 1$, are

$$[-1, \pm j, \mp j+1, \pm j+1, \mp j], \quad [-1, \pm j+1, \mp j-1, \pm j-1, \mp j+1],$$

$$[-1, \pm j-1, \pm j, \pm j, \pm j-1], \quad [\pm j, \pm j+1, \pm j, \mp j-1, \mp j-1],$$

$$[\pm j+1, \pm j+1, \pm j+1, \pm j+1, \pm j+1].$$

together with those arising by cyclic permutations. Since

$$\frac{2}{\pm j}-(\pm j+1)=-1, \quad \frac{2}{\pm j+1}-(\pm j+1)=\pm j,$$

$[\pm j+1, \pm j]$ follows from $[j, -1]$, and $[\pm j+1, \pm j+1]$ follows from $[\pm j+1, \pm j]$, by § 6. Hence relations (4) follow from (1), (2), (3).

For $p^n = 11$, an elementary proof is given in the *Bulletin of the American Mathematical Society*, Vol. IX., December, 1902.

For $p^n = 13$, the relations (4), with $\lambda \neq 0, 1$; $\mu \neq 0, 1$, are

$[-1, 2, 5, 3, 4]$, $[-1, 3, 7, 4, 6]$, $[-1, 4, 3, 5, 2]$, $[-1, 5, 9, 6, 8]$,
$[-1, 6, 4, 7, 3]$, $[-1, 7, 10, 8, 9]$, $[-1, 8, 6, 9, 5]$,
$[-1, 9, 8, 10, 7]$, $[-1, 10, -1, 11, 11]$, $[2, 2, 9, 10, 9]$,
$[2, 3, 8, 8, 3]$, $[2, 4, 2, 6, 6]$, $[2, 8, 7, 11, 10]$, $[2, 10, 11, 7, 8]$,
$[2, 11, 5, 5, 11]$, $[3, 3, 10, 5, 10]$, $[3, 6, 7, 9, 11]$,
$[3, 11, 9, 7, 6]$, $[4, 4, 8, 11, 8]$, $[4, 5, 7, 7, 5]$,
$[4, 9, 9, 4, 11]$, $[5, 6, 10, 10, 6]$,

together with the cyclic permutations. By* § 6,

$[2, 2] \sim [2, -1]$, $[3, 8] \sim [8, 7] \sim [7, 9] \sim [9, -1]$,
$[2, 4] \sim [4, 5] \sim [5, -1]$, $[2, 10] \sim [10, 6] \sim [6, -1]$,
$[10, 5] \sim [5, 6] \sim [6, 4] \sim [4, 1]$, $[4, 4] \sim [4, 3] \sim [3, 1]$,
$[3, 11] \sim [11, 9] \sim [9, 8] \sim [8, 1]$,
$[4, 9] \sim [9, 2] \sim [2, 5] \sim [5, 1]$.

But, by cyclic permutation of subscripts, $[2, 8]$ follows from $[8, 7]$, $[3, 6]$ from $[7, 9]$, and $[11, 10]$ from $[2, 8]$. Also $[2, 11] \sim [11, 10]$. Hence all the relations follow from (1), (2), (3).

For $p^n = 17$, relations (4), with $\lambda \neq 0, 1$; $\mu \neq 0, 1$, are

$[-1, 2, 12, 3, 11]$, $[-1, 3, 9, 4, 8]$, $[-1, 4, 14, 5, 13]$,
$[-1, 5, 6, 6, 5]$, $[-1, 6, 10, 7, 9]$, $[-1, 7, 13, 8, 12]$,
$[-1, 8, 4, 9, 3]$, $[-1, 9, 7, 10, 6]$, $[-1, 10, 11, 11, 10]$,
$[-1, 11, 3, 12, 2]$, $[-1, 12, 8, 13, 7]$, $[-1, 13, 5, 14, 4]$,

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* We employ $[3, 8]$ instead of $[2, 3]$, in view of § 3, since
$[2, 3] \sim [3, 3] \sim [2, 11] \sim [11, 10] \sim [10, 10] \sim [10, 11] \sim [11, 2] \sim [2, 3]$.

[−1, 14, −1, 15, 15], [2, 2, 6, 14, 6], [2, 3, 7, 12, 14],
[2, 4, 5, 10, 15], [2, 5, 2, 8, 8], [2, 7, 4, 4, 7],
[2, 10, 9, 15, 13], [2, 11, 13, 13, 11], [2, 13, 15, 9, 10],
[2, 14, 12, 7, 3], [2, 15, 10, 5, 4], [3, 3, 13, 9, 13],
[3, 4, 11, 6, 8], [3, 5, 5, 3, 10], [3, 8, 6, 11, 4],
[3, 14, 10, 10, 14], [3, 15, 7, 7, 15], [4, 6, 9, 11, 15],
[4, 10, 4, 12, 12], [4, 15, 11, 9, 6], [5, 8, 11, 12, 15],
[5, 9, 14, 7, 11], [5, 11, 7, 14, 9], [5, 12, 9, 9, 12],
[5, 15, 12, 11, 8], [6, 7, 8, 10, 13], [6, 12, 13, 14, 15],
[6, 13, 10, 8, 7], [6, 15, 14, 13, 12], [8, 9, 8, 14, 14],

together with those given by cyclic permutations. Relations arising by a cyclic permutation will be marked equal. Then

$[2, 2] \sim [2, -1]$, $[2, 3] \sim [3, 10] \sim [10, 4] \sim [4, -1]$,
$[2, 4] \sim [4, 7] \sim [7, 6] \sim [6, -1]$, $[2, 7] = [4, 7]$,
$[2, 5] = [8, 8] \sim [8, 5] \sim [5, 6] \sim [6, 1]$, $[8, 8] = [8, 2] \sim [2, 10]$,
$[2, 11] = [11, 13] \sim [13, 14] \sim [14, 9] \sim [9, 7] \sim [7, 1]$,
$[2, 13] \sim [13, 6] \sim [6, 10] \sim [10, 1]$,
$[2, 10] \sim [10, 5] = [2, 15] = [4, 2] \sim [2, 14]$,
$[3, 3] \sim [3, 9] \sim [9, 1]$, $[3, 4] \sim [4, 6] \sim [6, 2] \sim [2, 12] \sim [12, 1]$,
$[3, 5] = [5, 5] \sim [5, 9] \sim [9, -1]$, $[3, 8] \sim [8, 10] \sim [10, -1]$,
$[3, 14] \sim [14, 2] = [2, 3]$,
$[3, 15] \sim [15, 13] \sim [13, 10] \sim [10, 11] \sim [11, 1]$,
$[4, 6]$ from $[3, 4]$ above,
$[4, 10] = [10, 4] \sim [4, -1]$, $[4, 15] = [6, 4] \sim [4, 3] \sim [3, 8]$,
$[5, 8] \sim [8, 8] = [2, 5]$, $[5, 9] \sim [9, -1]$,
$[5, 11] = [11, 7] \sim [7, -1]$, $[5, 12] \sim [12, 15] \sim [15, 4] = [4, 6]$,
$[5, 15] = [8, 5] \sim [5, 6] \sim [6, 1]$, $[6, 7] = [8, 10] \sim [10, -1]$,
$[6, 12] = [12, 13] \sim [13, 13] = [2, 11]$,
$[6, 13] \sim [13, 2] = [2, 10]$, $[6, 15] \sim [15, 10] = [2, 15]$,
$[8, 9] \sim [9, 13] = [3, 3]$.

Hence all the relations (4) follow from (1), (2), (3).

9. Repeating the process indicated by (7), we get

$$[\lambda, \mu] \sim \left[\mu, \frac{2}{\mu} - \lambda\right] \sim \left[\frac{2}{\mu} - \lambda, \frac{\lambda\mu^2}{2-\lambda\mu}\right] \sim \left[\frac{\lambda\mu^2}{2-\lambda\mu}, \frac{(2-\lambda\mu)^2}{\lambda\mu^2}\right] \ldots .$$

Let t_m denote the m-th symbol in this series. Then

$$t_{2k} = [\rho^{1-k}\mu, \rho^k\lambda], \quad t_{2k+1} = [\rho^k\lambda, \rho^{-k}\mu], \quad \rho \equiv \frac{2}{\lambda\mu} - 1.$$

These formulæ are evidently true for $k = 1$ and $k = 2$, and, by simple induction, for general k. If $\rho = 0$, then $\frac{2}{\mu} - \lambda = 0$ and $[\lambda, \mu] \sim [\mu, 0]$ follows from (1), (2). For $p \neq 2$, we have $\rho \neq -1$; while $\rho \neq +1$, since $\lambda\mu \neq 1$. It follows that, if ρ belongs to one of the exponents $p^n - 1$, $\frac{1}{2}(p^n - 1)$, relation $[\lambda, \mu]$ is a consequence of a relation (4) with ± 1 as the second subscript, and hence a consequence of (1), (2), (3). Indeed, if ρ belongs to the exponent $p^n - 1$, we may determine k so that $\rho^k = \mu$; if ρ belongs to the exponent $\frac{1}{2}(p^n - 1)$, we may determine k so that ρ^k takes one of the two values $+\mu$, $-\mu$.

In view of § 3, the same result follows when λ, μ are replaced by the respective subscripts in any one of the pairs (5). Then $\rho \equiv \frac{2}{\lambda\mu} - 1$ is replaced by

$$\frac{\lambda\mu+\mu-2}{\lambda\mu-\mu}, \quad \frac{1+\lambda}{1-\lambda}, \quad \frac{1+\mu}{1-\mu}, \quad \frac{\lambda\mu+\lambda-2}{\lambda\mu-\lambda}.$$

We may apply the same argument to $\left[\mu, \frac{2}{\mu} - \lambda\right]$ that we applied to $[\lambda, \mu]$. Then ρ and the preceding four expressions are replaced by respectively

$$\frac{1}{\rho}, \quad \frac{2-\lambda\mu-\lambda\mu^2}{(2-\lambda\mu)(\mu-1)}, \quad \frac{1+\mu}{1-\mu}, \quad \frac{\mu-\lambda\mu+2}{\mu+\lambda\mu-2}, \quad \frac{\lambda\mu-\mu}{\lambda\mu+\mu-2}.$$

Proceeding similarly with $\left[\frac{2}{\lambda} - \mu, \lambda\right]$, from which follows $[\lambda, \mu]$, we have ρ and the four expressions replaced by

$$\frac{1}{\rho}, \quad \frac{\lambda\mu-\lambda}{\lambda\mu+\lambda-2}, \quad \frac{\lambda-\lambda\mu+2}{\lambda+\lambda\mu-2}, \quad \frac{1+\lambda}{1-\lambda}, \quad \frac{2-\lambda\mu-\lambda^2 u}{(2-\lambda\mu)(\lambda-1)}.$$

THEOREM.—*Relation* $[\lambda, \mu]$ *follows from* (1), (2), (3) *if any one of the following* expressions belongs to the exponent* p^n-1 *or* $\frac{1}{2}(p^n-1)$:—

$$\rho \equiv \frac{2}{\lambda\mu} - 1, \quad \tau = \frac{1+\lambda}{1-\lambda}, \quad \tau' = \frac{1+\mu}{1-\mu},$$

$$\alpha = \frac{\lambda\mu+\lambda-2}{\lambda\mu-\lambda}, \qquad \alpha' = \frac{\lambda\mu+\mu-2}{\lambda\mu-\mu},$$

$$\beta = \frac{\lambda-\lambda\mu+2}{\lambda+\lambda\mu-2}, \qquad \beta' = \frac{\mu-\lambda\mu+2}{\mu+\lambda\mu-2},$$

$$\gamma = \frac{2-\lambda\mu-\lambda^2\mu}{(2-\lambda\mu)(\lambda-1)}, \quad \gamma' = \frac{2-\lambda\mu-\lambda\mu^2}{(2-\lambda\mu)(\mu-1)}.$$

10. The preceding theorem suffices to prove readily that relations (4) all follow from (1), (2), (3), at least when $p^n < 49$, $p \neq 2$. We have only to verify that for no values of λ, μ do the expressions $\rho, \ldots, \gamma'$ all belong to exponents $< \frac{1}{2}(p^n-1)$. We recall (see § 9) that the values $\rho = \pm 1$ are excluded. The theorem is therefore evident for $p^n = 3$ or 5; likewise, for $p^n = 7, 9, 11, 23, 27$, or 47, the only marks belonging to exponents $< \frac{1}{2}(p^n-1)$ being ± 1. In general, it holds true when $\frac{1}{2}(p^n-1)$ is a prime number.

11. For $p^n = 13$, the only marks belonging to exponents < 6 are the solutions ± 1, ± 5; 1, 3, 9, of $X^4 \equiv 1$; $X^3 \equiv 1 \pmod{13}$. Let therefore ρ, τ, and τ' each take the values 5, 8, 3, 9. Then

$$\lambda\mu \equiv 9, 6, 7, 8; \quad \lambda \equiv 5, 8, 7, 6; \quad \mu \equiv 5, 8, 7, 6 \pmod{13}.$$

Since λ and μ take only the values ± 5, ± 6, we have $\lambda\mu \equiv 9$. Hence

$$(\lambda, \mu) = (5, -6), (-5, 6), (6, -5), (-6, 5).$$

For these values, $\alpha' \equiv 7, 0, 2, 3 \pmod{13}$, respectively. It remains to consider only $(\lambda, \mu) = (-6, 5)$: for it, $\alpha \equiv 7$; so that it also is excluded.

12. For $p^n = 17$, the numbers belonging to exponents < 8 are the solutions ± 1, ± 4 of $X^4 \equiv 1 \pmod{17}$. For ρ, τ, τ' equal to 4, -4, we get

$$\lambda\mu \equiv -3, 5; \quad \lambda \equiv 4, -4; \quad \mu \equiv 4, -4 \pmod{17}.$$

These congruences are contradictory; so that the theorem follows.

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* Other expressions may be derived, but are not needed below.

13. For $p^n = 19$, the numbers belonging to exponents < 9 are the solutions ± 1, ± 7, ± 8 of $X^6 \equiv 1 \pmod{19}$. Equating ρ, τ, τ' to 7, 8, 11, 12, in turn, we get

$$\lambda\mu \equiv 5, 15, 16, 6; \quad \lambda \equiv 15, 5, 4, 14; \quad \mu \equiv 15, 5, 4, 14.$$

For the values ± 4, ± 5 of λ, μ, we find $\lambda\mu \equiv 6$, or 16. Hence

$$(\lambda, \mu) = (-4, -4), (4, 4), (-5, -5), (5, 5).$$

For these values, $\alpha' \equiv 10, 11, 12, 9$, respectively. Hence $(\lambda, \mu) = (4, 4)$ or $(-5, -5)$. For these, $\beta' \equiv 10, 9$, respectively. Hence the theorem.

14. For $p^n = 25$, the $GF[p^n]$ will be defined by

$$j^2 \equiv 2 \pmod{5}.$$

The marks belonging to exponents < 12 are the solutions

$$(8) \quad \pm 1, \ \pm 2, \ \pm j, \ \pm 2j, \ \pm(2j+2), \ \pm(2j-2)$$

of $X^6 = 1$ and $X^8 = 1$. Giving these values (except ± 1) to τ or τ', we get the same values for λ or μ, in a different order. Giving them to ρ, we get

$$\lambda\mu = 4, 3, \ 2j+3, \ 3j+3, \ \pm(2j+4), \ \pm(2j+1), \ j+1, \ 4j+1.$$

The resulting pairs of values (λ, μ) are considered in five sets:

(a) $(-2, -2)$, $(2j+2, 2)$, $(-2j-2, -2)$, $(2j-2, 2)$, $(2j, j)$, $(-2j, -j)$, $(-j, -2j-2)$, $(j, -2j-2)$, $(-j, 2j+2)$, $(j, 2j-2)$, $(j, -2j+2)$, $(-j, 2j-2)$, $(2j, 2j)$, $(-2j, -2j)$, $(2j+2, -2j-2)$, $(2j+2, 2j-2)$, $(-2j-2, -2j+2)$, $(-2j+2, 2j-2)$;

(b) the pairs derived from those under (a) by interchanging the two marks of a pair;

(c) $(2, 2)$, $(-2j+2, -2)$, $(j, 2j+2)$, $(-j, -2j+2)$;

(d) the pairs derived from (c) by interchanging the two marks;

(e) $(j, -j)$, $(-j, j)$.

For the pairs (a), the function α has the respective values

$$0, \ 4j+4, \ 3j+4, \ 4j+2, \ 4j+2, \ j+2, \ j+4, \ 2j+1, \ 4j+2, \ 4j+4, \ j+2,$$
$$3j+1, \ 3j+1, \ 2j+1, \ 4j+3, \ 2j+1, \ j+1, \ j+3,$$

none of which are in the set (8). But, for the pairs (b), the function α' takes the same values in the same order.

For the pairs (c), the function β has the respective values

$$0, \quad j+4, \quad 4j+2, \quad j+2,$$

none of which are in the set (8). For the pairs (d), the function β' takes the same values.

Finally, for $(j, -j)$, $\gamma' = j+2$; for $(-j, j)$, $\gamma = j+2$.

15. For $p^n = 29$, the numbers belonging to exponents <14 are the roots of $X^4 \equiv 1$, $X^7 \equiv 1$. Excluding ± 1, these are

$$(9) \quad -4, \ -5, \ -6, \ 7, \ -9, \ 12, \ -12, \ 16.$$

Giving these values to ρ and to τ, we find that

$$\lambda\mu \equiv 9, \ 14, \ 17, \ 22, \ 7, \ 18, \ 13, \ 24;$$

$$\lambda, \ \mu \equiv 21, \ 16, \ 13, \ 8, \ 23, \ 12, \ 17, \ 6.$$

Hence we need examine only the values $\lambda, \ \mu \equiv 16, -6, 12, -12$, and of these only the ones making $\lambda\mu \equiv 14, 7, 28, 18$. Hence

$$(\lambda, \mu) = (16, 16), \ (-6, -6), \ (16, 12), \ (12, 16), \ (-6, -12),$$
$$(-12, -6).$$

For the first three, $\beta' \equiv 18, 11, 4$, respectively, none of which are in the set (9). For $(12, 16)$, $\beta \equiv 4$. For $(-6, -12)$, $\alpha' \equiv 0$. For $(-12, -6)$, $\alpha \equiv 0$. Hence the theorem is true.

16. For $p^n = 31$, the numbers belonging to exponents <15 are the solutions of $X^{10} \equiv 1$, $X^6 \equiv 1 \pmod{31}$. Excluding ± 1, these are

$$\pm 2, \ \pm 4, \ \pm 5, \ \pm 6, \ \pm 8, \ \pm 16.$$

Giving these values to τ, we get for λ the values

$$\pm 10, \ \pm 3, \ \pm 13, \ \pm 12, \ \pm 11, \ \pm 14,$$

no one of which is congruent to one in the previous set.

17. For $p^n = 37$, the numbers belonging to exponents <18 are the solutions of $X^{12} \equiv 1$, $X^9 \equiv 1$. Excluding ± 1, these are

$$(10) \quad 8, \ -10, \ -6, \ -11, \ -14; \quad 16, \ -3, \ 9, \ -4, \ 10, \ 12, \ 7.$$

Giving these values to τ, we find for λ the respective values

$$9, \ -7, \ -6, \ 16, \ 4, \ -10, \ 2, \ -14, \ 14, \ -16, \ -2, \ 10.$$

Hence $\lambda, \mu \equiv 9, -6, 16, -14, -10$, or 10. Giving the values (10) to ρ, we find for $\lambda\mu$ the respective values

$$-8,\ 8,\ 7,\ -15,\ -3,\ 11,\ -1,\ 15,\ -13,\ 17,\ 3,\ -9.$$

The resulting sets of values (λ, μ) fall into three groups:

(*a*) $(9, 9)$, $(9, -14)$, $(-6, -6)$, $(-6, 16)$, $(16, 16)$, $(-14, 10)$, $(10, -10)$, $(-14, -14)$;

(*b*) those derived from the sets (*a*) by interchange of elements;

(*c*) $(-14, -10)$, $(-10, -14)$.

For the pairs (*a*), the function α has the respective values

$$-7,\ -12,\ 13,\ -12,\ 15,\ 3,\ 19,\ 22.$$

For the pairs (*b*), the function α' has the same respective values. For $(-14, -10)$, $\beta' \equiv 0$. For $(-10, -14)$, $\beta \equiv 0$.

18. For $p^n = 41$, the numbers belonging to exponents < 20 are the solutions of $X^{10} \equiv 1$, $X^8 \equiv 1$. Excluding ± 1, these are

$$(11)\quad \pm 3,\ \pm 9,\ \pm 14,\ \pm 4,\ \pm 10,\ \pm 16,\ \pm 18.$$

Giving these values to τ, we find for λ the values

$$21,\ 2,\ 9,\ -9,\ 20,\ -2,\ 17,\ -12,\ 12,\ -17,\ -16,\ -18,\ 16,\ 18.$$

Hence $\lambda, \mu \equiv \pm 9, \pm 16, \pm 18$. Observing which of the products $\lambda\mu$ of these numbers make ρ equal to one of the set (11), we get

$$\lambda\mu \equiv -1,\ 10,\ \text{or}\ 21.$$

The resulting sets of values (λ, μ) fall into three groups:

(*a*) $(9, 9)$, $(-9, -9)$, $(9, 16)$, $(-9, -16)$, $(-16, -16)$, $(16, -18)$;

(*b*) those derived from (*a*) by interchange of elements;

(*c*) $(16, 16)$, $(-16, 18)$, $(18, -16)$.

For the pairs (*a*), the function α' has the respective values

$$24,\ 19,\ 7,\ 30,\ 6,\ 6.$$

For the pairs (*b*), the function α has the same respective values. For $(16, 16)$, $\beta' \equiv 14$. For $(-16, 18)$, $\beta' \equiv -15$. For $(18, -16)$, $\beta \equiv -15$. Since none of these values occur in the set (11), the theorem follows.

19. For $p^n = 43$, the numbers belonging to exponents <21 are the solutions of $X^{14} \equiv 1$, $X^6 \equiv 1$. Excluding ± 1, these are

$$(12) \quad \pm 2, \ \pm 4, \ \pm 6, \ \pm 7, \ \pm 8, \ \pm 11, \ \pm 16, \ \pm 21.$$

Giving these values to τ, we find for λ the values

$$-14, 3, -8, 16, 13, 10, -10, -13, -4, -11, 8, -16, 11, 4, -3, 14.$$

Hence $\lambda, \mu \equiv \pm 4, \pm 8, \pm 11, \pm 16$. Observing which of the products $\lambda\mu$ of these numbers make ρ equal to one of the set (12), we get

$$\lambda\mu \equiv 11, -2, \text{ or } 4.$$

The resulting sets (λ, μ) fall into three groups:

(a) $(4, -8)$, $(-4, 8)$, $(-11, 8)$, $(11, 16)$, $(-11, -16)$, $(16, 16)$;

(b) those derived from (a) by interchanging elements;

(c) $(-8, 11)$, $(11, -8)$.

For the pairs (a), the function α' has the respective values

$$-9, 20, -9, 20, -5, -15.$$

For the pairs (b), the function α has the same respective values. For $(-8, 11)$, $\beta \equiv -14$. For $(11, -8)$, $\beta' \equiv -14$. Since none of these values occur in the set (12), the theorem follows.

20. For $n = 1$, the $GF[p^n]$ is the field of integers taken modulo p. We may then write $S_\lambda = S^\lambda$, where $S = S_1$. Relations (1), (2), (3) then reduce to the following:—

$$T^2 = I, \quad S^p = I, \quad (ST)^3 = I, \quad (S^\tau T S^{2/\tau} T)^2 = I,$$

where $\tau = 1, \ldots, p-1$. For $\tau = 1$ or 2, the last relation becomes

$$STS^2TSTS^2T = STS^2 . S^{-1}TS^{-1} . S^2T = (ST)^3 = I.$$

For $\tau = -1$ or -2, the relation likewise reduces to (2). Hence, for $p^n = 3$ or 5, the group is generated by S and T subject to the relations

$$T^2 = I, \quad S^p = I, \quad (ST)^3 = I.$$

The relation given by $\tau = p - t$ follows from the inverse of that given by $\tau = +t$; the relation given by $\tau = 2/t$ follows from that given by $\tau = t$. Hence we may restrict τ to the values $3, \ldots, p-3$, retaining only one of the positive residues (necessarily in the set $3, \ldots, p-3$) of each quadruple $t, -t, 2/t, -2/t$. Equalities occur

among the latter only when $t^2 \equiv 2$ or $t^2 \equiv -2 \pmod{p}$, the four reducing to two distinct residues in each case. For $p = 8q+1$, $+2$ and -2 are both quadratic residues of p; for $p = 8q+5$, $+2$ and -2 are both non-residues; for $p = 8q+3$, $+2$ is a non-residue and -2 a residue; for $p = 8q+7$, $+2$ is a residue and -2 a non-residue. Hence, for $p = 8q+1$, there are $\frac{1}{4}(p-5-4)+2 = \frac{1}{4}(p-1)$ values of τ; for $p = 8q+5$, there are $\frac{1}{4}(p-5)$ values of τ; for $p = 8q+3$ or $8q+7$, there are $\frac{1}{4}(p-5-2)+1 = \frac{1}{4}(p-3)$ values of τ. Examples occur in § 1.

For $n>1$, analogous results are readily found by employing the theorem of § 62 of *Linear Groups*.

21. For $p^n = 3^2$, the group G_{360} is generated by T and S_λ,

$$\lambda = 0,\ \pm 1,\ \pm j,\ \pm j+1,\ \pm j-1 \quad [j^2 \equiv -1 \pmod{3}],$$

subject to the relations (1), (2), and

$$(13) \quad (S_j T S_j T)^2 = I, \quad (S_{j+1} T S_{-j+1} T)^2 = I.$$

It is readily seen that G_{360} can be generated by three operators T, S_1, and S_j, subject only to the relations

$$(14) \quad \begin{cases} T^2 = I, \quad S_1^3 = I, \quad S_j^3 = I, \quad S_1 S_j = S_j S_1, \quad (S_1 T)^3 = I, \\ (S_j T S_j T)^2 = I, \quad (S\, S_1 T S_1 S_j^{-1} T)^2 = I. \end{cases}$$

These relations evidently follow from the preceding. The inverse is proved to be true by defining

$$S_0 = I, \quad S_{-1} = S_1^{-1}, \quad S_{-} = S_j^{-1}, \quad S_{j+1} = S_j S_1, \quad S_{j-1} = S_j S_1^{-1},$$

$$S_{-j-1} = S_1^{-1} S_j^{-1}, \quad S_{-j+1} = S_1 S_j^{-1}.$$

Again, G_{360} can be generated by the operators E_1, E_2, E_3, E_4, subject to the generational relations

$$(15) \quad \begin{cases} E_1^3 = I, \quad E_2^2 = E_3^2 = E_4^2 = I, \\ (E_1 E_2)^3 = (E_2 E_3)^3 = (E_3 E_4)^3 = I, \\ (E_1 E_3)^2 = (E_1 E_4)^2 = (E_2 E_4)^2 = I. \end{cases}$$

Indeed, if we make the following definitions:

$$E_1 = S_1, \quad E_2 = T, \quad E_3 = TS_{-j}TS\,T, \quad E_4 = S_j E_3,$$

relations (15) all follow from (1), (2), and (13). This is evident except possibly in the following cases :—

$$\begin{aligned} E_4^2 &= S_j TS_{-j} . TS_j TS \ . TS_{-j} TS\ T \\ &= S_j TS_{-j} . S_{-j} TS_{-j} T . TS_{-j} TS_j T = (S_j TS_j T)^2 = I, \\ (E_2 E_3)^3 &= S_{-j} . TS_j TS_j . S_j TS\ T . S_{-j} TS_j T \\ &= S_{-j} . S_{-j} TS_{-j} T . TS_{-j} TS_{-j} . S_{-j} TS\ T = (S_j TS_j T)^2 = I, \\ (E_1 E_3)^2 &= S_1 TS_{-j} TS_j . TS_1 T . S_{-j} TS_j T \\ &= S_1 TS_{-j} T . S_{j-1} TS_{-j-1} T . S_j T \\ &= S_1 TS_{-j} T . TS_{j+1} TS_{-j+1} . S_j T \quad [\text{by } (13_2)] \\ &= (S_1 T)^3 = I. \end{aligned}$$

$$E_1 E_4 = S_1 S_j E_3 = S_j E_1 E_3 = S_j E_3 E_1^2 = E_4 E_1^2, \quad (E_1 E_4)^3 = I.$$

Inversely, relations (1), (2), (13) follow from (15) if

$$S_0 = I, \quad S_1 = E_1, \quad S_j = E_4 E_3, \quad S_{j+1} = E_1 E_4 E_3, \quad S_{j-1} = E_1^2 E_4 E_3,$$

$$T = E_2,\ S_{-1} = E_1^2,\ S_{-j} = E_3 E_4,\ S_{-j-1} = E_3 E_4 E_1^2,\ S_{-j+1} = E_3 E_4 E_1.$$

By the earlier result, it suffices to verify relations (14).

The isomorphism of G_{360} with the alternating group on six letters follows by noting that relations (1), (2), (13) are all satisfied by the substitutions

$$\begin{aligned} S_1 &= (123), & S &= (456), & S_{j+1} &= (123)(456), \\ S_{-1} &= (132), & S_{-j} &= (465), & S_{-j-1} &= (132)(465), \\ T &= (12)(34), & S_{j-1} &= (132)(456), & S_{-j+1} &= (123)(465). \end{aligned}$$

It follows otherwise by noting that (15) are satisfied by

$$E_1 = (123), \quad E_2 = (12)(34), \quad E_3 = (12)(45), \quad E_4 = (12)(56).$$

Generational Relations of an Abstract Simple Group of Order 4080.

By L. E. Dickson. Received and read December 11th, 1902.

Introduction.

1. In an earlier paper in these *Proceedings*, Vol. xxxv., pp. 292–305, the writer investigated, for the case $p > 2$, the abstract group G simply isomorphic with the group Γ of all linear fractional transformations of determinant unity in the $GF[p^n]$. The present paper deals with the case $p = 2$, when the group Γ is of order $2^n(2^{2n}-1)$, and is simple if $n > 1$. Use is made of the transformations

$$(1) \quad T: \; z' = \frac{1}{z}; \quad S_\lambda: \; z' = z+\lambda.$$

For $p = 2$, $n = 2$, the field is defined by the primitive irreducible congruence

$$i^2 \equiv i+1 \pmod 2.$$

The group Γ of order 60 is generated by

$$A = S_i T \quad \text{and} \quad B = S_{i+1};$$

a complete set of generational relations for G is*

$$(2) \quad A^5 = I, \quad B^2 = I, \quad (AB)^3 = I.$$

Setting $\alpha = AB$, $\beta = B$, whence $A = \alpha\beta$, $B = \beta$, we obtain a second set of generational relations for G:

$$(2') \quad \alpha^3 = I, \quad \beta^2 = I, \quad (\alpha\beta)^5 = I.$$

For $p = 2$, $n = 3$, the field is defined by the primitive irreducible congruence

$$i^3 \equiv i+1 \pmod 2.$$

The group Γ of order 504 is generated by

$$A = S_i T \quad \text{and} \quad B = S_{i+1};$$

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* Dickson, *Linear Groups*, § 281, A being replaced by its inverse.

a complete set of generational relations for G is*

$$(3) \quad A^9 = I, \quad B^2 = I, \quad (AB)^3 = I, \quad (A^3BA^5B)^2 = I.$$

As generators of Γ may be taken $\alpha = TS_{i+1}$ and $\beta = S_i$; the resulting complete set of generational relations for G is†

$$(3') \quad \alpha^7 = I, \quad \beta^2 = I, \quad (\alpha\beta)^3 = I, \quad (\alpha^3\beta\alpha^5\beta\alpha^3\beta)^2 = I.$$

Since the sets of relations (2) and (3) are simpler than the respective sets (2′) and (3′), it is natural to seek as generators of Γ, for the $GF[2^4]$, two operators A and B of respective periods 17 and 2, rather than two operators α and β of respective periods 15 and 2. Some general results bearing on this point are reserved for a later communication to the Society.

For $p = 2$, $n = 4$, the field is defined by the primitive irreducible congruence

$$i^4 \equiv i + 1 \pmod 2.$$

It will be shown that the group $\mathbf{\Gamma}$ of order 4080 is generated by

$$A = S_i T \quad \text{and} \quad B = S_{i+1},$$

and that a complete set of generational relations for the isomorphic group G_{4080} is

$$(4) \qquad A^{17} = I, \quad B^2 = I, \quad (AB)^3 = I,$$

$$(5) \quad (A^3BA^7B)^2 = I, \quad (A^4BA^{12}B)^2 = I, \quad (A^6BA^9B)^2 = I.$$

From these follow at once the relations

$$(5') \quad (A^5BA^{13}B)^2 = I, \quad (A^8BA^{11}B)^2 = I, \quad (A^{10}BA^{14}B)^2 = I.$$

From (4) alone follow

$$(5'') \quad (ABA^2B)^2 = I, \quad (A^{15}BA^{16}B)^2 = I.$$

Interchanging the two exponents in any one of the eight relations (5), (5′), (5″), we obtain a relation equivalent to it in virtue of (4). Hence‡ there exists a relation

$$(A^\xi BA^\eta B)^2 = I, \quad \text{for} \quad \xi = 1, 2, \ldots, 16.$$

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* Dickson, *Bulletin Amer. Math. Soc.*, January, 1903.

† Dickson, *ibid.*; Burnside and Fricke, *Math. Ann.*, Vol. LII.; de Séguier, *Journal de Mathématiques*, S. 5, t. VIII. (1902), p. 267.

‡ A like relation follows from (3), where now $\xi = 1, \ldots, 8$.

By means of Lagrange's interpolation formula, we find that

$$\eta \equiv 6\xi + 9\xi^3 + \xi^5 + 8\xi^7 + 9\xi^9 + 15\xi^{11} + 12\xi^{13} + 10\xi^{15} \pmod{17}.$$

In fact, this is the analytic representation of the substitution*

$$(1\ 2)(3\ 7)(4\ 12)(5\ 13)(6\ 9)(8\ 11)(10\ 14)(15\ 16).$$

The relation is thus less simple than in the case $p > 2$.

First Set of Generational Relations of G_{4080}.

2. The abstract G_{4080} is generated by the operators

$$T,\ S_\lambda \quad (\lambda = 0, i, i^2, \ldots, i^{14}, i^{15} = 1),\dagger$$

subject only to the generational relations‡

$$(6) \quad T^2 = I, \quad S_0 = I, \quad S_\lambda S_\mu = S_{\lambda+\mu} \quad (\lambda, \mu \text{ any marks}),$$

$$(7) \quad S_\lambda T S_\mu T S_{(\lambda-1)/(\lambda\mu-1)} T S_{-(\lambda\mu-1)} T S_{(\mu-1)/(\lambda\mu-1)}\ T = I$$

$$(\lambda, \mu \text{ any marks such that } \lambda\mu \neq 1).$$

Relation (7) will be designated by either of the symbols

$$(7') \quad (\lambda, \mu), \quad \left(\lambda,\ \mu,\ \frac{\lambda-1}{\lambda\mu-1},\ -\lambda\mu+1,\ \frac{\mu-1}{\lambda\mu-1}\right).$$

If, in any particular relation (7), we permute the subscript cyclically, we obtain a relation, still of type (7), which follows at once from that particular relation and relations (6). The relations (7) with $\lambda = 0$ or 1, or with $\mu = 0$ or 1, all reduce in virtue of (6) to

$$(8) \quad (S_1 T)^3 = I.$$

If, of relations (7) with $\lambda \neq 0, 1$; $\mu \neq 0, 1$, we retain only one of the set arising by cyclic permutation of the subscripts, we obtain the following :—

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* *Annals of Math.*, Ser. 1, Vol. XI., p. 70.

† $i^4 \equiv i + 1$, $i^5 \equiv i^2 + i$, $i^6 \equiv i^3 + i^2$, $i^7 \equiv i^3 + i + 1$, $i^8 \equiv i^2 + 1$, $i^9 \equiv i^3 + i$, $i^{10} \equiv i^2 + i + 1$, $i^{11} \equiv i^3 + i^2 + i$, $i^{12} \equiv i^3 + i^2 + i + 1$, $i^{13} \equiv i^3 + i^2 + 1$, $i^{14} \equiv i^3 + 1$, $i^{15} \equiv 1$.

‡ E. H. Moore. *Cf.* Dickson, *Linear Groups*, pp. 300–302.

$(i^3+i,\ i^3+i,\ i^2+i,\ i^2+i,\ i^3+i)$,
$(i^2+i+1,\ i^2+i+1,\ i^2+i+1,\ i^2+i+1,\ i^2+i+1)$,
$(i+1,\ i+1,\ i^3+1,\ i^2,\ i^3+1)$,
$(i^2+1,\ i^2+1,\ i^3+i^2+1,\ i+1,\ i^3+i^2+1)$,
$(i^3+1,\ i^3+1,\ i^3+i^2+i+1,\ i^3+i^2,\ i^3+i^2+i+1)$,
$(i^3+i^2,\ i^3+i^2,\ i^3,\ i^3+i^2+i,\ i^2)$,
$(i^3+i^2+i,\ i^3+i^2+i,\ i^3,\ i^3+i,\ i^3)$,

$(i,\ i+1,\ i^3+i,\ i^2+i+1,\ i^3+i^2)$,
$(i,\ i^2,\ i^3+i,\ i^3+1,\ i^3+i)$,
$(i,\ i^2+i,\ i^3+i^2,\ i^3+i^2+1,\ i^3+i^2+i+1)$,
$(i+1,\ i^2+i,\ i^3+i,\ i^3+i+1,\ i^3)$,
$(i+1,\ i^2+i+1,\ i^3+i^2+1,\ i^3,\ i^2)$,
$(i^3,\ i^2+1,\ i^3,\ i^2+i,\ i^3+i^2+i+1)$,

$(i,\ i,\ i^3+i^2+i,\ i^2+1,\ i^3+i^2+i)$,
$(i^2,\ i^2,\ i^3+i+1,\ i,\ i^3+i+1)$,
$(i^3,\ i^3,\ i,\ i^3+i^2+1,\ i)$,
$(i^3+i,\ i^3+i,\ i^2+1,\ i^3+1,\ i^2+1)$,
$(i^3+i+1,\ i^3+i+1,\ i^3+i^2,\ i^3,\ i^3+i^2)$,
$(i^3+i^2+1,\ i^3+i^2+1,\ i^3+i,\ i^3+i^2+i+1,\ i^3+i)$,
$(i^3+i^2+i+1,\ i^3+i^2+i+1,\ i+1,\ i^3+i+1,\ i+1)$,

$(i,\ i^2+1,\ i^3+i^2+i+1,\ i^3+i+1,\ i^2+i+1)$,
$(i^2+i+1,\ i^3+i^2+i,\ i^3+i+1,\ i^3+i^2+1,\ i^3+1)$,
$(i+1,\ i^2+1,\ i^2+i,\ i^3+i^2+i,\ i^3+i^2)$,
$(i^2,\ i^2+i+1,\ i^3+i^2+i+1,\ i^3+i^2+i,\ i^3+i)$,
$(i^3+1,\ i^3+i+1,\ i^2+i,\ i^3+i^2+1,\ i^3+i^2+i)$,
$(i^2+1,\ i^2+i+1,\ i^3,\ i^3+1,\ i^3+i^2)$,

together with the twelve derived from the last twelve by reversing the order of the subscripts.* The new twelve relations may be dropped, since they may be derived by inverting the first twelve and applying (6).

These relations may be separated into sets of three in various ways, such that from two of a set and relations (6) and (8) follows the third of that set. For example, $(i^2+1,\ i^2+1)$ may be written

$$S_{i^2+1}TS_{i^2+1}TS_{i^3+i^2+1}TS_i \cdot S_1TS_{i^3+i^2+1}T = I.$$

We may replace $TS_{i^3+i^2+1}TS_i$ by $S_iTS_{i^3}TS_{i^3}T$, in view of $(i^3,\ i^3)$, and then replace TS_1T by S_1TS_1, in view of (8). There results the relation $(i^2+1,\ i^2+i+1)$. Again, from $(i^2+1,\ i^2+1)$ and $(i+1,\ i^2+1)$, we get

$$\begin{aligned} I &= S_{i^2+1}TS_{i^2+1}TS_{i^3+i^2+1} \cdot TS_{i+1}TS_{i^3+i^2} \cdot S_1T \\ &= S_{i^2+1}TS_{i^2+1}TS_{i^3+i^2+1} \cdot S_{i^2+1}TS_{i^2+i}TS_{i^3+i^2+i}T \cdot S_1T. \end{aligned}$$

Replacing TS_1T by S_1TS_1 and then transforming by S_1, we get $(i^2,\ i^2+1)$. In this way we obtain the sets on the opposite page.

Hence relations (7) all follow from (6), (8), and

(9) $(i,\ i+1),\ (i+1,\ i+1),\ (i+1,\ i^2+1),\ (i,\ i^2+i),\ (i^2+i,\ i^2+i),$
$(i^2+i+1,\ i^2+i+1).$

3. The final relations (6) all follow from†

$$(10)\quad \begin{cases} S_0 = I, \quad S_\mu^2 = I, \quad S_{i+1}S_\mu = S_{i+1+\mu} \quad (\mu = 1,\ i,\ \ldots,\ i^{14}), \\ S_1S_{i^2} = S_{i^2+1}, \quad S_1S_{i^3} = S_{i^3+1}, \quad S_1S_{i^3+i^2} = S_{i^3+i^2+1}, \quad S_{i^2}S_{i^3} = S_{i^3+i^2}. \end{cases}$$

Indeed, inverting these relations, we get

$$S_\mu S_{i+1} = S_{i+1+\mu}, \quad S_{i^2}S_1 = S_{i^2+1}, \quad S_{i^3}S_1 = S_{i^3+1}, \quad \ldots.$$

In particular, $\quad S_{i+1}S_1 = S_i, \quad S_1S_{i+1} = S_i.$

Hence $$S_1S_i = S_iS_1 = S_{i+1},$$

$$\left.\begin{aligned} S_iS_\tau = S_{i+1}S_1S_\tau = S_{i+1}S_{\tau+1} = S_{\tau+i} \\ S_1S_{\tau+i} = S_1S_iS_\tau = S_{i+1}S_\tau = S_{\tau+i+1} \end{aligned}\right\} \quad (\tau = i^2,\ i^3,\ i^3+i^2).$$

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* The number of relations is thirty-eight in agreement with *Linear Groups*, § 279. It was verified that in them every mark λ is followed directly by every mark μ for which $\lambda\mu \neq 1$, the first subscript being regarded as following the last.

† This section is not essential in the later parts of the paper.

$(i, i+1)$, (i, i), (i, i^2+1);
$(i, i+1)$, (i^3, i^3), $(i+1, i^2+i)$;
$(i, i+1)$, $(i+1, i+1)$, $(i+1, i^2+i+1)$;
$(i, i+1)$, (i, i^2), $(i+1, i^2+1)$;
$(i, i+1)$, (i, i^2+i), $(i^3+i^2+i+1, i^3+i^2+i+1)$;
(i, i^2+1), (i, i^2+i), (i^3+i^2+1, i^3+i^2+1);
(i, i^2+1), $(i+1, i^2+1)$, (i^3+1, i^3+i+1);
(i, i^2+1), $(i+1, i^2+i+1)$, (i^2, i^2+1);
$(i+1, i^2+1)$, $(i+1, i^2+i+1)$, (i^3+i+1, i^3+i^2+i);
$(i+1, i^2+1)$, (i^2, i^2+1), (i^2+1, i^2+1);
$(i+1, i^2+i)$, $(i+1, i^2+i+1)$, (i^3+i+1, i^3+i+1);
(i^2, i^2+1), (i^3+i, i^3+i), (i^2, i^2+i+1);
(i^2, i^2+i+1), (i^2, i^2), $(i^3+i^2+i+1, i^3+i^2+i+1)$;
(i^2, i^2+i+1), (i^2+i+1, i^3+i^2+i), (i^3+i^2+1, i^3+i^2+1);
(i^2+1, i^2+i+1), (i^3+i+1, i^3+i+1), (i^2+i+1, i^3+i^3+i).

(i, i^2), (i^3, i^3+i+1), (i^3+i^2+i, i^3+i^2+i);
(i, i^2), (i, i^2+1), (i^2+i+1, i^3+i^2+i);
(i, i^2), $(i+1, i^2+i+1)$, (i^3+1, i^3+i+1);
(i, i^2), $(i+1, i^2)$, (i^3+1, i^3+i+1);
(i, i^2), (i^2, i^2), (i^2, i^2+1);
(i, i^2+i), (i, i), (i^3+i^2, i^3+i^2);
(i, i^2+i), (i^3+1, i^3+i+1), (i^3+1, i^3+1);
(i, i^2+i), (i^3+1, i^3+i+1), $(i+1, i^2+i)$;
$(i+1, i^2+1)$, (i^2+1, i^2+i+1), (i^3+1, i^3+1);
$(i+1, i^2+i)$, $(i+1, i+1)$, (i^3+i, i^3+i);
$(i+1, i^2+i)$, (i^3+1, i^3+i+1), (i^3+i^2+i, i^3+i^2+i);
(i^2, i^2+1), (i^2+1, i^2+i+1), (i^3+i^2, i^3+i^2);
(i^2+1, i^2+i+1), (i^3, i^3), (i^2+1, i^2+1);
(i^2, i^2+i+1), (i^2+i+1, i^3+i^2+i), (i^2+1, i^2+i+1);

Hence $$S_1 S_\mu = S_{\mu+1},$$

for every μ. Then

$$S_i S_{t+1} = S_i S_1 S_t = S_{i+1} S_t = S_{t+i+1};$$

so that $$S_i S_\mu = S_{\mu+i},$$

for every μ. The other cases follow similarly.

The group G_{4080} is generated by T and the S_λ subject to the relations $T^2 = I$, (8), (9), *and* (10).

Second Set of Generational Relations of G_{4080}.

4. The commutative group formed by the sixteen operators

$$S_\lambda \quad (\lambda = 0, i, i^2, \ldots, i^{15} = 1),$$

subject to the final relations (6), is generated by four operators*

$$(11) \quad S_j \quad (j = 1, i+1, i^2+i, i^3),$$

subject only to the generational relations

$$(12) \quad S_j^2 = I, \quad S_j S_k = S_k S_j \quad (j, k = 1, i+1, i^2+i, i^3).$$

Indeed, from (6), follow (12) and

$$(13) \quad S_0 = I, \; S_i = S_1 S_{i+1}, \; S_{i^2} = S_1 S_{i+1} S_{i^2+i}, \; S_{i^2+1} = S_{i+1} S_{i^2+i}, \; \ldots,$$

which express every S_μ in terms of the generators (11). Inversely, from (12) and (13), the latter serving to define the S_μ other than (11), follow relations (6).

The group G_{4080} is generated by T, S_j $(j = 1, i+1, i^2+i, i^3)$, subject to the relations $T^2 = I$, (8), (9) *and* (12), *provided the operators S_μ in* (9) *are expressed in terms of the generators by means of relations* (13).

Third Set of Generational Relations of G_{4080}.

5. From relation (i, i), we get

$$S_i (TS_i)^3 = S_{i^3+i^2} TS_{i^2+1} TS_{i^3+i^2}.$$

But, from (i^2+1, i^2+i+1), we have

$$TS_{i^3+i^2} TS_{i^2+1} T = S_{i^2+1} TS_{i^3} TS_{i^2+i+1}.$$

---

* For the sake of the later application, these are chosen instead of the more symmetrical generators S_j $(j = 1, i, i^2, i^3)$.

Hence $$(TS_i)^4 = S_{i^3+1}TS_{i^3}TS_{i^3+i+1},$$

$$(TS_i)^8 = S_{i^3+1}TS_{i^3}TS_iTS_{i^3}TS_{i^3+i+1}.$$

But $$TS_{i^3}TS_iT = S_{i^3}TS_iTS_{i^3+i^2+1} \text{ by } (i^3, i^3).$$

Hence $$(TS_i)^8 = S_1TS_iTS_{i^2+1}TS_{i^3+i+1}.$$

Writing the square of the second member and twice applying

$$TS_iTS_{i^2+1}T = S_{i^2+i+1}TS_{i^3+i+1}TS_{i^3+i^2+i+1},$$

which follows from (i, i^2+1), we get

$$(TS_i)^{16} = S_iS_{i^3}TS_{i^3+i+1}TS_iTS_{i^3+i+1}TS_{i^3}.$$

The second member equals S_iT by (i^2, i^2). Hence*

$$(14) \quad (S_iT)^{17} = I.$$

6. Setting $$S_iT = A, \quad S_{i+1} = B,$$

we get $$B^2 = I, \quad BA = S_1T,$$

from (6). Applying also (8) and (14), we obtain relations (4). For the concrete group Γ relations (5) hold, and

$$(15) \quad \begin{cases} S_1 = A^{10}BA^7BA^{10}, & S_{i^3+i} = A^9BA^2BA^9, \\ S_{i^3} = A^{-1}BA^8BA^9BA, & T = BA^{11}BA^7BA^9B. \end{cases}$$

To show that A and B, subject only to the relations (4) and (5), generate the group G_{4080}, it suffices to prove that the five operators T and S_j $(j = 1, i+1, i^2+i, i^3)$, defined by (15) and $S_{i+1} = B$, satisfy the relations given in the theorem of § 4.

7. Applying (4) and (5_1), the first relation (5), we get

$$T^2 = BA^{11}.BA^7BA^3.BA^7BA^9B = BA^{11}.A^{-3}BA^{-7}B.BA^7BA^9B = I.$$

Evidently S_{i+1}, S_{i^2+i}, S_{i^3} are of period 2 by (4), while

$$S_1^2 = A^{10}.BA^7BA^3.BA^7BA^{10}$$
$$= A^{10}.A^{-3}BA^{-7}B.BA^7BA^{10} = I \quad [\text{by } (5_1)].$$

---

* This may also be shown by means of the fractional group.

$$(S_iT)^8 : z' = \frac{(i^3+i^2+i)z + i^3 + i + 1}{(i^3+i+1)z + i^3 + i + 1}, \quad (S_iT)^9 : z' = \frac{(i^3+i+1)z + i^3 + i + 1}{(i^3+i+1)z + i^3 + i^2 + i}.$$

Hence $$(S_iT)^9 = (S_iT)^{-8}.$$

A third proof uses the canonical form of S_iT.

The condition for the identity $BAT = S_1$ is

$$BA.BA^{11}BA^7BA^9B.A^7BA^{10}BA^7 = I.$$

Replacing BA^9BA^7 by $A^{-6}BA^{-9}B.A$, in view of (5_3), and thrice replacing BAB by $A^{-1}BA^{-1}$, in view of (4), the condition reduces to

$$A^{-1}BA^9BA^6BA^9BA^7 = I,$$

a consequence of (5_3). Hence

$$BA = S_1T^{-1} = S_1T, \quad (S_1T)^3 = I.$$

To show that $$BS_1 = S_1B,$$

it suffices to prove that $$AT = BATB.$$

The condition is

$$BA.BA^{11}BA^7BA^9B.B.BA^8BA^{10}BA^6B.A^{-1} = I.$$

Replacing BAB by $A^{-1}BA^{-1}$, and transforming by A^2,

$$A^{-3}BA^{10}BA^7BA^9BA^8BA^{10}BA^6BA = I.$$

Replacing $A^{-3}BA^{10}B$ by BA^7BA^3, and transforming by BAB,

$$BA^6BA^9ABA^9BA^8BA^{10}BA^5 = I.$$

Replacing BA^6BA^9 by $A^8BA^{-6}B$, and BAB by $A^{-1}BA^{-1}$,

$$A^8BA^{-7}BA^8BA^8BA^{11}.A^{-1}BA^5 = I.$$

Replacing BA^8BA^{11} by A^6BA^9B, and then $BA^{-7}BA^{14}$ by A^3BA^7B, and $BA^{-1}B$ by ABA, we obtain an identity.

To show that $$BS_{i^2+i} = S_{i^2+i}B,$$

we transform $(BS_{i^2+i})^2$ by A^{-6} and twice replace A^6BA^9B by its inverse, and get

$$\begin{aligned}(BA^8BA^{-4}BA^3)^2 &\equiv A^{-3}B\,(BA^3BA^7ABA^{-4})^2\,BA^3\\ &\equiv A^{-3}B\,(A^{-7}BA^{-4}BA^{-5})^2\,BA^3,\end{aligned}$$

by (5), and $$BAB = A^{-1}BA^{-1}.$$

Replacing $BA^{-12}BA^{-4}$ by its inverse, we obtain I.

To show that $$BS_{i^3} = S_{i^3}B,$$

we note that $(BS_{i^3})^2 = (ABA^9BA^9BA)^2$ [by (4)]

$$\begin{aligned}&= ABA^9B.S_{i^2+i}.BA^9BA = ABA^9S_{i^2+i}A^9BA\\ &= ABA^{18}BA^2BA^{18}BA = I \quad [\text{by (4)}].\end{aligned}$$

We next show that

$$(16) \quad S_{i^2} S_{i^2+i} = A^3BA^9BA^3.$$

The condition for this identity is

$$A^{-1}BA^8BA^9BA^{10}BA^2BA^6BA^8BA^{-3} = I.$$

Replacing BA^2B by $A^{-1}BA^{-2}BA^{-1}$, in view of (4), and BA^9BA^9B by $A^{-2}BA^8BA^8BA^{-2}$, in view of $(BS_{i^2+i})^2 = I$, we get

$$A^{-1}BA^6BA^8BA^8BA^{-4}BA^5BA^8BA^{-3} = I.$$

Replacing $BA^{-4}BA^5$ by $A^{12}BA^4B$, and transforming by A^7,

$$A^9BA^6BA^8BA^3BA^{12}BA^4 = I.$$

Replacing A^9BA^6B and $BA^{12}BA^4$ by their inverses, we get

$$BA^{-6}.BA^{-1}BA^{-1}B.A^5B = BA^{-6}.A.A^5B = I.$$

The second member of (16) is of period 2 in view of (5_3). Hence S_{i^2} is commutative with S_{i^2+i}. Next,

$$\begin{aligned} S_{i^2}S_1 &= A^{-1}BA^8BAA^8BA^{11}BA^7BA^{10} \\ &= A^{-1}BA^8BA.BA^6BA^9.A^7BA^{10} \quad [\text{by } (5_3)] \\ &= A^{-1}BA^7BA^6BA^{11} \quad [\text{by } (4_3)]; \end{aligned}$$

$$\begin{aligned} (S_{i^2}S_1)^2 &= A^{-1}BA^7BA^6BA^9ABA^7BA^6BA^9A^2 \\ &= A^{-1}BA^7.A^8BA^{-6}B.ABA^7.A^8BA^{-6}B.A^2 \\ &= A^{-1}BA^{-2}BA^{-7}BA^{-3}BA^{-6}BA^2 \quad [\text{by } (4_3)] \\ &= A^{-1}BA^{-2}.A^3BA^7B.BA^{-6}BA^2 \\ &= A^{-1}BABABA^2 = I \quad [\text{by } (4)]. \end{aligned}$$

Hence S_{i^2} is commutative with S_1. Next,

$$\begin{aligned} S_{i^2+i}S_1 &= A^9BA^2.BA^2B.A^7BA^{10} \\ &= A^9BA^2.A^{-1}BA^{-2}BA^{-1}.A^7BA^{10} \\ &= A^8BA^{-3}BA^6BA^{10}; \end{aligned}$$

$$\begin{aligned} (S_{i^2+i}S_1)^2 &= A^8BA^{-3}BA^5BA^{-4}BA^6BA^{10} \\ &= A^8BA^{-3}.A^4BA^{-5}B.BA^6BA^{10} \\ &= A^8BABABA^{10} = A^8A^{-1}A^{10} = I. \end{aligned}$$

Hence S_{i^2+i} is commutative with S_1. Hence relations (12) are all satisfied. Before considering relations (9), certain auxiliary results are derived.

8. Defining S_i, S_{i^2}, ... by formulæ (13), we get

$$S_{i^2+1} = S_{i+1} S_{i^2+i} = BA^9 BA^2 BA^9,$$

$$S_{i^2} T = S_{i^2+i} S_i T = A^9 BA^2 BA^{10},$$

$$S_{i^2+1} T S_{i^2} = BA^9 BA^2 BA^{-1} BA^{-2} BA^8 = BA^9 BA^4 BA^9,$$

upon twice replacing $BA^{-1}B$ by ABA. Also

$$S_{i^2} T S_{i^2+i} = (S_{i^2+1} T S_{i^2})^{-1} S_{i+1} = A^8 BA^{-4} BA^8.$$

9. We may write $(i+1, i+1)$ in the form

$$S_{i+1} . TS_i . S_1 T . S_{i^2} . S_1 T . S_{i^2} T . S_{i^2} . S_1 T = I.$$

Supplying the values of the factors, the relation becomes

$$BA^7 BA^9 BABA A^9 BA^2 BA^9 BA^8 BA^9 BABA = I.$$

Twice replacing $BABA$ by $A^{-1}B$, we get

$$BA^7 BA^8 BA^9 BA^2 BA^9 BA^8 BA^8 B = I.$$

Since B is commutative with

$$A^9 BA^2 BA^9 \equiv S_{i^2+i},$$

the relation reduces to an identity upon applying (4).

The left number of $(i, i+1)$ is

$$\begin{aligned}
&S_i T . S_{i+1} . TS_i . S_{i^2} . TS_{i^2} . S_{i+1} . TS_{i^2} S_{i^2} T \\
&\quad = ABA^{-2} BA^8 BA^9 BA^8 BA^{-2} BA^8 B . TS_{i^2} S_{i^2} T \\
&\quad = ABA^{-2} BA^8 BA^9 B . S_{i^2+i}^{-1} B . TS_{i^2} S_{i^2} T \\
&\quad = ABA^{-2} BA^8 BA^9 S_{i^2+i}^{-1} . TS_{i^2} . S_{i^2} T \\
&\quad = ABA^{-2} BA^6 BA^8 . A^7 BA^{-2} BA^8 . S_{i^2} T \\
&\quad = ABA^{-2} BA^6 . ABA^8 BA . A^8 S_{i^2} T \\
&\quad = ABA^{-2} B . BA^{-3} BA^{-7} . A^9 S_{i^2} T \\
&\quad = ABA^{-5} BABA^8 BA^9 BA . T \\
&\quad = ABA^{-6} BA^7 BA^9 BAT = ATAT \\
&\quad = BS_1 BS_1 = I.
\end{aligned}$$

Relation (i, i^2+i) may be written

$$S_i T . S_{i^2+i} TS_{i^2} . S_{i^2} . TS_1 . S_{i^2} . S_{i^2} TS_{i^2+1} . S_{i^2} . S_i T = I.$$

Substituting, and transforming by A^{-2}, it becomes

$$A^{12} BA^4 B (A^8 B)^4 A^9 BA^9 BA^{-4} B (A^9 B)^3 = I.$$

Replacing $A^{12}BA^4B$ by its inverse, transforming by BA^8, replacing the new front product $A^5BA^{13}B$ by its inverse, and transforming by BA^8, we get

$$A^{13}BA^5A^{-2}BA^8BA^8BA^9BA^9BA^{-4}BA^9B = I.$$

Replacing $A^{-2}BA^8BA^8B$ by $BA^9BA^9BA^2$, in view of

$$(BS_{i^2+i})^2 = I,$$

and replacing $A^{13}BA^5B$ by its inverse, and transforming by BA^8B, we get

$$BA^{-1}A^5BA^{13}BA^9BA^{11}BA^9BA^{-4} = I.$$

Replacing $A^5BA^{13}B$ by its inverse, and $BA^{-1}B$ by ABA,

$$ABA^5BA^4BA^{11}BA^9BA^{-4} = I.$$

Transforming by A^{10}, and replacing BA^9BA^6 by its inverse,

$$A^8BA^{-7}A^{12}BA^4BA^5BA^8B = I.$$

Replacing $A^{12}BA^4B$ by its inverse, and transforming by A^{11},

$$A^{-3}BA^{-7}B.A^{-4}BA^{10}.BA^8BA^{11} = I.$$

Replacing the first and third products by their inverses,

$$BA^7BA^{-1}BA^{-1}BA^9B = BA^7.A.A^9B = I.$$

We may write $(i+1, i^2+1)$ thus:

$$S_{i+1}.TS_1.S_{i^2}TS_{i^2+i}.TS_i.(S_i.S_{i^2}T)^2 = I,$$

$$BA^{-1}BA^8BA^{-4}BA^8A^{-1}(A^{-1}BA^8BA^9BA^{10}BA^2BA^{10})^2 = I,$$

$$ABA^9BA^{-4}BA^7(A^{-1}BA^8BA^9BA^9BA^{-2}BA^9)^2 = I.$$

In the expanded square occurs the factor

$$A^{-2}BA^8BA^8B = BA^9BA^9BA^2,$$

as remarked in the preceding paragraph. The relation becomes

$$ABA^9BA^{-4}BA^6BA^8BA^9BABA^9BA^{11}BA^9BA^{-2}BA^9 = I.$$

Replacing BAB by $A^{-1}BA^{-1}$, and then BA^8BA^{11} by its inverse, and transforming by A, we get

$$BA^9BA^{-4}BA^6BA^8BA^{14}BABA^{-2}BA^{10} = I.$$

Replacing BAB by $A^{-1}BA^{-1}$, and $BA^{-3}BA^{10}$ by its inverse,

$$BA^9BA^{-4}BA^5BA^8BA^3BA^3B = I.$$

Transforming by BA^4, replacing $A^5BA^{-4}B$ and BA^8BA^7 by their

inverses, the relation becomes

$$BA^4BABABA^{-3}B = BA^4A^{-1}A^{-3}B = I.$$

Consider $(i^2+i,\ i^2+i)$, viz.,

$$(TS_{i^2+i})^5 = I.$$

Now $$(TS_{i^2+i})^2 = TS_i . S_{i^2}TS_{i^2+i} = A^7BA^{-4}BA^8,$$

$$(TS_{i^2+i})^4 = A^7BA^{-4}.BA^{-2}B.A^{-4}BA^8$$

$$= A^7BA^{-4}.ABA^2BA.A^{-4}BA^8.$$

The original condition thus becomes

$$BA^{11}BA^7BA^9B.A^9BA^2BA^9.A^7BA^{-3}BA^2BA^{-3}BA^8 = I.$$

Replacing B^1A^6B by ABA, and transforming by BA^{-6},

$$BA^7BA^9BA^9BA^3BA^{-2}BA^2BA^{-3}BA^8BA^{-6} = I.$$

Replacing BA^8BA^{-6} by its inverse, and transforming by BA^8B,

$$BA^{16}BA^9BA^9BA^3BA^{-2}BA^2BA^3 = I.$$

Replacing $BA^{16}B$ by ABA, and transforming by ABA^7,

$$A^3BA^9BA^3BA^{-2}BA^2BA^4BA^7 = I.$$

Replacing BA^2B by $A^{-1}BA^{-2}BA^{-1}$, we get

$$A^3BA^9BA^3BA^{-3}BA^{-2}BA^3BA^7 = I.$$

Replacing BA^3BA^7 by its inverse, we get

$$(A^3BA^9BA^3)B(A^3BA^9BA^3)^{-1}B = I \quad [\text{by (16)}].$$

Consider $(i^2+i+1,\ i^2+i+1)$, viz.,

$$W^5 = I,$$

where $$W = S_{i^2+i+1}T,$$

$$W^2 = S_{i+1}.S_{i^2}TS_{i^2+i}.S_1T = BA^8BA^{-4}BA^8BA,$$

$$W^4 = BA^8BA^{-4}BA^8.A^{-1}BA^{-1}.A^8BA^{-4}BA^8BA,$$

$$W^5 = W^4S_{i^2+i}S_iT = W^4BA^9BA^2BA^{10}$$

$$= BA^8BA^{-4}BA^7BA^7BA^{-4}BA^7BA^8BA^2BA^{10}.$$

Transforming both members of $W^5 = I$ by A^6, replacing $A^{-6}BA^8B$ by its inverse, and then transforming by AB,

$$ABA^{10}BA^{-1}A^3BA^7BA^7BA^{-4}BA^7BA^8BA^2 = I.$$

Replacing A^3BA^7B by its inverse, and $BA^{-1}B$ by ABA,

$$ABA^{11}BA^{-1}A^{12}BA^4BA^{-4}BA^7BA^8BA^2 = I.$$

Replacing $A^{12}BA^4B$ by its inverse, then $BA^{-1}B$ by ABA, then BAB by $A^{-1}BA^{-1}$, then BA^6BA^9 by its inverse, we get

$$ABA^{12}BA^4BA^{-6}BA^{-1}BA^2 = I.$$

Replacing $A^{12}BA^4B$ by its inverse, and $BA^{+1}B$ by ABA, we obtain an identity. Hence the theorem is proved.

On Perpetuants. *By* J. H. Grace. Communicated December 11th, 1902. Received December 16th, 1902. Revised March 4th, 1903.

1. In a recent paper I proved that perpetuants of unit degree in each quantic involved are of the form

$$(ab)^\lambda (ac)^\mu \dots (ak)^\rho (ah)^\sigma (al)^\tau,$$

where $\tau \geqslant 1$, $\sigma \geqslant 2$, $\rho \geqslant 4$, ...

It is easily seen by the methods there explained that the form

$$(ab)^\lambda (bc)^\mu \dots (kh)^\sigma (hl)^\tau,$$

with the same conditions imposed on the exponents, is equally suited to the expression of perpetuants.

In the present paper I shall find the general form of a perpetuant when all the letters do not refer to different quantics. The results lead incidentally to the generating functions discovered by MacMahon and Stroh.

2. *Statement of Results for one Quantic.*

The symbols $a_1, a_2, a_3, \dots$ all referring to the same quantic, the general form of a perpetuant is

$$(a_1a_2)^{2\alpha_1} (a_2a_3)^{\alpha_2} (a_3a_4)^{\alpha_3} \dots (a_ra_{r+1})^{\alpha_r},$$

wherein

$$a_1 \geqslant a_2, \quad a_2 \geqslant a_3+2^{r-3}, \quad a_3 \geqslant a_4+2^{r-4}, \ \ldots, a_{r-2} \geqslant a_{r-1}+2, \quad a_{r-1} \geqslant a_r+1,$$

$$a_r > 0.$$

Hence, writing

$$\begin{aligned} a_r &= 1+\xi_r, \\ a_{r-1} &= 2+\xi_r+\xi_{r-1}, \\ a_{r-2} &= 4+\xi_r+\xi_{r-1}+\xi_{r-2}, \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ a_2 &= 2^{r-2}+\xi_r+\xi_{r-1}+\ldots+\xi_2, \\ 2a_1 &= 2^{r-1}+2\,(\xi_r+\xi_{r-1}+\ldots+\xi_2+\xi_1), \end{aligned}$$

all the ξ's are zero or positive integers.

The weight ϖ is given by

$$\begin{aligned} \varpi &= 2a_1+a_2+\ldots+a_{r-1}+a_r \\ &= (2^r-1)+2\xi_1+3\xi_2+\ldots+(r+1)\,\xi_r, \end{aligned}$$

and accordingly the number of perpetuants of weight ϖ and degree i is the coefficient of z^{ϖ} in

$$\frac{z^{2^{i-1}-1}}{(1-z^2)\,(1-z^3)\ldots(1-z^n)}.$$

3. *Result for Two Quantics.*

If the letters a refer to one form and the letters b to the other, the general perpetuant of degree i in the coefficients of the first and j in the coefficients of the second is

$$(a_1a_2)^{2\alpha_1}\,(a_2a_3)^{\alpha_2}\ldots(a_{i-1}a_i)^{\alpha_{i-1}}\,(a_ib_1)^{\alpha_i}\,(b_1b_2)^{\beta_1}\ldots(b_{j-1}b_j)^{\beta_{j-1}},$$

with the conditions

$$\beta_{j-1} > 0, \quad \beta_{j-2} \geqslant \beta_{j-1}+1, \quad \beta_{j-3} \geqslant \beta_{j-2}+2, \quad \ldots, \quad \beta_2 \geqslant \beta_3+2^{j-4},$$

$$\beta_1 \geqslant \beta_2+2^{j-3},$$

$$a_i > \beta_1+2^{j-2}, \quad a_{i-1} \geqslant 2^j, \quad a_{i-2} \geqslant a_{i-1}+2^j, \quad a_{i-3} \geqslant a_{i-2}+2^{j+1},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$a_3 \geqslant a_4+2^{j+i-5}, \quad a_2 \geqslant a_3+2^{j+i-4}, \quad a_1 \geqslant a_2.$$

We may therefore write

$$\begin{aligned}
\beta_{j-1} &= 1+\zeta_j,\\
\beta_{j-2} &= 2+\zeta_j+\zeta_{j-1},\\
&\cdots \quad \cdots \quad \cdots\\
\beta_3 &= 2^{j-4}+\zeta_j+\zeta_{j-1}+\ldots+\zeta_4,\\
\beta_2 &= 2^{j-3}+\zeta_j+\ldots+\zeta_4+\zeta_3,\\
\beta_1 &= 2^{j-2}+\zeta_j+\ldots+\zeta_2,\\
\alpha_i &= 2^{j-1}+\zeta_j+\zeta_{j-1}+\ldots+\zeta_2+\zeta_1,\\
\alpha_{i-1} &= 2^j+\xi_{i-1},\\
\alpha_{i-2} &= 2^{j+1}+\xi_{i-1}+\xi_{i-2},\\
\alpha_{i-3} &= 2^{j+2}+\xi_{i-1}+\xi_{i-2}+\xi_{i-3},\\
&\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
\alpha_4 &= 2^{j+i-5}+\xi_{i-1}+\xi_{i-2}+\ldots+\xi_4,\\
\alpha_3 &= 2^{j+i-4}+\xi_{i-1}+\ldots+\xi_4+\xi_3,\\
\alpha_2 &= 2^{j+i-3}+\xi_{i-1}+\ldots+\xi_4+\xi_3+\xi_2,\\
2\alpha_1 &= 2^{j+i-2}+2\xi_{i-1}+\ldots+2\xi_4+2\xi_3+2\xi_2+2\xi_1.
\end{aligned}$$

The total weight is then

$$\varpi = (2^{j+i-1}-1)+(2\xi_1+3\xi_2+\ldots+i\xi_{i-1})+(\zeta_1+2\zeta_2+\ldots+j\zeta_j),$$

and, all the ξ's and ζ's being zero or positive integers, the generating function is

$$\frac{z^{2^{j+i-1}-1}}{(1-z^2)(1-z^3)\ldots(1-z^j)(1-z)(1-z^3)\ldots(1-z^j)}.$$

The result for any number of quantities can now be written down at once.

4. The proof of the foregoing statement will now be given. It depends on simple principles which can be best explained by the consideration of covariants of degrees 3 and 4.

As in the previous paper, I shall make frequent use of the fact that, if

$$(ab)^\lambda\,(bc)^\mu\,(cd)^\nu\ldots$$

be a covariant in n symbols, then

$$(bc)^\mu\,(cd)^\nu\ldots$$

is a covariant in $(n-1)$ symbols, and to this apply methods of re-

duction that are found before the problem of covariants in n symbols is undertaken.

It is, however, to be remarked that, if in the original covariant b and c are equivalent symbols, they are not to be so considered in the form

$$(bc)^{\mu}(cd)^{\nu}\ldots,$$

because the interchange of b and c would alter a factor we are leaving out of account in the subsidiary reduction.

There are two methods of reduction—

(i.) By means of identities of the type

$$(bc)+(ca)+(ab)=0.$$

(ii.) By interchange of two equivalent symbols.

All reductions possible by the aid of (i.) were obtained in the previous paper; accordingly, if equivalent letters are interchanged as much as possible, and the results of the first paper taken into account, it would seem a fair inference that every possible reduction is obtained. However, I shall not insist on this point in the present paper.

To facilitate such interchanging it is essential that all letters belonging to the same quantic should occur together in the symbolical forms considered, viz.,

$$(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu}\ldots;$$

for example, in the discussion of covariants of degree 2 in the coefficients of one quantic, and degree 1 in those of a second quantic, it would be inadvisable to take a and c as the equivalent symbols in

$$(ab)^{\lambda}(bc)^{\mu}.$$

The natural procedure is to take the equivalent symbols to be either a and b or b and c, and, in fact, both these cases will be considered.

5. *Covariants of Degree* 3.

Every covariant of degree 3 can be expressed as an aggregate of covariants

$$(ab)^{\lambda}(bc)^{\mu},$$

and various cases arise according as the letters belong to the same or different quantics. These cases will now be treated in order.

(I.) a, b, c refer to different quantics, *i.e.*, $a \neq b \neq c$.

As was shown in the previous paper, one covariant of each weight is reducible in virtue of the fundamental identity; it is convenient to

take that one to be that in which the exponent of (ab) is unity, so that the perpetuants are

$$(ab)^{\lambda}\,(bc)^{\mu},$$

wherein $\mu \geqslant 1$, $\lambda \geqslant 2$.

(II.) $a,\ b = c$.

Here, in addition to using the identity, we have to interchange b and c. Suppose that $\lambda+\mu = w$; then we can express all covariants of weight w in terms of

$$(ab)^{\lambda}\,(ac)^{\mu},$$

$$(ac)^{\lambda}\,(ab)^{\mu},$$

wherein $\lambda \geqslant \mu$.

The case $\lambda = \mu$ can only arise when w is even, and then we have an identity expressing $(bc)^{w}$ in terms of the above covariants; so that $(ab)^{\frac{1}{2}w}\,(ac)^{\frac{1}{2}w}$ can be expressed in terms of the forms

$$(ab)^{\lambda}\,(ac)^{\mu},$$

$$(ac)^{\lambda}\,(ab)^{\mu},$$

$$(bc)^{w},$$

where $\lambda > \mu$.

If w be odd, the identity we have used is useless for our present purpose, because, b and c being equivalent symbols, $(bc)^{w}$ vanishes, and on the other side the terms cancel out two and two. In other words, when w is even we need a reducing identity and there is one at hand, while when w is odd we need no reduction and the reducing identity is nugatory—no opportunity of reduction is lost.

Finally, since b and c are equivalent,

$$(ab)^{\lambda}\,(ac)^{\mu} = (ac)^{\lambda}\,(ab)^{\mu},$$

and hence the perpetuants are

$$(ab)^{\lambda}\,(ac)^{\mu},$$

when $\lambda > \mu$. Then, replacing (ac) by $(ab)+(bc)$, we can express

$$(ab)^{\lambda}\,(ac)^{\mu}$$

in terms of the set $(ab)^{\lambda}\,(bc)^{\mu}$,

and the inequality $\lambda > \mu$ persists, for (bc) cannot arise to a greater power than (ac) originally occurred.

Thus, when $b = c$ the perpetuants are given by

$$\lambda \geqslant \mu+1, \quad \mu \geqslant 1.$$

(III.) $a = b, c.$

By interchanging a and b we can express

$$(ab)^\lambda (bc)^\mu$$

when λ is odd, in terms of forms for which λ is even, and greater than its original value.

Also by means of the identity we may reject the case in which $\lambda = 1$, but this reduction is included in the former; hence the perpetuants are given by $\mu \geqslant 1$, λ even and $\geqslant 2$.

(IV.) $a = b = c.$

By means of Jordan's lemma we can express all products of (bc), (ca), (ab) in terms of products

$$(ab)^\lambda (bc)^\mu,$$

$$(bc)^\lambda (ca)^\mu,$$

$$(ca)^\lambda (ab)^\mu,$$

where $\lambda \geqslant 2\mu$, and, since the symbols are equivalent, the forms

$$(ab)^\lambda (bc)^\mu, \quad (bc)^\lambda (ca)^\mu, \quad (ca)^\lambda (ab)^\mu$$

are identical.

Hence all covariants except

$$(ab)^\lambda (bc)^\mu \quad (\lambda \geqslant 2\mu)$$

have been reduced.

Again, by the usual method when λ is odd, we can express

$$(ab)^\lambda (bc)^\mu$$

in terms of products in which λ is even and greater than its original value, so that for an irreducible form λ is even and equal to or greater than 2μ.

6. *Covariants of Degree* 4.

The typical form is now

$$(ab)^\lambda (bc)^\mu (cd)^\nu,$$

and we have to discuss seven cases, of which the first two contain the essential point of the present method.

(I.) $a = b = c = d.$

We first reduce the form $(bc)^\mu (cd)^\nu$,

in which c, d are equivalent to each other but not to b. Thus, by § 5 (II.),

$$\nu \geqslant 1, \quad \mu \geqslant \nu + 1.$$

We shall now show that we may take λ even and equal to or greater than 2μ. For clearness of exposition it is advisable to consider the reduction of forms

$$(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu}$$

in a definite order, viz., $(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu}$

is taken before $(ab)^{\lambda'}(bc)^{\mu'}(cd)^{\nu'}$,

when the first of the differences

$$\lambda-\lambda', \quad (\lambda+\mu)-(\lambda'+\mu'), \quad (\lambda+\mu+\nu)-(\lambda'+\mu'+\nu')$$

which does not vanish is positive.

When λ is odd we can at once express the form in terms of forms previously considered; so we may assume that λ is even in what follows.

Again, $(ab)^{\lambda}(bc)^{\mu}$ can be expressed linearly in terms of the sets

$$(ab)^{\alpha}(bc)^{\beta},$$
$$(bc)^{\alpha}(cd)^{\beta},$$
$$(ca)^{\alpha}(ab)^{\beta},$$

where $\alpha \geqslant 2\beta$. Hence $(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu}$

is a linear combination of the sets

$$(ab)^{\alpha}(bc)^{\beta}(cd)^{\nu},$$
$$(bc)^{\alpha}(ca)^{\beta}(ca)^{\nu},$$
$$(ca)^{\alpha}(ab)^{\beta}(cd)^{\nu}.$$

In the second of these sets we replace (cd) by $(ca)+(ad)$, and in the third we replace it by $(cb)+(bd)$.

The three sets are then expressed in terms of the sets

$$(ab)^{\alpha}(bc)^{\beta}(cd)^{\nu},$$
$$(bc)^{\alpha}(ca)^{\beta}(ad)^{\nu},$$
$$(ca)^{\alpha}(ab)^{\beta}(bd)^{\nu},$$

and other forms. In these others, however, the sum of the exponent of factors involving a, b, c only is greater than $\alpha+\beta$ or $\lambda+\mu$; so they have been previously considered by hypothesis.

Further, the three sets just written are equivalent, since $a=b=c$, and hence for a perpetuant we must have

$$\nu \geqslant 1, \quad \mu \geqslant \nu+1, \quad \lambda \geqslant 2\mu \text{ and even.}$$

[If the relation $\mu \geqslant \nu+1$ were disturbed in the course of the

second reduction, we should reduce $(bc)^{\mu}(cd)^{\nu}$ again, which would lead to μ being increased, and the resulting covariant would have appeared before.]

Cor.—The above considerations are sufficient to show that whenever a, b, c are equivalent and

$$(ab)^{\lambda}(bc)^{\mu}P,$$

where P involves other symbols, is a perpetuant then λ is even and equal to or greater than 2μ.

(II.) $a, b = c = d$.

Just as in (I.), we must have

$$\nu \geqslant 1, \quad \mu \geqslant \nu+1.$$

It will now be shown that $\lambda \geqslant \mu+2$.

In fact, if $\lambda+\mu = w$, the product

$$(ab)^{\lambda}(bc)^{\mu}$$

can be expressed linearly in terms of the products

$$\begin{aligned}
&(ab)^{w}, \quad (ab)^{w-1}(ac), \quad \ldots, \quad (ab)^{w-r}(ac)^{r},\\
&(ac)^{w}, \quad (ac)^{w-1}(ab), \quad \ldots, \quad (ac)^{w-r}(ab)^{r},\\
&(bc)^{w}, \quad (bc)^{w-1}(ac), \quad \ldots, \quad (bc)^{w-p}(ac)^{p},
\end{aligned}$$

provided $\qquad 2(r+1)+p+1 = w+1 \text{ or } w+2, \qquad (A)$

the alternative being settled by the fact that, w and p being given, r must be a positive integer.

In the first two of the three sets of products written above the difference between the exponents of (ab) and (ac), being at least equal to $w-2r$, is greater than p in all cases for $w-2r > p$ by the relation (A).

Now, in the form $\qquad (bc)^{w-p}(ac)^{p}(ca)^{\nu},$

unless $p > 1$, the form $\qquad (ac)_{p}(ca)^{\nu}$

can be reduced in such a way that the exponent of (ac) becomes two at least—this follows from the results for the third degree—and thus the sum of the exponents of the factors involving a, b, c only has been increased; so the covariants that arise have been discussed already.

Thus we can express the covariant

$$(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu}$$

in terms of forms already discussed and the two sets

$$(ab)^{\alpha}(ac)^{\beta}(cd)^{\nu},$$

$$(ac)^{\alpha}(ab)^{\beta}(cd)^{\nu},$$

where $\alpha > \beta+1$.

Next, the second set can be expressed by means of

$$(ca)^{\alpha}(cb)^{\beta}(cd)^{\nu},$$

where we have still $\alpha > \beta+1$, and, using $(cd) = (cb)+(bd)$, this set in turn can be expressed in terms of the set

$$(ca)^{\alpha}(cb)^{\beta}(bd)^{\nu}$$

and forms already considered.

Further, the set $(ab)^{\alpha}(ac)^{\beta}(cd)^{\nu}$

can be expressed by means of

$$(ab)^{\alpha}(bc)^{\beta}(cd)^{\nu}$$

where $\alpha > \beta+1$ still.

Hence the only forms we need retain are

$$(ab)^{\alpha}(bc)^{\beta}(cd)^{\gamma},$$

$$(ca)^{\alpha}(cb)^{\beta}(cd)^{\gamma},$$

wherein $\gamma \geqslant 1$, $\beta \geqslant \gamma+1$, $\alpha \geqslant \beta+2$.

Finally, these two sets are equivalent, since b and c can be interchanged; so for the perpetuants of degree 4 in this case we have

$$(ab)^{\lambda}(bc)^{\mu}(cd)^{\nu},$$

where $\nu \geqslant 1$, $\mu \geqslant \nu+1$, $\lambda \geqslant \mu+2$.

Cor.—By using just the same argument we can show that, if

$$(ab)^{\lambda}(bc)^{\mu}P$$

be a perpetuant, and the symbols b, c are equivalent, then we may take

$$\lambda-\mu \geqslant \kappa,$$

where κ is the smallest possible exponent of (bc) in a perpetuant involving all the letters in

$$(ab)^{\lambda}(bc)^{\mu}P$$

except a, the letters b, c being now regarded as not equivalent, although this, as may be seen from the results, makes no difference to the minimum exponent.

(III.) $a, b, c = d$.

In this case we must have

$$\nu \geqslant 1, \quad \mu \geqslant \nu+1,$$

and a, b, c being all different, the only condition to be satisfied by λ is that for a perpetuant type, viz., $\lambda \geqslant 4$.

(IV.) $a = b, \ c = d$.

Here $\nu \geqslant 1$, $\mu \geqslant \nu+1$, and, finally, λ, in addition to being 4 at least, must be even.

(V.) $a = b, c, d$.

Here $\nu \geqslant 1$, $\mu \geqslant 2$, $\lambda \geqslant 4$ and even.

(VI.) $a = b = c, d$.

Here $\nu \geqslant 1$, $\mu \geqslant 2$, and, by (I.) of this section, Cor., λ must be even and $\geqslant 2\mu$.

(VII.) $a, b = c, d$.

Here $\nu \geqslant 1$, $\mu \geqslant 2$, and, by Cor. (II.) of this section, $\lambda \geqslant \mu+2$.

7. We shall now complete the theory for a single form.

If the covariant be $(ab)^\lambda (bc)^\mu (cd)^\nu P$,

then to reduce $(bc)^\mu (cd)^\nu P$

we have to use the result for one degree lower, and remember that in such reductions b is not equivalent to $c, d, \ldots$.

Thus, for example, consider the covariant

$$(ab)^\lambda (bc)^\mu (cd)^\nu (de)^\rho$$

of degree 5, in which $a = b = c = d = e$.

In the reduction of $(bc)^\mu (cd)^\nu (de)^\rho$,

we have $b, c = d = e$.

Thus $\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant \nu+2$.

Finally, by § 6, (I.), Cor., λ is even and at least equal to 2μ.

Hence the general type is

$$(ab)^{2\alpha_1} (bc)^{\alpha_2} (ca)^{\alpha_3} (de)^{\alpha_4},$$

wherein $\alpha_4 \geqslant 1$, $\alpha_3 \geqslant \alpha_4+1$, $\alpha_2 \geqslant \alpha_3+2$, $\alpha_1 \geqslant \alpha_2$.

8. Next consider $(ab)^\lambda (bc)^\mu (cd)^\nu (de)^\rho$

with $a, b = c = d = e$.

We have, as before,

$$\rho \geqslant 1, \quad \nu = \rho + 1, \quad \mu \geqslant \nu + 2,$$

and, by § 6, (II.), Cor., $\lambda - \mu \geqslant 4$,

for 4 is the least value of the first exponent in any perpetuant of degree 4.

Hence the type is $(ab)^{a_1}(bc)^{a_2}(cd)^{a_3}(de)^{a_4}$,

wherein $a_4 \geqslant 1$, $a_3 \geqslant a_4 + 1$, $a_2 \geqslant a_3 + 2$, $a_1 \geqslant a_2 + 4$.

9. Take next $(ab)^\lambda (bc)^\mu (cd)^\nu (de)^\rho (ef)^\sigma$,

in which all the symbols are equivalent.

We have first

$$\sigma \geqslant 1, \quad \rho \geqslant \sigma + 1, \quad \nu \geqslant \rho + 2, \quad \mu \geqslant \nu + 4,$$

and, by § 6, (I.), Cor., λ is even and $\geqslant 2\mu$.

In the same case, if a were unlike the other symbols, the condition for λ would be replaced by

$$\lambda \geqslant \mu + 8.$$

After this the general theorem for perpetuants of a single form and that for perpetuants of two forms, but unit degree in one, are self-evident.

10. To develop the theory for two forms we proceed step by step with the parallel case of three forms. the degree in the extra form being unity throughout, and the symbol corresponding to that form coming first in the expression.

The method will become evident on consideration of some examples of covariants of degree 5 of the type

$$(ab)^\lambda (bc)^\mu (cd)^\nu (de)^\rho.$$

(I.) $a = b = c = d, e$.

Here, from § 6, (VII.),

$$\rho \geqslant 1, \quad \nu \geqslant 2, \quad \mu \geqslant \nu + 2,$$

and, by § 6, (I.) Cor., λ is even and $\geqslant 2\mu$.

Thus the general type of perpetuant is

$$(ab)^{2a_1}(bc)^{a_2}(cd)^{a_3}(de)^{a_4},$$

where $a_4 \geqslant 1$, $a_3 \geqslant 2$, $a_2 \geqslant a_3 + 2$, $a_1 \geqslant a_2$.

(II.) $a = b = c,\ d = e.$

Hence, by § 5, (III.),

$$\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant 4,$$

and, by § 6, (I.), Cor., λ is even and $\geqslant 2\mu$.

The type is therefore expressed by

$$a_4 \geqslant 1, \quad a_3 \geqslant a_4+1, \quad a_2 \geqslant 4, \quad a_1 \geqslant a_2.$$

(III.) $a = b = c,\ d,\ e.$

By the type reduction for degree 4,

$$\rho \geqslant 1, \quad \nu \geqslant 2, \quad \mu \geqslant 4.$$

Finally, λ is even and $\geqslant 2\mu$.

The type is given by

$$a_4 \geqslant 1, \quad a_3 \geqslant 2, \quad a_2 \geqslant 4, \quad a_1 \geqslant a_2.$$

(IV.) $a = b,\ c = d = e.$

By § 6, (II.), $\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant \nu+2.$

Finally, λ is even, and by the type reduction at least 8; thus we have as the general perpetuant

$$(ab)^{2a_1}\,(bc)^{a_2}\,(cd)^{a_3}\,(de)^{a_4},$$

wherein $a_4 \geqslant 1,\ a_3 \geqslant a_4+1,\ a_2 \geqslant a_3+2,\ a_1 > 4.$

(V.) $a,\ b = c,\ d = e.$

By § 6, (III.), $\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant 4,$

and, by § 6, (II.), Cor., $\lambda-\mu \geqslant 4.$

Hence the complete conditions are

$$\lambda \geqslant \mu+4, \quad \mu \geqslant 4, \quad \nu \geqslant \rho+1, \quad \rho \geqslant 1.$$

(VI.) $a,\ b = c = d = e.$

By § 6, (II.), $\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant \nu+2,$

and, finally, $\lambda \geqslant \mu+4.$

(VII.) $a,\ b = c = d,\ e.$

By § 6, (VII.), $\rho \geqslant 1, \quad \nu \geqslant 2, \quad \mu \geqslant \nu+2,$

and $\lambda \geqslant \mu+4.$

(VIII.) $a, b, c = d = e.$

$$\rho \geqslant 1, \quad \nu \geqslant \rho+1, \quad \mu \geqslant \nu+2, \quad \lambda \geqslant 8.$$

11. The complete theory for two forms now presents no difficulty. Consider, for example,

$$(ab)^{\lambda}\,(bc)^{\mu}\,(cd)^{\nu}\,(de)^{\rho}\,(ef)^{\sigma},$$

wherein $a = b = c, \quad d = e = f.$

By the reduction of

$$(bc)^{\mu}\,(cd)^{\nu}\,(de)^{\rho}\,(ef)^{\sigma} \quad (b, c, d = e = f),$$

we have $\sigma \geqslant 1, \quad \rho \geqslant \sigma+1, \quad \nu \geqslant \rho+2, \quad \mu \geqslant 8$ (§ 9, VIII.),

and, by § 6, (I.), Cor., λ is even and equal to or greater than 2μ.

Thus the general form is

$$(ab)^{2\alpha_1}\,(bc)^{\alpha_2}\,(cd)^{\alpha_3}\,(de)^{\alpha_4}\,(ef)^{\alpha_5},$$

where $\alpha_5 \geqslant 1, \ \alpha_4 \geqslant \alpha_5+1, \ \alpha_3 \geqslant \alpha_4+2, \ \alpha_2 \geqslant 8, \ \alpha_1 \geqslant \alpha_2.$

In general for two forms the conditions are like those for perpetuants of unit degree in the second form until we come to the first factor involving two letters belonging to the second form. The exponent of this is determined from the type theory, and then the remaining conditions are written down by the aid of § 6, (II.), Cor., until we come to the first exponent of all.

It will be noticed that the simplest perpetuant of any given degree is of a form which is quite independent of the symbols being alike or different.

On the Calculation of the Finite Equations of a Continuous Group. *By* H. F. Baker. Received and Read December 11th, 1902.

The following remark which was judged too symbolical to be included in a previous note (*Proc. Lond. Math. Soc.*, Vol. xxxiv., p. 91) seems worth making. It obtains the solution of a system of equations, there shown to have unique solutions, by means of linear equations only.

If E denote, as before, the matrix whose general element is

$$E_{\rho\sigma} = \sum_{\tau} c_{\sigma\tau\rho} e_\tau \quad (\rho, \sigma, \tau = 1, \ldots, r),$$

and Δ denote the matrix

$$\Delta = 1 + E + \frac{E^2}{2!} + \frac{E^3}{3!} + \ldots,$$

the series $F = \Delta - 1 - \frac{1}{2}(\Delta-1)^2 + \frac{1}{3}(\Delta-1)^3 - \ldots$

converges when $e_1, \ldots, e_r$ are small enough, since then the roots of the determinantal equation

$$|\Delta - 1 - \phi| = 0$$

will be of modulus less than unity; if D denote the operator

$$D = e_1 \frac{\partial}{\partial e_1} + \ldots + e_r \frac{\partial}{\partial e_r},$$

we have $D\Delta = E\Delta.$

and hence
$$\begin{aligned} DF &= E\Delta - E\Delta(\Delta-1) + E\Delta(\Delta-1)^2 - \ldots \\ &= E\Delta\left[1 + (\Delta-1)\right]^{-1} \\ &= E, \end{aligned}$$

namely, $$D(F-E) = 0;$$

as then $F-E$ is zero when $e_1, \ldots, e_r$ are all zero, it is always zero; namely, for values of $e_1, \ldots, e_r$ in the neighbourhood of zero, we have

$$E = \Delta - 1 - \tfrac{1}{2}(\Delta-1)^2 + \tfrac{1}{3}(\Delta-1)^3 - \ldots.$$

Now, let $$e''_\sigma = \psi_\sigma(e, e') \quad (\sigma = 1, \ldots, r)$$

be the finite equations of the group in question, with canonical

variables e and e'', and canonical parameters e'; then, by the so-called exponential theorem,

$$\Delta_{e''} = \Delta_{e'} \Delta_e,$$

and hence $E'' = \Delta_{e'} \Delta_e - 1 - \frac{1}{2}(\Delta_{e'} \Delta_e - 1)^2 + \frac{1}{3}(\Delta_{e'} \Delta_e - 1)^3 - \dots$.

Here everything on the right side is directly calculable from the constants of structure. When the group has no special infinitesimal transformations, we know that among the r^2 linear forms in $e_1'', \dots, e_r''$ which constitute the elements of the matrix E'', there are r linearly independent; hence by solution of only linear equations we obtain the finite equations of the group. When there are $r-m$ special infinitesimal transformations we do not in this way obtain more than m independent equations; but the solution can be completed by quadratures, as explained in the note referred to.

The formula $E = \Delta - 1 - \frac{1}{2}(\Delta - 1)^2 + \frac{1}{3}(\Delta - 1)^3 - \dots$

may be proved in another way, without differentiation, by using (1) the general formula for the sum of such a series of matrices, (2) the fact that, if $\theta_1, \theta_2, \dots$ be the roots of the determinantal equation

$$|E - \theta| = 0,$$

the roots of

$$|\Delta - \phi| = 0$$

are given by

$$\phi = e^{\theta};$$

in fact to every invariant factor of the first equation there is one of the latter of the same exponent. The formula is only an algebraical formula of inversion involving the roots of two related algebraical equations.

On the Integration of Linear Differential Equations.

By H. F. Baker. Received and Read December 11th, 1902.

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Part I.

Let u_1, u_2, ... each denote square matrices of the same number of rows and columns whose elements are analytic functions of the complex variable t; let barriers be made joining the singularities of these functions to $t = \infty$, so that within the star region bounded thereby each function is developable about every point in an ordinary power series; let t_0 be an arbitrary fixed point in the region, and Qu_i denote the matrix of which any element is obtained by integrating the corresponding element of u_i from t_0 to t along a path lying within the region; further, let a_1, a_2, ... be such constants that the series

$$1 + a_1 t + \frac{a_2 t^2}{2!} + \frac{a_3 t^3}{3!} + \ldots$$

converges for all finite values of t, however great, and let the symbols $Qu_1 Qu_2$, $Qu_1 Qu_2 Qu_3$, ... denote $Q(u_1 Qu_2)$, $Q\left[u_i Q(u_2 Qu_3)\right]$, ..., so that each integration denoted by Q affects all that follow it in any symbol.

Then form the series of matrices

$$\nabla = 1 + a_1 Qu_1 + a_2 Qu_1 Qu_2 + a_3 Qu_1 Qu_2 Qu_3 + \ldots,$$

defining a single matrix of which any element is the infinite series of corresponding elements, one from each term of the series ∇. It can be shown, as in *Proc. Lond. Math. Soc.*, Vol. xxxiv., 1902, pp. 354, 359,

that each of these infinite series converges over the whole of the infinite star region described. The matrix ∇ thus exists, and is finite and developable over the whole of this region.

Such a matrix as ∇ may be called a *matrizant;* we shall restrict the term to the case when each of the matrices $u_1, u_2, \ldots$ is the same matrix u, and each of the constants $a_1, a_2, \ldots$ is unity, and shall denote the matrizant in that case by $\Omega(u)$, or sometimes, in more detail, by $\Omega^{t,t_0}(u)$. Its most fundamental property is that

$$\frac{d}{dt}\Omega(u) = u\Omega(u),$$

so that each of its columns furnishes a set of solutions of the system of linear differential equations

$$\frac{dx_i}{dt} = u_{i1}x_1 + \ldots + u_{in}x_n \quad (i = 1, \ldots, n),$$

and, since for $t = t_0$ we have

$$\Omega(u) = 1,$$

these sets of solutions are independent.

There are various other properties, partly of a formal nature, which must be clearly stated.

1. It is necessary to repeat the process for summing a series of matrices in order to show clearly how the infinite series of logarithms which may arise in the terms of a matrizant may disappear in the sum. If $f(t)$ denote an integral function

$$f(t) = 1 + a_1\frac{t}{1!} + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + \ldots,$$

and M be any matrix satisfying the equation

$$M^\mu = \lambda + \lambda_1 M + \ldots + \lambda_{\mu-1}M^{\mu-1},$$

and no equation of lower order, and the roots of the algebraic equation in θ,

$$\theta^\mu = \lambda + \lambda_1\theta + \ldots + \lambda_{\mu-1}\theta^{\mu-1}$$

—say, $\theta_1, \theta_2, \ldots, \theta_\mu$—be all different, the series

$$\nabla = 1 + a_1\frac{M}{1!} + a_2\frac{M^2}{2!} + a_3\frac{M^3}{3!} + \ldots,$$

wherein each term is to be reduced to a polynomial in M of order

$\mu-1$, will when rearranged in powers of M have for sum a form

$$\nabla = C_0 + C_1 M + \ldots + C_{\mu-1} M^{\mu-1},$$

in which the single quantities C_0, C_1, ..., $C_{\mu-1}$ have the values obtained by expressing the identity

$$f(\theta) = C_0 + C_1\theta + \ldots + C_{\mu-1}\theta^{\mu-1}$$

for all the roots θ_1, ..., θ_μ of a supposed irreducible equation

$$\theta^\mu = \lambda + \lambda_1\theta + \ldots + \lambda_{\mu-1}\theta^{\mu-1};$$

this gives at once (*Proc. Lond. Math. Soc.*, Vol. XXXIV., pp. 114, 359)

$$\nabla = \frac{D_0}{D} + \frac{D_1}{D} M + \ldots + \frac{D_{\mu-1}}{D} M^{\mu-1},$$

where

$$D = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \theta_1 & \theta_2 & \ldots & \theta_\mu \\ \ldots & \ldots & \ldots & \\ \theta_1^{\mu-1} & \theta_2^{\mu-1} & \ldots & \theta_\mu^{\mu-1} \end{vmatrix}$$

is the product of the differences of the roots θ_1, ..., θ_μ, and D_{i-1} is the determinant obtained from D by replacing the elements of the i-th row of D by $f(\theta_1)$, ..., $f(\theta_\mu)$.

This formula yields at once the corresponding formula when the roots θ_1, ..., θ_μ are not all different; for instance, by subtracting the first two columns of every one of the determinants D, D_0, ..., $D_{\mu-1}$, dividing by $\theta_2-\theta_1$, and proceeding to the limit, we obtain the case $\theta_2 = \theta_1$. The quantities $C_i = D_i/D$, which, in the case of unequal roots, are linear functions of $f(\theta_1)$, ..., $f(\theta_\mu)$, become in general linear functions of these quantities and their differential coefficients in regard to θ_1, ..., θ_μ; the coefficients in these linear functions are the same whatever be the form of the function f, and depend only on the matrix M; they may therefore be calculated from a suitably chosen simple form of $f(t)$, in particular $(1-t)^{-1}$. Thus we prove the formula of Frobenius and Stickelberger (references in Bromwich, *Proc. Camb. Phil. Soc.*, Vol. XI., 1901, p. 79), that, if the inverse matrix $(\theta-M)^{-1}$ be arranged in the form

$$(\theta-M)^{-1} = \sum_{i=1}^{\mu} \left[\frac{M_i}{\theta-\theta_i} + \frac{N_i}{(\theta-\theta_i)^2} + \ldots + \frac{K_i}{(\theta-\theta_i)^{\epsilon_i}} \right],$$

where M_i, N_i, ..., K_i are matrices, then

$$\nabla = \sum_{i=1}^{\mu} \left[f(\theta_i) M_i + f'(\theta_i) N_i + \ldots + \frac{f^{(\epsilon_i-1)}(\theta_i)}{(\epsilon_i-1)!} K_i \right].$$

One of the simplest cases of $\Omega(u)$ is when u has the form $t^{-1}M$, where M is a matrix of constants; then

$$Qu = M\log t/t_0, \quad Qu\,Qu = M^2Q(t^{-1}\log t/t_0) = \frac{M^2}{2}(\log t/t_0)^2, \quad \ldots,$$

or, if $\lambda = \log t/t_0$,

$$\Omega(u) = 1 + \lambda M + \frac{\lambda^2}{2!}M^2 + \frac{\lambda^3}{3!}M^3 + \ldots;$$

thus, if the matrices of constants M_i, N_i, ... be determined from the matrix M, we have, putting $f(\theta) = e^{\lambda\theta}$,

$$\Omega(t^{-1}M) = \sum_{i=1}^{\mu}\left\{ M_i\left(\frac{t}{t_0}\right)^{\theta_i} + N_i\left(\frac{t}{t_0}\right)^{\theta_i}\log\frac{t}{t_0} + \ldots \right.$$
$$\left. \ldots + K_i\left(\frac{t}{t_0}\right)^{\theta_i}\frac{1}{(\epsilon_i-1)!}\left(\log\frac{t}{t_0}\right)^{\epsilon_i-1}\right\},$$

the exponent $\epsilon_i - 1$ of the highest power of the logarithm corresponding to the root θ_i being one less than the multiplicity of this root in the equation satisfied by M, namely, than the exponent of the first invariant factor of the matrix $M-\theta$ corresponding to this root.

2. A second property to be remarked is that, if M be a matrix of constants, we have

$$M\Omega(u)\,M^{-1} = \Omega(MuM^{-1}).$$

3. If $s = \phi(t)$ be uniquely reversible, in a form $t = \phi_1(s)$, or we confine t to a region about t_0 for which this is so, and if $t_0 = \varphi(h_0)$, and

$$\Omega^{t,t_0}[u(t)] = F(t),$$

then

$$F(s) = \Omega^{t,h_0}\left\{ u[\varphi(t)]\frac{d\phi(t)}{dt}\right\}.$$

If, for instance,

$$s = \phi(t) = \frac{at+b}{ct+d}, \quad t_0 = h_0 = \frac{at_0+b}{ct_0+d}, \quad \Omega(u) = F(t),$$

then

$$F\left(\frac{at+b}{ct+d}\right) = \Omega\left[\frac{ad-bc}{(ct+d)^2}\,u\left(\frac{at+b}{ct+d}\right)\right];$$

it may happen that a matrix of constants, μ, can be found such that

$$\mu^{-1}u(t)\,\mu = \frac{ad-bc}{(ct+d)^2}\,u\left(\frac{at+b}{ct+d}\right);$$

then

$$F\left(\frac{at+b}{ct+d}\right) = \mu^{-1}\Omega(u)\,\mu.$$

4. We clearly have

$$\Omega^{t,t_0}(u) = \Omega^{t,t_1}(u)\,\Omega^{t_1,t_0}(u); \quad [\Omega^{t,t_0}(u)]^{-1} = \Omega^{t_0,t}(u).$$

We have, however,

$$[\Omega^{t,t_0}(u)]^{-1} = \overline{\Omega^{t,t_0}(-\bar{u})},$$

where $\bar{u}$ denotes the matrix obtained from u by interchange of rows and columns, and the same transposition is to be carried out after $\Omega(-\bar{u})$ has been calculated.

For, if $U = \Omega(u)$,

$$\frac{dU^{-1}}{dt} = -U^{-1}\frac{dU}{dt}U^{-1} = -U^{-1}u,$$

and, if $V = \Omega(-\bar{u})$, $W = \bar{V}$,

$$\frac{dV}{dt} = -\bar{u}V, \quad \frac{dW}{dt} = -Wu,$$

and hence

$$\frac{d}{dt}(WU) = \frac{dW}{dt}U + W\frac{dU}{dt} = -WuU + WuU = 0,$$

while, for $t = t_0$, we have $W = 1$, $U = 1$;

thus, as stated, $U^{-1} = W$,

or $[\Omega(u)]^{-1} = \Omega(-\bar{u}) = 1 - Qu + Q(Qu.u) - Q[Q(Qu.u)u] + \ldots,$

where each term is formed from the preceding by multiplying by u on the right and then integrating the product, or, if v_n denote the general term,

$$v_{n+1} = -Q(v_n u).$$

5. If u, h be arbitrary matrices of the same order, the latter of non-vanishing determinant, and a star region be constructed within which the elements of both the matrices

$$u, \quad huh^{-1} + \frac{dh}{dt}h^{-1}$$

are everywhere developable, we have

$$\Omega\left(huh^{-1} + \frac{dh}{dt}h^{-1}\right) = h\Omega(u)h_0^{-1},$$

where h_0 is the value of h at $t = t_0$. For, if

$$v = huh^{-1} + \frac{dh}{dt}h^{-1},$$

$$\frac{d}{dt}[\Omega^{-1}(u)\,h^{-1}\Omega(v)]$$

$$= -\Omega^{-1}(u)\,uh^{-1}\Omega(v) - \Omega^{-1}(u)\,h^{-1}\frac{dh}{dt}h^{-1}\Omega(v) + \Omega^{-1}(u)\,h^{-1}v\Omega(v)$$

$$= -\Omega^{-1}(u)\,h^{-1}\left\{ huh^{-1} + \frac{dh}{dt}h^{-1} - v \right\}\Omega(v) = 0,$$

and for $t = t_0$ we have

$$\Omega^{-1}(u)\,h^{-1}\Omega(v) = h_0^{-1}.$$

Taking $$h = \Omega(w), \quad huh^{-1} = \sigma,$$

we obtain $$\Omega(w+\sigma) = \Omega(w)\,\Omega[\Omega^{-1}(w)\,\sigma\Omega(w)].$$

6. Two particular cases of § 5 seem worth remark. First, if $w_1, w_2, \ldots, w_\mu$ be each single functions of t, and $A_1, \ldots, A_\mu$ be matrices of constants of the same order of which any two are commutable, so that $A_\rho A_\sigma = A_\sigma A_\rho$, and the star region be suitably constructed, we have

$$\Omega(w_1A_1 + \ldots + w_\mu A_\mu) = \Omega(w_1A_1)\,\Omega(w_2A_2)\ldots\Omega(w_\mu A_\mu).$$

For, if $$\sigma = w_2A_2 + \ldots + w_\mu A_\mu,$$

we have $$\Omega^{-1}(w_1A_1)\,\sigma\Omega(w_1A_1) = \sigma.$$

Second, if $A_1 = 1$, we have

$$\Omega(w_1+u) = e^{Qw_1}\Omega(u),$$

u being an arbitrary matrix, a result obtainable by multiplying the series

$$\Omega(u) = 1 + Qu + QuQu + \ldots,$$

$$e^{Qw_1} = 1 + Qw_1 + Qw_1Qw_1 + \ldots .$$

7. If ξ denote a matrix of which any column consists of the elements of one of n linearly independent sets of solutions of the system of linear differential equations

$$\frac{dx_i}{dt} = u_{i1}x_1 + \ldots + u_{in}x_n,$$

so that $$\frac{d\xi}{dt} = u\xi,$$

then, by the *cogredient* substitution, wherein h is an arbitrary matrix for which $|h| \neq 0$,

$$\eta = h\xi,$$

we obtain the system

$$\frac{d\eta}{dt} = v\eta, \quad v = huh^{-1} + \frac{dh}{dt}h^{-1},$$

and, by the *contragredient* substitution, wherein again $|k| \neq 0$,

$$\zeta = k(\bar{\xi})^{-1},$$

we obtain the linear system

$$\frac{d\xi}{dt} = w\zeta, \quad w = -k\bar{u}k^{-1} + \frac{dk}{dt}k^{-1}.$$

If the matrices u, v, w be assigned beforehand, these equations give for the forms of the matrices h, k, by §§ 4, 5, the equations

$$h = \Omega(v)h_0\Omega^{-1}(u), \quad k = \Omega(w)k_0\overline{\Omega(u)},$$

where the constant matrices h_0, k_0 are arbitrary.

8. Thus any two linear systems can be transformed into one another either cogrediently or contragrediently. Now a single linear equation

$$\lambda_n\frac{d^n y}{dt^n} + \lambda_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + \lambda_0 y = 0$$

can be reduced to a linear system in various ways; taking here

$$x_1 = y, \quad x_2 = \frac{dy}{dt}, \quad \ldots, \quad x_n = \frac{d^{n-1}y}{dt^{n-1}}$$

for the elements of a column of a matrix ξ, whose n columns are obtained by putting y equal to the constituents of a fundamental system of integrals for the single linear equation, we have what we may for the present call the special linear system

$$\frac{d\xi}{dt} = u\xi,$$

where in the first $(n-1)$ rows of u every element is zero, except that one immediately to the right of the diagonal element, which is unity, while the n-th row consists of the elements

$$-\frac{\lambda_0}{\lambda_n}, \quad -\frac{\lambda_1}{\lambda_n}, \quad \ldots, \quad -\frac{\lambda_{n-1}}{\lambda_n}.$$

Any linear system can then, by § 7, be reduced to a special linear system, and so to a single linear equation, arbitrarily given, either cogrediently or contragrediently, the necessary explicit form of the

matrix h or k used in the transformation requiring a knowledge of the solutions of the systems to be connected by the transformation.

If, however, we be given a linear system, which may be special, and require to transform this to some single linear equation, say contragrediently, by suitable choice of a matrix k, the equation

$$wk = -k\bar{u} + \frac{dk}{dt},$$

wherein w has the form just explained, arising for a special linear system, shows that the elements of k satisfy equations

$$k_{i+1,j} = \frac{dk_{i,j}}{dt} - (u_{j,1}k_{i,1} + \ldots + u_{j_n}k_{i_n})$$
$$(j = 1 \ldots n;\ i = 1 \ldots n-1),$$

namely, that the $(i+1)$-th row of k is determined from the i-th by the rule

$$k^{(i+1)} = \frac{dk^{(i)}}{dt} - uk^{(i)}.$$

Conversely, if we take the first row of k arbitrarily and determine the subsequent rows by this rule from the first row, it is easily seen that w has the form for a special linear system, namely, the elements of any column of the matrix ζ satisfy equations

$$\frac{dz_1}{dt} = z_2,\ \frac{dz_2}{dt} = z_3,\ \ldots,\ \frac{dz_{n-1}}{dt} = z_n,\quad \frac{dz_n}{dt} = w_{n1}z_1 + \ldots + w_{nn}z_n.$$

In particular, if the original system (u) is special, derived from

$$\lambda_n \frac{d^n x}{dt^n} + \lambda_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + \lambda_0 x = 0,$$

and we take $\quad k_{11} = 0 = k_{12} = \ldots = k_{1n-1},\quad k_{1n} = -\dfrac{1}{\lambda_n}$,

it can be shown that the (w) system belongs to the so-called Lagrange adjoint equation

$$\lambda z - \frac{d}{dt}(\lambda_1 z) + \frac{d^2}{dt^2}(\lambda_2 z) + \ldots + (-1)^n \frac{d^n}{dt^n}(\lambda_n z) = 0.$$

Obviously the most general contragredient transformation

$$\zeta = k\,(\bar{\xi})^{-1}$$

may be obtained by first applying a particular contragredient transformation $\zeta_0 = k_0 (\bar{\xi})^{-1}$, and then a cogredient transformation $\zeta = k k_0^{-1} \zeta_0$. For instance, when we are transforming from a special linear system we may take the special contragredient transformation to be that to Lagrange's adjoint equation; or we may take $k_0 = 1$, though in that case the intermediate system will not be a special system. Also, as the equations of a contragredient transformation

$$wk + k\bar{u} = \frac{dk}{dt}, \quad k = \Omega(w)\, k_0 \overline{\Omega(u)},$$

are equivalent to

$$u\bar{k} + \bar{k}\bar{w} = \frac{d\bar{k}}{dt}, \quad \bar{k} = \Omega(u)\, \bar{k}_0\, \Omega(w),$$

the inverse transformation, as applied to the new system to return to the original, is obtained by changing the rows of k into columns; in the case when we are transforming from a special system (u) to a special system (w), this gives the incidental result that, if the rows of a matrix k of non-vanishing determinant are determined from the first (arbitrary) row by the relation

$$k^{(i+1)} = \frac{dk^{(i)}}{dt} - uk^{(i)},$$

then the columns of this matrix are determined from the first column by the relation

$$k^{(j+1)} = \frac{dk^{(j)}}{dt} - wk^{(j)},$$

the special matrices u, w being connected by the equation

$$wk + k\bar{u} = \frac{dk}{dt}.$$

This explains the reciprocal relations of an equation, and its adjoint; for instance, for $n = 3$, the forms of the special matrices u, w and two forms of the matrix k are respectively

$$u = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\lambda}{\lambda_3} & -\frac{\lambda_1}{\lambda_3} & -\frac{\lambda_2}{\lambda_3} \end{Bmatrix}, \quad w = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\mu}{\mu_3} & -\frac{\mu_1}{\mu_3} & -\frac{\mu_2}{\mu_3} \end{Bmatrix},$$

$$k = \begin{Bmatrix} 0 & 0 & -\dfrac{1}{\lambda_3} \\ 0 & \dfrac{1}{\lambda_3} & -\dfrac{\lambda_2+\lambda_3'}{\lambda_3^2} \\ -\dfrac{1}{\lambda_3} & \dfrac{\lambda_2-2\lambda_3'}{\lambda_3^2} & \dfrac{\lambda_1+\lambda_2'+\lambda_3''}{\lambda_3^2} - \dfrac{(\lambda_2-2\lambda_3')(\lambda_2-\lambda_3')}{\lambda_3^3} \end{Bmatrix}$$

$$= - \begin{Bmatrix} 0 & 0 & -\dfrac{1}{\mu_3} \\ 0 & \dfrac{1}{\mu_3} & \dfrac{\mu_2-2\mu_3'}{\mu_3^2} \\ -\dfrac{1}{\mu_3} & -\dfrac{\mu_2+\mu_3}{\mu_3^2} & \dfrac{\mu_1-\mu_2'+\mu_3''}{\mu_3^2} - \dfrac{(\mu_2-2\mu_3')(\mu_2-\mu_3')}{\mu_3^3} \end{Bmatrix}.$$

Precisely similar remarks may be made as to cogredient transformations; in fact the relation

$$h = \Omega(v)\, h_0 \Omega^{-1}(u)$$

is the same as

$$h = \Omega(v)\, h_0 \overline{\Omega(-\bar{u})},$$

and is a cogredient equation starting from equations in which u is replaced by $-\bar{u}$, that is, from the system obtained from the original by the particular contragredient transformation $\zeta_0 = (\bar{\xi})^{-1}$.

9. It has been remarked that there are various ways of reducing a single linear equation to a linear system. We explain now a way which is of great importance in the sequel of this paper.

If the single linear equation be

$$\frac{d^n y}{dt^n} = \frac{P_{n-1}}{\phi_n}\frac{d^{n-1}y}{dt^{n-1}} + \frac{P_{n-2}}{\phi_{n-1}\phi_n}\frac{d^{n-2}y}{dt^{n-2}} + \frac{P_{n-3}}{\phi_{n-2}\phi_{n-1}\phi_n}\frac{d^{n-3}y}{dt^{n-3}} + \ldots + \frac{P}{\phi_1\phi_2\ldots\phi_n}y,$$

and we put, with $y' = dy/dt$, &c.,

$$x_1 = y, \quad x_2 = \phi_1 y', \quad x_3 = \phi_1\phi_2 y'', \quad x_4 = \phi_1\phi_2\phi_3 y''', \quad \ldots$$

$$\ldots, \quad x_n = \phi_1\phi_2\ldots\phi_{n-1}y^{(n-1)},$$

then $\dfrac{dx_1}{dt} = \dfrac{x_2}{\phi_1}$, $\dfrac{dx_2}{dt} = \dfrac{\phi_1'}{\phi_1}x_2 + \dfrac{x_3}{\phi_2}$, $\dfrac{dx_3}{dt} = \left(\dfrac{\phi_1'}{\phi_1} + \dfrac{\phi_2'}{\phi_2}\right)x_3 + \dfrac{x_4}{\phi_3}$, ...

$$\ldots, \quad \frac{dx_{n-1}}{dt} = \left(\sum_{r=1}^{n-2}\frac{\phi_r'}{\phi_r}\right)x_{n-1} + \frac{x_n}{\phi_{n-1}},$$

and $$\frac{dx_n}{dt} = \left(\sum_{r=1}^{n-1} \frac{\phi_r'}{\phi_r}\right) x_n + \frac{P}{\phi_n} x_1 + \frac{P_1}{\phi_n} x_2 + \frac{P_2}{\phi_n} x_3 + \ldots + \frac{P_{n-1}}{\phi_n} x_n ;$$

so that, if for brevity $$H_m = \sum_{r=1}^{m} \frac{\phi_r'}{\phi_r},$$

$$\frac{dx}{dt} = \left\{ \begin{array}{cccccccc} 0 & \frac{1}{\phi_1} & 0 & 0 & \ldots & 0 & 0 \\ 0 & H_1 & \frac{1}{\phi_2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & H_2 & \frac{1}{\phi_3} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & H_{n-2} & \frac{1}{\phi_{n-1}} \\ \frac{P}{\phi_n} & \frac{P_1}{\phi_n} & \frac{P_2}{\phi_n} & \frac{P_3}{\phi_n} & \ldots & \frac{P_{n-2}}{\phi_n} & \frac{P_{n-1}}{\phi_n} + H_{n-1} \end{array} \right\} x.$$

Various cases of this formula to be referred to are—

(*a*) When every one of the functions of t denoted by $P, P_1, \ldots, P_{n-1}, \phi_1, \ldots, \phi_n$ is an integral polynomial, and no one of $\phi_1, \ldots, \phi_n$ has a multiple factor, though any two or more of them may have common factors; then each of the rational functions

$$\frac{1}{\phi_s} = \sum_c \frac{1/\phi_s'(c)}{t-c_s}, \quad H_m = \sum_{r=1}^{m} \sum_c \frac{1}{t-c_r}, \quad \frac{P_i}{\phi_n} = (1, t) + \sum_c \frac{P_i(c_n)/\phi_n'(c_n)}{t-c_n}$$

occurring in the matrix has only poles of the first order for finite values of t, and the linear system has a form

$$\frac{dx}{dt} = \left[A + A_1 t + \ldots + A_\mu t^\mu + \sum_i C \frac{1}{t-c} \right] x,$$

wherein A, A_1, ..., A_μ, C are matrices of constants and the summation refers to all the poles c which arise. The value of μ is zero and $A = 0$ when ϕ_n is of greater dimension than any of $P, P_1, \ldots, P_{n-1}$, and otherwise μ is equal to the difference of the dimensions of ϕ_n and the highest dimension of any of these.

(*b*) When each of the functions $P, P_1, \ldots, P_{n-1}$ is an integral polynomial and $\phi_1 = \phi_2 = \ldots = \phi_n = \phi$ is an integral polynomial without repeated roots.

(*c*) When $P, \ldots, P_{n-1}$ are again polynomials, ϕ_n is a polynomial

without repeated roots, and $\phi_1 = \phi_2 = \ldots = \phi_{n-1} = 1$. Then each of the functions $H_1, \ldots, H_{n-1}$ is zero. And further simplification arises when the dimensions of $P, P_1, \ldots, P_{n-1}$ are equal to or less than that of ϕ_n.

(*d*) When $P, P_1, \ldots, P_{n-1}$ are analytical functions for which $t-c$ is not a singularity, and each of $\phi_1, \ldots, \phi_n$ is either unity or $t-c$. In the ordinary case in which no one of $P, P_1, \ldots, P_{n-1}$ vanishes at $t = c$ and ϕ_n is $t-c$, the polynomial with matrix coefficients, $A + A_1 t + \ldots + A_\mu t^\mu$ does not enter at all in the resulting linear system.

Of these (*a*), (*b*), (*c*) are intimately related, while (*d*) includes cases in which the coefficients in the linear differential equation are algebraical functions of t.

As a particular example of (*a*) we may take the equation

$$(t+1)\,t^3 y''' - [(a_2+b_2)\,t + a_2]\,t^2 y'' - [(a_1+b_1)\,t + a_1]\,ty' - [(a+b)\,t + a]\,y = 0,$$

which, with $\phi_3 = t\,(t+1), \quad \phi_2 = t, \quad \phi_1 = t,$

$$P = a\,(t+1) + bt, \quad P_1 = a_1\,(t+1) + b_1 t, \quad P_2 = a_2\,(t+1) + b_2 t,$$

leads to $$\frac{dx}{dt} = \left[\frac{1}{t}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ a & a_1 & a_2+2 \end{pmatrix} + \frac{1}{t+1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & b_1 & b_2 \end{pmatrix}\right] x.$$

As an example of (*b*) we may take

$$y'' = \frac{At+B}{t\,(t-1)}\,y' + \frac{Ct^2+Dt+E}{t^2\,(t-1)^2}\,y,$$

which, with

$$\phi_2 = \phi_1 = t\,(t-1), \quad P = Ct^2 + Dt + E, \quad P_1 = At + B,$$

leads to

$$\frac{dx}{dt} = \left[\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} + \frac{1}{t}\begin{pmatrix} 0 & -1 \\ -E & 1-B \end{pmatrix} + \frac{1}{t-1}\begin{pmatrix} 0 & 1 \\ C+D+E & A+B+1 \end{pmatrix}\right] x.$$

As an example of (*c*) we may take

$$y'' = y'\left[\lambda + \sum_{r=1}^{\sigma} \frac{\lambda_r}{t-c_r}\right] + y\left[\mu + \sum_{r=1}^{\sigma} \frac{\mu_r}{t-c_r}\right],$$

which, with $$\phi_2 = \prod_{r=1}^{\sigma} (t-c_r), \quad \phi_1 = 1,$$

$$P_1 = \phi_1 \left[\lambda + \sum_{r=1}^{\sigma} \frac{\lambda_r}{t-c_r}\right], \quad P = \phi_2 \left[\mu + \sum_{r=1}^{\sigma} \frac{\mu_r}{t-c_r}\right],$$

leads to $$\frac{dx}{dt}\left[\begin{pmatrix} 0 & 1 \\ \mu & \lambda \end{pmatrix} + \sum_{r=1}^{\sigma} \frac{1}{t-c_r} \begin{pmatrix} 0 & 0 \\ \mu_r & \lambda_r \end{pmatrix}\right] x.$$

As an example of (d) we may take the same equation, putting now

$$x_1 = y, \quad x_2 = (t-c_1)\, y';$$

then we find

$$\frac{dx}{dt} = \left[\begin{pmatrix} 0 & 0 \\ \sum_{r=1}^{\sigma} \mu_r & \lambda \end{pmatrix} + (t-c_1) \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix} + \frac{1}{t-c_1} \begin{pmatrix} 0 & 1 \\ 0 & \lambda_1+1 \end{pmatrix} + \sum_{r=2}^{\sigma} \frac{1}{t-c_r} \begin{pmatrix} 0 & 0 \\ \mu_r(c_r-c_1) & \lambda_r \end{pmatrix} \right] x,$$

which takes a simpler form in the case of an equation of common occurrence for which $\lambda = 0$, $\mu = 0$, and $\sum_{r=1}^{\sigma} \mu_r = 0$; in that case the equation is, in the usual phraseology, *regular* at infinity.

As another example of (d), of the greatest importance for our purpose, we may take

$$y^{(n)} = \frac{p_{n-1}+tQ_{n-1}}{t} y^{(n-1)} + \frac{p_{n-2}+tQ_{n-2}}{t^2} y^{(n-2)} + \ldots + \frac{p+tQ}{t^n} y,$$

where p, p_1, ..., p_{n-1} are constants, and Q, Q_1, ..., Q_{n-1} are developable about $t = 0$ in positive integral powers; putting

$$\phi_n = \phi_{n-1} = \ldots = \phi_1 = t, \quad P_i = p_i + tQ_i,$$

we obtain $$\frac{dx}{dt} = \left(\frac{A}{t} + V\right) x$$

where $$A = \begin{Bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ p & p_1 & p_2 & p_3 & \ldots & p_{n-1}+n-1 \end{Bmatrix}$$

is a matrix of constants and V is a matrix of which all the elements

are zero, except those in the last row, which are

$$Q,\ Q_1,\ \ldots,\ Q_{n-1}.$$

It will be seen below that the algebraic equation $|A-\rho|=0$ is the so-called index equation at $t=0$.

10. In connexion with these examples the obvious fact seems worth remarking at once that, if a new variable s can be found such that each of the logarithms

$$\log(t-c_1),\quad \log(t-c_2),\quad \ldots,\quad \log(t-c_\sigma)$$

is a single-valued analytic function of s, for a certain range of the latter, then, for a suitable corresponding range of t, every integral of the linear system

$$\frac{dx}{dt}=\left[A+A_1t+\ldots+A_\mu t^\mu+\sum_{r=1}^{\sigma}(t-c_r)^{-1}C_r\right]x$$

is a single-valued function of s; for, if

$$\log(t-c_r)=\psi_r(s),$$

so that

$$t=c_r+\exp\psi_r(s)=\psi(s),\text{ say,}$$

the system is

$$\frac{dx}{ds}=\left[\psi'(s)\left[A+A_1\psi+\ldots+A_\mu\psi^\mu\right]+\sum_{r=1}^{\sigma}\psi_r'(s)\,C_r\right]x$$

$$=ux,\text{ say,}$$

and each of the terms

$$Qu=\int_{s_0}^{s}u\,ds,\quad QuQu=\int_{s_0}^{s}u\,ds\int_{s_0}^{s}u\,ds,\quad \ldots$$

in the matrizant solution

$$x=\Omega(u)\,x_0=(1+Qu+QuQu+\ldots)\,x_0$$

is a single-valued analytic function of s about the properly chosen value s_0.

A very particular case of this is the well known theorem that the independent and dependent variables of a hypergeometric differential equation are single-valued functions of a variable s, which is the quotient of two integrals of a certain hypergeometric equation for which $\alpha=\frac{1}{2},\ \beta=\frac{1}{2},\ \gamma=1$; in fact, if

$$\kappa=\int_0^{\frac{1}{2}\pi}d\theta\,(1-t\sin^2\theta)^{-\frac{1}{2}},\quad \kappa'=\int_0^{\frac{1}{2}\pi}d\theta\left[1-(1-t)\sin^2\theta\right]^{-\frac{1}{2}},$$

the expressions for $\log t$, $\log(1-t)$ unambiguously in terms of $q = e^{i\pi s} = e^{-\pi\kappa'/\kappa}$ are given in Jacobi's *Fundamenta Nova*. But the suggestion above, into the utility of which this is not the place to enter, applies to a linear system

$$\frac{dx}{dt} = wx,$$

in which w has poles of higher order, say

$$w = A + \ldots + A_\mu t^\mu + \Sigma\left[C_r(t-c_r)^{-1} + \ldots + C_{h_r}(t-c_r)^{-h_r}\right];$$

a very particular case is Bessel's equation

$$y'' = -\frac{1}{t}y' + \frac{n^2-t^2}{t^2}y,$$

which, by $x_1 = y$, $x_2 = ty'$, gives the system

$$\frac{dx}{dt} = \left[-t\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \frac{1}{t}\begin{pmatrix}0 & 1\\ n^2 & 0\end{pmatrix}\right]x,$$

solved by single-valued functions of $s = \log t$.

11. Some remarks should be made as to the direct evaluation of $\Omega(a\omega + \ldots + a_r\omega_r)$ as a method for the integration of a linear system

$$\frac{dx}{dt} = (a\omega + a_1\omega_1 + \ldots + a_r\omega_r)x,$$

in which $\omega_1, \ldots, \omega_r$ are functions of t, and $a, a_1, \ldots, a_r$ are matrices of constants of the same order, no two of which are commutable (§ 6).

If
$$\phi_0 = \int_{t_0}^{t}\omega\,dt, \quad \phi_{01} = \int_{t_0}^{t}\omega\,dt\int_{t_0}^{t}\omega_1\,dt,$$
$$\phi_{011} = \int_{t_0}^{t}\omega\,dt\int_{t_0}^{t}\omega_1\,dt\int_{t_0}^{t}\omega_1\,dt, \quad \ldots,$$
we have $\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$$\Omega(a\omega + \ldots + a_r\omega_r) = 1 + a\phi_0 + \ldots + a_r\phi_r + (a^2\phi_{00} + aa_1\phi_{01} + a_1a\phi_{10} + \ldots) + \ldots$$
$$\ldots + (\Sigma a^{\lambda_0}a_1^{\lambda_1}\ldots\phi_{00\ldots11\ldots}) + \ldots,$$

where the suffix of the ϕ in the general term is exactly similar to the exponents $\lambda_0, \lambda_1, \ldots$ of the matrix product $a^{\lambda_0}a_1^{\lambda_1}\ldots$. Of the products $a\ldots a_r$, a^2, aa_1, $a_1a\ldots$, all are necessarily expressible linearly with numerical coefficients in terms of, say, N of them, where N is at most equal to the square of the order of any one of the matrices. Thus, when what we may call the multiplication table of the

matrices $a, a_1, \ldots$ is known, namely, the law by which any product of powers of them can be expressed in terms of N of them, the linear system may be regarded as integrated in explicit terms, the problem of integration breaking up into the two problems of ascertaining this multiplication table and of finding the properties of the sums of the infinite series of functions of t of the form $\phi_0 \ldots \phi_{01} \ldots$ which arise as the coefficients of the N fundamental matrices.

In particular the last example but one of § 9 shows that for any single differential equation of the second order with rational coefficients, which is regular at its singular points including $t = \infty$, the solution can be expressed by sums of series of elementary functions of the form

$$\phi_0 = Q(t-c)^{-1}, \quad \phi_{01} = Q(t-c)^{-1}Q(t-c_1)^{-1},$$
$$\phi_{10} = Q(t-c_1)^{-1}Q(t-c)^{-1}, \ldots .$$

Consider, for instance, a linear system

$$\frac{dx}{dt} = (a\omega + a_1\omega_1)x$$

wherein the matrices of any, the same, order satisfy the equations

$$a^2 = 0, \quad aa_1 = a, \quad a_1a = 0, \quad a_1^2 = a_1,$$

which we represent by

	a	a_1
a	0	a
a_1	0	a_1

;

then we have

$$\Omega(a\omega + a_1\omega_1) = 1 + a[\phi_0 + \phi_{01} + \phi_{011} + \phi_{0111} + \ldots] + a_1[\phi_1 + \phi_{11} + \phi_{111} + \phi_{1111} + \ldots]$$
$$= 1 + aQ[\omega e^{Q\omega_1}] + a_1[e^{Q\omega_1} - 1],$$

where Q denotes integration from t_0 to t. An example of such matrices is given by

$$a = \begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix},$$

which arise by $x_1 = e^{-\frac{1}{2}Q\omega}y, \quad x_2 = \frac{1}{\omega}e^{-\frac{1}{2}Q\omega}\frac{dy}{dt}$

from the equation

$$\frac{d^2y}{dt^2} = \left[\omega + \omega_1 + \frac{1}{\omega}\frac{d\omega}{dt}\right]\frac{dy}{dt} - \frac{1}{4}\omega(\omega + 2\omega_1)y;$$

and an infinite number of such pairs of matrices can be derived by linear transformation of the dependent variables from the system in which

$$a = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

c being an arbitrary constant. But the integration in finite terms applies to two matrices of any order having the same multiplication table.

Another example is the linear system

$$\frac{dx}{dt} = \begin{Vmatrix} 0 & \omega_1 & \omega_2 & \omega_3 \\ -\omega_1 & 0 & -\omega_3 & \omega_2 \\ -\omega_2 & \omega_3 & 0 & -\omega_1 \\ -\omega_1 & -\omega_2 & \omega_1 & 0 \end{Vmatrix} x = (i\omega_1 + j\omega_2 + k\omega_3)\,x$$

where i, j, k are matrices satisfying

$$i^2 = j^2 = k^2 = -1, \quad jk = -kj = i, \quad ki = -ik = j, \quad ij = -ji = k,$$

and the solution is expressible by four series of functions ϕ in the form $\Omega(i\omega_1 + j\omega_2 + k\omega_3) = A + iA_1 + jA_2 + kA_3$

where, for instance, A is

$$A = 1 - \phi_{11} - \phi_{22} - \phi_{33} - \phi_{123} + \phi_{132} - \phi_{231} + \phi_{321} - \phi_{312} + \phi_{132} + \dots.$$

The systematic study of linear systems from this point of view breaks up into two independent problems: (1) the determination of all irreducible types of multiplication tables of sets of matrices of the same order, a problem akin to that of the enumeration of types of discontinuous groups; (2) the investigation of the properties of the functions represented by such series of repeated integrations as those denoted above by $A, A_1 \dots$. These series converge for all finite values of t in the suitably chosen star region, and a first approximation to their investigation is the determination of their character near the corners of the region. It is to this determination for a wide class of cases that the second part of this paper is devoted.

Part II.

12. It has been remarked in § 9 that the single linear equation

$$y^{(n)} = \frac{P_{n-1}}{t} y^{(n-1)} + \frac{P_{n-2}}{t^2} y^{(n-2)} + \ldots + \frac{P}{t^n} y,$$

wherein $P, \ldots, P_{n-1}$ are developable about $t = 0$ in a series of positive integral powers of t, leads to a linear system of the n-th order

$$\frac{dx}{dt} = \left(\frac{A}{t} + V\right) x$$

wherein A is a matrix of constants and V is a matrix capable of development for the neighbourhood of $t = 0$ in a form

$$A_1 + A_2 t + A_3 t^2 + \ldots$$

wherein $A_1, A_2, A_3, \ldots$ are matrices of constants.

Independently now of whether the system is so derived from a single linear equation or not, we proceed to consider the character about $t = 0$ of the matrizant

$$\Omega\left(\frac{A}{t} + V\right)$$

wherein A, V are as in the description just given.

13. We assume the following theorem of algebra:—Let M be any matrix of constants, say of n rows and columns, which may be of zero determinant; let θ be any root of the determinantal equation $|M - \rho| = 0$, of multiplicity l; let $\rho - \theta$ be of multiplicity l_1 in the highest common factor in regard to ρ of the first minors of the determinant $|M - \rho|$, of multiplicity l_2 in the highest common factor of the minors of $(n-2)$ rows and columns, and so on; let $\epsilon_1 = l - l_1$, $\epsilon_2 = l_1 - l_2$, &c.; so that, if the minors of $n-r$ rows and columns do not all vanish when $\rho = \theta$, we have $\epsilon_r = l_{r-1}$ and

$$(\rho - \theta)^l = (\rho - \theta)^{\epsilon_1} (\rho - \theta)^{\epsilon_2} \ldots (\rho - \theta)^{\epsilon_r},$$

the factors $(\rho - \theta)^{\epsilon_1}$, $(\rho - \theta)^{\epsilon_2}$, ... being what are called the invariant factors of the matrix $M - \rho$ for the root θ. They are the same for this as for the matrix $\mu^{-1}(M - \rho)\mu = \mu^{-1} M \mu - \rho$, wherein μ is any matrix of the same order as M of non-vanishing determinant; the exponents $\epsilon_1, \epsilon_2, \ldots$ are non-vanishing positive integers known to satisfy the inequalities $\epsilon_1 \geqslant \epsilon_2 \geqslant \epsilon_3 \ldots \geqslant \epsilon_r > 0$. If $\epsilon_1', \epsilon_1'', \ldots$ be, like ϵ_1 for θ, the first of these respectively for the other roots $\theta', \theta'', \ldots$ of $|M - \rho| = 0$, the equation satisfied by M is of the form

$$(M - \theta)^{\epsilon_1} (M - \theta')^{\epsilon_1'} (M - \theta'')^{\epsilon_1''} \ldots = 0.$$

The important theorem is —rows, each of n elements, l rows in all, linearly independent of each other, to be denoted each row by a single letter such as $x_1, x_2, \ldots$, can be found to satisfy the following l sets each of n linear equations :—

$$\begin{array}{llll} (M-\theta)\,x_1 = 0, & (M-\theta)\,x_2 = x_1, & \ldots, & (M-\theta)\,x_{\epsilon_1} = x_{\epsilon_1-1}, \\ (M-\theta)\,y_1 = 0, & (M-\theta)\,y_2 = y_1, & \ldots, & (M-\theta)\,y_{\epsilon_2} = y_{\epsilon_2-1}, \\ \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \ldots & \ldots \\ (M-\theta)\,z_1 = 0, & (M-\theta)\,z_2 = z_1, & \ldots, & (M-\theta)\,z_{\epsilon_r} = z_{\epsilon_r-1}, \end{array}$$

and can be chosen in such a way that the most general solution of the set of n linear equations expressed by $(M-\theta)x = 0$ is a linear function of the rows $x_1, y_1, \ldots, z_1$; the most general solution of the set of equations $(M-\theta)^2 x = 0$ is a linear function of the rows $x_1 \ldots z_1, x_2 \ldots z_2$, and so on. Further, if l' be the multiplicity of the root θ', similar l' rows $x'_1, x'_2, \ldots, y'_1, \ldots$ can be chosen to satisfy the corresponding sets of linear equations for the root θ', and similar rows for the remaining roots $\theta'', \ldots$, and the whole number of $n = l + l' + l'' + \ldots$ such rows can be chosen to be linearly independent of one another, so that the matrix of n rows and columns of which the elements of any *column* are constituted by the elements of these rows is of non-vanishing determinant. Let this matrix, when its columns in order are formed from the rows $x_1, x_2, \ldots, x_{\epsilon_1}, y_1, \ldots, y_{\epsilon_2}, \ldots, z_{\epsilon_r}, x'_1, \ldots, z'_1, \ldots$, be denoted by μ. It is then another way of stating the above equations to say that $\mu^{-1} M \mu = m$ where m is a matrix, called the canonical form of M, constituted as follows :—It has zero everywhere, save in the diagonal and the $n-1$ places immediately to the right of the diagonal. The diagonal consists, first, of the root θ, l times repeated; then of the root θ', l' times repeated; and so on. The $n-1$ elements to the right of the diagonal, which we shall in future denote by $a_{i,\,i+1}$ for $i = 1, \ldots, (n-1)$, consist, first, of $\epsilon_1 - 1$ unities, then a zero, then $\epsilon_2 - 1$ unities, then a zero, and so on, there being a zero in the l-th row, then $\epsilon'_1 - 1$ unities, and so on for the roots in turn. If we form the conditions for the equations $M\mu = \mu m$, in fact, they will be seen to be the equations above. The matrix m has the same roots and invariant factors as M, and the rows satisfying for it the equations corresponding to those above for M, viz., $(M-\theta)x = 0, \ldots$, are in turn $\xi_1 = (1000\ldots), \xi_2 = (010\ldots), \ldots$; so that the matrix μ belonging to m is the matrix unity.

14. Now let the roots of the determinantal equation $|A-\rho|=0$ be arranged as follows:—Let θ_1 be the root, or one of them, whose real part is greater than for any other root; let $\theta_2, \theta_3, \ldots, \theta_{s_1}$ be all the remaining roots which differ from θ_1 by integers, so chosen that their real parts are in descending order; thus we shall have $\theta_1-\theta_2=m_1$, $\theta_2-\theta_3=m_2, \ldots$, these being positive integers or zero, and $s_1=1$ being supposed included. Now from the remaining roots of $|A-\rho|=0$ choose that one of the largest real parts, or one of them, say θ_{s_1+1}, and let those follow this root which differ from θ_{s_2+1} by integers, as before, in descending order of real parts down to θ_{s_2}. If this rule be continued, it arranges all the roots.

Now choose a matrix μ as in § 13, such that

$$\mu^{-1}A\mu=\begin{Bmatrix}\theta_1 & a_{12} & . & . \\ . & \theta_2 & a_{23} & . \\ . & . & . & . \end{Bmatrix}=a, \text{ say,}$$

wherein $a_{12}, a_{23}, \ldots$ are either unity or zero, according to the invariant factors, but in particular $a_{s_1, s_1+1}, a_{s_2, s_2+1}, \ldots$ are certainly zero, and let $\mu^{-1}V\mu=v$, so that

$$\mu^{-1}\Omega\left(\frac{A}{t}+V\right)\mu=\Omega(\mu^{-1}A\mu.t^{-1}+\mu^{-1}V\mu)=\Omega\left(\frac{a}{t}+v\right).$$

We first investigate the character of the matrix

$$\xi=\Omega\left(\frac{a}{t}+v\right)=\Omega(u), \text{ say.}$$

15. Consider, to this end, a matrix of constants defined as follows:—

$$\gamma=\begin{Bmatrix}\gamma_1 & . & . & . \\ . & \gamma_2 & . & . \\ . & . & \gamma_3 & . \\ . & . & . & . \end{Bmatrix}$$

wherein γ_1 is of s_1 rows and columns, its diagonal being the diagonal of γ, all elements in the first s_1 rows and columns of γ other than those belonging to γ_1 being zero; and γ_2 of (s_2-s_1) rows and columns, its diagonal being the diagonal of γ, all elements in the (s_1+1)-th, ..., s_2-th rows and columns of γ other than those of γ_2 being

zero; and so on. Further, γ_1 is of the form

$$\gamma_1 = \begin{Bmatrix} 0 & c_{12} & c_{13} & \dots & c_{1s_1} \\ 0 & 0 & c_{23} & \dots & c_{2s_1} \\ \vdots & \cdot & \cdot & \dots & \cdot \end{Bmatrix}$$

where $c_{12}, c_{13}, c_{23}, \dots$ are constants to be further defined below, and $\gamma_2, \gamma_3, \dots$ have similar forms; thus in γ all elements in and to the left of the diagonal are zero.

It is manifest that, if m be a positive integer,

$$\gamma^m = \begin{Bmatrix} \gamma_1^m & \cdot & \cdot \\ \cdot & \gamma_2^m & \cdot \\ \cdot & \cdot & \cdot \end{Bmatrix};$$

hence, if λ denote $\log t/t_0$, we have

$$\Omega\left(\frac{\gamma}{t}\right) = \begin{Bmatrix} \nabla\left(\frac{\gamma_1}{t}\right) & \cdot & \cdot \\ \cdot & \nabla\left(\frac{\gamma_2}{t}\right) & \cdot \\ \cdot & \cdot & \cdot \end{Bmatrix}$$

where $$\nabla\left(\frac{\gamma_1}{t}\right) = 1 + \gamma_1\lambda + \frac{\gamma_1^2\lambda^2}{2!} + \dots, \dots.$$

Since the only root of the determinantal equation $|\gamma_1 - \rho| = 0$ is zero, there is a power $\gamma_1^{k_1}$ which is identically zero, k_1 at most s_1, and so for the others; thus $\nabla\left(\frac{\gamma_1}{t}\right)$ contains only a finite number of powers of λ, at most up to $\lambda^{s_1-1}, \dots$. In particular when in γ_1 every one of the constants c_{ij} other than $c_{12}, c_{23}, \dots$, in which $j = i+1$, is zero, so that γ_1 is of the form

$$\gamma_1 = \begin{Bmatrix} 0 & c_{12} & 0 & \cdot \\ \cdot & 0 & c_{23} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{Bmatrix},$$

we have

$$\gamma_1^2 = \begin{Bmatrix} 0 & 0 & c_{12}c_{23} & \cdot & \cdot \\ 0 & 0 & 0 & c_{23}c_{34} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{Bmatrix},$$

$$\gamma_1^3 = \begin{Bmatrix} 0 & 0 & 0 & c_{12}c_{23}c_{34} & \cdot & \cdot \\ 0 & 0 & 0 & 0 & c_{23}c_{34}c_{45} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{Bmatrix}, \dots,$$

$$\text{and} \quad \nabla\left(\frac{\gamma_1}{t}\right) = \left\{ \begin{array}{ccccc} 1 & c_{12}\lambda & \frac{1}{2!}c_{12}c_{23}\lambda^2 & \frac{1}{3!}c_{12}c_{23}c_{34}\lambda^3 & . \\ 0 & 1 & c_{23}\lambda & \frac{1}{3!}c_{23}c_{34}\lambda^2 & . \\ 0 & 0 & 1 & c_{34}\lambda & . \\ 0 & 0 & 0 & 1 & . \\ . & . & . & . & . \end{array} \right\}.$$

We know, however, that a matrix of non-vanishing determinant σ can be found to put $\sigma^{-1}\gamma\sigma$ into a form δ with constants d_{ij}, in place of c_{ij}, in which all but the elements $d_{i,i+1}$ are zero, and each of these is either zero or unity; this would give

$$\sigma^{-1}\Omega\left(\frac{\gamma}{t}\right)\sigma = \Omega\left(\frac{\delta}{t}\right) = \left\{ \begin{array}{ccc} \nabla\left(\frac{\delta_1}{t}\right) & . & . \\ . & \nabla\left(\frac{\delta_2}{t}\right) & . \\ . & . & . \end{array} \right\}$$

where $\nabla(\delta_1/t)$, ..., are of the simple form above in terms of the constants d_{12}, d_{23}, ...

In addition to the matrix $\Omega\left(\frac{\gamma}{t}\right)$ consider now the matrix

$$\Theta = \left\{ \begin{array}{ccc} \theta_1 & . & . \\ . & \theta_2 & . \\ . & . & . \end{array} \right\},$$

consisting only of the roots θ_1, θ_2, ... of $|A-\rho| = 0$, written in order in the diagonal, so that

$$\Omega\left(\frac{\Theta}{t}\right) = \left\{ \begin{array}{ccc} (t/t_0)^{\theta_1} & . & . \\ . & (t/t_0)^{\theta_2} & . \\ . & . & . \end{array} \right\}.$$

If we form the product

$$\Omega\left(\frac{\Theta}{t}\right)\Omega\left(\frac{\gamma}{t}\right),$$

it will be the same as $\Omega\left(\frac{\gamma}{t}\right)$, save that every element in the i-th

row is multiplied by $(t/t_0)^{\theta_i}$. But we have the relation (§ 5)

$$\Omega\left(huh^{-1}+\frac{dh}{dt}h^{-1}\right)=h\Omega(u)h_0^{-1},$$

giving, if $h=\Omega(w)$,

$$\Omega(w)\,\Omega(u)=\Omega\left[w+\Omega(w)\,u\Omega^{-1}(w)\right];$$

herein put $w=\Theta/t$, $u=\gamma/t$; then

$$\Omega\left(\frac{\Theta}{t}\right)\Omega\left(\frac{\gamma}{t}\right)=\Omega\left[\frac{\Theta}{t}+\frac{1}{t}\,\Omega\left(\frac{\Theta}{t}\right)\gamma\Omega^{-1}\left(\frac{\Theta}{t}\right)\right];$$

the general element of

$$\Omega(\Theta/t)\,\gamma\Omega^{-1}(\Theta/t)$$

is
$$\begin{aligned}\left[\Omega(\Theta t^{-1})\,\gamma\Omega^{-1}(\Theta t^{-1})\right]_{pr}&=\sum_{q=1}^{n}\left[\Omega(\Theta t^{-1})\right]_{pq}\left[\gamma\Omega^{-1}(\Theta t^{-1})\right]_{qr}\\&=(t/t_0)^{\theta_p}\left[\gamma\Omega^{-1}(\Theta t^{-1})\right]_{pr}\\&=(t/t_0)^{\theta_p}\sum_{s=1}^{n}\gamma_{ps}\left[\Omega^{-1}(\Theta t^{-1})\right]_{sr}\\&=(t/t_0)^{\theta_p}\,(t/t_0)^{-\theta_r}\,\gamma_{pr}=(t/t_0)^{\theta_p-\theta_r}\,\gamma_{pr},\end{aligned}$$

of which however, by the definition of the matrix γ, the element γ_{pr} is zero when $\theta_p-\theta_r$ is not an integer, and is, in fact, zero when $r\leqslant p$; so that when $\gamma_{pr}=c_{pr}$ is not zero $\theta_p-\theta_r$ is a positive integer or zero; thus on the whole

$$\Omega\left(\frac{\Theta}{t}\right)\Omega\left(\frac{\gamma}{t}\right)=\Omega(\phi)$$

where
$$\phi=\frac{1}{t}\begin{Bmatrix}\theta_1 & c_{12}(t/t_0)^{\theta_1-\theta_2} & c_{13}(t/t_0)^{\theta_1-\theta_2} & \cdot\\ \cdot & \theta_2 & c_{23}(t/t_0)^{\theta_2-\theta_3} & \cdot\\ \cdot & \cdot & \cdot & \cdot\end{Bmatrix},$$

all places in the matrix, other than those in the diagonal, which were filled by zeros in the matrix γ being here also filled by zeros, and each of the exponents $\theta_1-\theta_2$, $\theta_1-\theta_3$, ..., which occur being either zero or a positive integer.

16. Consider now the differential equation for a matrix η expressed by

$$\frac{d\eta}{dt}=\left(\frac{a}{t}+v\right)\eta-\eta\phi,$$

equivalent to a system of n^2 linear equations for the elements of η, of which, if $\frac{a}{t}+v=u$, the general one is

$$\frac{d\eta_{ij}}{dt}=u_{i1}\eta_{1j}+\ldots+u_{in}\eta_{nj}-(\eta_{i1}\phi_{1j}+\ldots+\eta_{in}\phi_{nj})$$
$$(i,j=1,2,\ldots,n);$$

if g be any particular form of η satisfying this equation, the matrix

$$g\Omega(\phi)$$

satisfies

$$\frac{d}{dt}[g\Omega(\phi)]=\frac{dg}{dt}\Omega(\phi)+g\phi\Omega(\phi)=(ug-g\phi)\Omega(\phi)+g\phi\Omega(\phi)$$
$$=u[g\Omega(\phi)];$$

and is therefore of the form

$$g\Omega(\phi)=\Omega(u)\,g_0,$$

where g_0 is a matrix of constants, being the value at t_0 of the matrix g, or its continuation; thus

$$\Omega\left(\frac{a}{t}+v\right)=g\Omega(\phi)g_0^{-1}.$$

We proceed to show that when the constants c_{12}, c_{13}, c_{23}, ... in ϕ are suitably chosen there exists such a matrix g, reducing to unity when $t=0$, and expressible about $t=0$ as an ordinary power series

$$1+g_1t+g_2t^2+\ldots,$$

wherein g_1, g_2, ... are matrices, which is convergent for sufficiently small values of t. Its continuation to all values in the star region is then given by the equation

$$g=\Omega\left(\frac{a}{t}+v\right)g_0\Omega^{-1}(\phi).$$

For this purpose we write down the differential equations to be satisfied by the elements of the columns of the matrix η in greater detail; let the row of elements constituting the first column of η be called x, those the second, third, ... columns respectively y, z, ...; next, those constituting the (s_1+1)-th column be called X, those the (s_1+2)-th column be called Y, and so on; then, taking account of

the form of the matrix ϕ, we have

$$\frac{dx}{dt} = \left(\frac{\alpha}{t} + v\right) x - \frac{1}{t}\,\theta_1 x,$$

$$\frac{dy}{dt} = \left(\frac{\alpha}{t} + v\right) y - \frac{1}{t}\left[\theta_2 y + c_{12}(t/t_0)^{\theta_1-\theta_2}\, x\right],$$

$$\frac{dz}{dt} = \left(\frac{\alpha}{t} + v\right) z - \frac{1}{t}\left[\theta_3 z + c_{23}(t/t_0)^{\theta_2-\theta_3}\, y + c_{13}(t/t_0)^{\theta_1-\theta_3}\, x\right],$$

... ...

$$\frac{dX}{dt} = \left(\frac{\alpha}{t} + v\right) X - \frac{1}{t}\,\theta_{s_1+1} X,$$

$$\frac{dY}{dt} = \left(\frac{\alpha}{t} + v\right) Y - \frac{1}{t}\left[\theta_{s_1+2} Y + c_{s_1+1,\, s_1+2}(t/t_0)^{\theta_{s_1+1}-\theta_{s_1+2}} X\right],$$

... ...

We proceed then to establish in turn (1) the existence of formal solutions of these in the form of power series in t, (2) the convergence of these series.

17. First as to the formal solution. The equation for the row x, when we put

$$v = \alpha_1 + \alpha_2 t + \alpha_3 t^2 + \ldots,$$

where α_1, α_2, ... are matrices of constants, and assume

$$x = x_0 + t x_1 + t^2 x_2 + \ldots,$$

where x_0, x_1, x_2, ... denote rows of elements, gives

$$t x_1 + 2t^2 x_2 + 3t^3 x_3 + \ldots + m t^m x_m + \ldots$$
$$= (\alpha - \theta_1 + \alpha_1 t + \alpha_2 t^2 + \ldots)(x_0 + t x_1 + t^2 x_2 + \ldots);$$

hence, equating coefficients of the same powers of t,

$$(\alpha - \theta_1)\, x_0 = 0, \quad (\alpha - \theta_1 - 1)\, x_1 + \alpha_1 x_0 = 0,$$

... ...

$$(\alpha - \theta_1 - m)\, x_m + \alpha_1 x_{m-1} + \alpha_2 x_{m-2} + \ldots + \alpha_m x_0 = 0,$$

... ...

Of these the first is clearly satisfied by

$$x_0 = (1000\ldots),$$

while, since there is no root of $|\alpha - \rho| = 0$ exceeding θ_1 by a positive

integer, each of the following rows x_1, x_2, ... is given, without ambiguity, by the formula

$$x_m = -(\alpha-\theta_1-m)^{-1}(\alpha_1 x_{m-1}+\ldots+\alpha_m x_0).$$

Similarly substituting in the equation for y a series

$$y = y_0+ty_1+t^2y_2+\ldots,$$

where y_0, y_1, ... are rows of elements, we have

$$ty_1+2t^2y_2+\ldots = (\alpha-\theta_2+\alpha_1 t+\alpha_2 t^2+\ldots)(y_0+ty_1+t^2y_2+\ldots)$$
$$-c_{12}(t/t_0)^{\theta_1-\theta_2}(x_0+tx_1+t^2x_2+\ldots);$$

here $\theta_1-\theta_2$ is zero or a positive integer, say k_1, since otherwise the equation for X, to be considered presently, is the next equation. For distinctness consider the cases separately.

(*a*) When $\theta_1-\theta_2=0$ the term a_{12} in the matrix α may be zero or unity, according to the invariant factors; in either case the value

$$y_0 = (0100\ldots)$$

gives
$$(\alpha-\theta_2)y = a_{12}x_0;$$

taking then $c_{12}=a_{12}$, the terms in t^0 vanish and subsequent terms are given, after equating coefficients of t^m, without ambiguity by the equation

$$(\alpha-\theta_2-m)y_m+\alpha_1 y_{m-1}+\ldots+\alpha_m y_0-a_{12}x_m = 0.$$

(*b*) When $\theta_1-\theta_2=k_1>0$ the term a_{12} in the matrix α is zero, and the same value of y_0 as in the previous case gives

$$(\alpha-\theta_2)y_0 = 0,$$

reducing the coefficient of t^0 to zero; for the coefficient of t^m in which $m<k_1$ we have, as before, without ambiguity,

$$y_m = -(\alpha-\theta_2-m)^{-1}(\alpha_1 y_{m-1}+\ldots+\alpha_m y_0);$$

but the term in t^{k_1} is a critical term, giving, since $\theta_2+k_1=\theta_1$,

$$(\alpha-\theta_1)y_{k_1}+\alpha_1 y_{k_1-1}+\ldots+\alpha_{k_1}y_0-c_{12}t_0^{-k_1}x_0 = 0,$$

wherein the determinant $|\alpha-\theta_1|$ is zero and the inverse matrix $(\alpha-\theta_1)^{-1}$ unmeaning. But, since $\theta_1\neq\theta_2$, it follows, in virtue of the way in which the roots have been arranged, that θ_1 is not equal to any other root of $|\alpha-\rho|=0$; thus the only diagonal element of $\alpha-\theta_1$ which vanishes is the first, and every element of y_{k_1} except its first is determined by this equation without ambiguity when c_{12} is

assigned. Take the first element of y_{k_1} zero for simplicity, and take the disposable constant c_{12} so that $c_{12}t^{-k_1}$ = first element of $\alpha_1 y_{k_1-1}+\ldots+\alpha_{k_1}y_0$, and then determine the other elements of y_{k_1}. For the coefficient of t^m in which $m > k_1$ no such difficulty arises, the row y_m being determined without ambiguity by means of

$$y_m = -(\alpha-\theta_2-m)^{-1}[\alpha_1 y_{m-1}+\ldots+\alpha_m y_0-c_{12}t_0^{-k_1}x_{m-k_1}],$$

there being no root of $|\alpha-\rho|=0$ of the form $\theta_2+m=\theta_1+m-k$.

To make the argument still clearer consider in similar detail the equation for the column z; putting $z=z_0+tz_1+t^2_{z_2}+\ldots$, the equation to be satisfied identically in regard to t is

$$tz_1+2t^2z_2+\ldots=(\alpha-\theta_3+\alpha_1 t+\alpha_2 t^2+\ldots)(z_0+tz_1+t^2z_2+\ldots)$$
$$-c_{23}(t/t_0)^{\theta_2-\theta_3}(y_0+ty_1+t^2y_2+\ldots)-c_{13}(t/t_0)^{\theta_1-\theta_3}(x_0+tx_1+t^2x_2+\ldots).$$

(*a*) When $\theta_1=\theta_2=\theta_3$, the constants a_{12}, a_{23} in the matrix α may each be either zero or unity, but in any case the row

$$z_0 = (00100\ldots)$$

gives $$(\alpha-\theta_3)\,z_0 = a_{23}y_0.$$

Take then $c_{23}=a_{23}$, $c_{13}=0$; any row z_m for $m=1, 2, \ldots$ is determined without ambiguity by the equation

$$z_m = -(\alpha-\theta_3-m)^{-1}(\alpha_1 z_{m-1}+\ldots+\alpha_m z_0-a_{23}y_m).$$

(*b*) When $\theta_1-\theta_2=0$, $\theta_2-\theta_3=k_2>0$, the constant a_{12} in the matrix α is zero or unity, and the constant a_{23} is zero; the same value as before for z_0 gives $(\alpha-\theta_3)\,z_0=0$, and the equation for z_m, for $m=1, 2, \ldots$, namely,

$$0=(\alpha-\theta_3-m)\,z_m+\alpha_1 z_{m-1}+\ldots+\alpha_m z_0-c_{23}t_0^{-k_2}y_{m-k_2}-c_{13}t_0^{-k_2}x_{m-k_2},$$

wherein the terms in x_{m-k_2}, y_{m-k_2} are to be omitted if $m<k_2$; considering then this value of m, there are two roots, viz., θ_1 and θ_2 equal to θ_3+k_2 and two of the diagonal elements of $\alpha-\theta_3-m$ are zero, namely, the first and second, all others being other than zero; then, when c_{13}, c_{23} are assigned, all elements of z_m after the second are determined by this equation without ambiguity. By equating first and second elements in the equation to zero, we obtain respectively, since $m=k_2$,

$$a_{12}z_m^{(2)}+(\alpha_1 z_{m-1}+\ldots+\alpha_m z_0)^{(1)}-c_{13}t_0^{-k_2}=0,$$
$$(\alpha_1 z_{m-1}+\ldots+\alpha_m z_0)^{(2)}-c_{23}t_0^{-k_2}=0;$$

thus when $a_{12} = 1$ we can take $c_{13} = 0$; but in any case c_{13}, c_{23} can be chosen so that these equations are satisfied.

(*c*) When $\theta_1 - \theta_2 = k_1 > 0$, $\theta_2 - \theta_3 = 0$, the constant a_{12} in the matrix α is zero, and the constant a_{23} is zero or unity; but the same value as before for z_0 gives $(\alpha - \theta_3) z_0 = a_{23} y_0$, and, taking $c_{23} = a_{23}$, the coefficient of t^0 vanishes; the equation arising by the coefficient of t^m,

$$0 = (\alpha - \theta_3 - m) z_m + \alpha_1 z_{m-1} + \ldots + \alpha_m z_0 - c_{23} y_m - c_{13} t_0^{-k_1} x_{m-k_1},$$

wherein for $m < k_1$ the last term is to be omitted, is only critical for values of m for which $\theta_3 + m$ is a root, namely, for $m = k$; considering this value of m, every diagonal term of $\alpha - \theta_3 - m$ is other than zero except the first, and so every element of z_m except the first is definite; we can then take the first element of z_m zero and choose c_{13} so that $c_{13} t_0^{-k_1} =$ first element of $\alpha_1 z_{m-1} + \ldots + \alpha_m z_0 - a_{23} y_m$.

(*d*) Lastly, when $\theta_1 - \theta_2 = k_1 > 0$, $\theta_2 - \theta_3 = k_2 > 0$, each of the constants a_{12}, a_{23} in the matrix α is zero. As before, the value $z_0 = (00100\ldots)$ gives $(\alpha - \theta_3) z_0 = 0$; but the general equation for $m > 0$,

$$0 = (\alpha - \theta_3 - m) z_m + \alpha_1 z_{m-1} + \ldots + \alpha_m z_0 - c_{23} t_0^{-k_2} y_{m-k_2} - c_{13} t_0^{-k_1 - k_2} x_{m-k_1-k_2},$$

wherein $y_{m-k_2} = 0$ for $m < k_2$ and $x_{m-k_1-k_2} = 0$ for $m < k_1 + k_2$, is critical for both the values $m = k_2$, $m = k_1 + k_2$ for which $\theta_3 + m$ is a root, respectively θ_2 and θ_1.

When $m = k_2$ only the second diagonal element of $\alpha - \theta_3 - m$ is zero; we may take the second element of $z_{k_2} = 0$, and choose $c_{23} t_0^{-k_2} =$ second element of $(\alpha_1 z_{k_2-1} + \ldots + \alpha_{k_2} z_0)$, the other elements of z_{k_2} being determined without ambiguity.

When $m = k_1 + k_2$ only the first element of the diagonal of $\alpha - \theta_3 - m$ is zero; we may take the first element of $z_{k_1+k_2} = 0$, and choose

$$c_{13} t_0^{-k_1-k_2} = \text{first element of } (\alpha_1 z_{k_1+k_2-1} + \ldots + \alpha_{k_1+k_2} z_0 - c_{23} t_0^{-k_2} y_{k_1}).$$

A precisely similar argument applies for immediately succeeding columns. Consider now the $(s_1 - 1)$-th column; puttting

$$X = X_0 + t X_1 + t^2 X_2 + \ldots,$$

the equation $$t \frac{dX}{dt} = (a + vt) X - \theta_{s_1+1} X$$

gives

$$t X + 2t^2 X_2 + \ldots = (\alpha - \theta_{s_1+1} + \alpha_1 t + \alpha_2 t^2 + \ldots)(X_0 + t X_1 + t^2 X_2 + \ldots),$$

and we take $$X_0 = (0\ldots010\ldots),$$

wherein the (s_1-1)-th element is unity, satisfying $(\alpha-\theta_{s_1+1})X_0 = 0$; while, as there is no positive integer m such that $\theta_{s_1+1}+m$ is a root of $|\alpha-\rho| = 0$, no other critical terms arise.

For the (s_1+2)-th column Y we have to satisfy

$$tY_1+2t^2Y_2+\ldots = (\alpha-\theta_{s_1+2}+a_1t+a_2t^2+\ldots)(Y_0+tY_1+t^2Y_2+\ldots)$$
$$-c_{s_1+1,\,s_1+2}(t/t_0)^{\theta_{s_1+1}-\theta_{s_1+2}}(X_0+tX_1+t^2X_2+\ldots).$$

(*a*) If $\theta_{s_1+1}-\theta_{s_1+2} = 0$, the constant $a_{s_1+1,\,s_1+2}$ in the matrix α is zero or unity, but the row

$$Y_0 = (0\ldots010\ldots),$$

with unity in the (s_1+2)-th place, satisfies

$$(\alpha-\theta_{s_1+2})Y_0 = a_{s_1+1,\,s_1+2}X_0;$$

we take then $$c_{s_1+1,\,s_1+2} = a_{s_1+1,\,s_1+2},$$

and, as there is no positive integer m for which $\theta_{s_1+2}+m = \theta_{s_1+1}+m$ is a root of $|\alpha-\rho| = 0$, no other critical terms arise.

(*b*) When $\theta_{s_1+1}-\theta_{s_1+2} = l_1 > 0$, the constant $a_{s_1+1,\,s_1+2} = 0$, the same value of Y_0 as before gives $(\alpha-\theta_{s_1+2})Y_0 = 0$; the general equation

$$0 = (\alpha-\theta_{s_1+2}-m)Y_m+\mathfrak{a}_1Y_{m-1}+\ldots+\mathfrak{a}_mY_0-c_{s_1+1,\,s_1+2}t_0^{-l_1}X_{m-l_1},$$

wherein $X_{m-l_1} = 0$ for $m < l_1$, is critical only for $m = l_1$, and in the matrix $\alpha-\theta_{s_1+2}-l_1 = \alpha-\theta_{s_1+1}$ the only element of the diagonal which vanishes is the (s_1+1)-th; we may take the (s_1+1)-th element of Y_m zero, and, the (s_1+1)-th element of X_0 being unity, determine $c_{s_1+1,\,s_1+2}$ so that

$$c_{s_1+1,\,s_1+2}\,t_0^{-l_1} = (s_1+1)\text{-th element of } (\mathfrak{a}_1Y_{l_1-1}+\ldots+\mathfrak{a}_{l_1}Y_0);$$

the other elements of $Y_m = Y_{l_1}$ are then determined without ambiguity.

The same examination in detail can be continued. It is, however, sufficiently clear that in all cases the constant c_{ij} can be chosen so as to give a perfectly definite expansion in powers of t for every element of every column of the matrix η in such a way that for $t = 0$ the matrix $\eta = 1$, that is, has unity for every diagonal element and zero for every other element.

18. In regard now to the convergence of the series which have been determined for the various columns. Each of the n differ-

ential equations is of the form

$$t\frac{dy}{dt} = (\beta + \alpha_1 t + \alpha_2 t^2 + \ldots)y - x,$$

wherein y denotes the column to be determined, x is a numerical multiple of a column previously determined, as we may suppose by a convergent series, or a sum of a finite number of such, and β is a matrix of zero determinant. The numerical multipliers in x, namely, the constants $c_{12}, c_{13}, c_{23}, \ldots$, are determined with the early terms of the series for y, certainly of finite rank, since the equation $|\beta - \rho| = 0$ cannot have two roots whose difference is not finite; thus the question of convergence relates only to the series

$$y - (y_0 + ty_1 + \ldots + t^{N-1}y_{N-1}) = t^N y_N + t^{N+1}y_{N+1} + \ldots,$$

where the general coefficient on the right hand is determined by an equation of the form

$$y_m = (m-\beta)^{-1}[\alpha_1 y_{m-1} + \ldots + \alpha_m y_0 - x_m],$$

in which x_m and $y_0, \ldots, y_{N-1}$ are given and N is such that for $m > N-1$ the determinant of $m-\beta$ is not zero.

Now let the rows of real positive elements $Y_0, \ldots, Y_{N-1} X_m$ and matrices of real positive elements A_i, H be determined so that

$$|y_0| < Y_0, \quad |y_1| < Y_1, \quad \ldots, \quad |y_{N-1}| < Y_{N-1}, \quad |\alpha_i| < A_i$$

for $i = 1, 2, \ldots, \infty$, and, for $m \geqslant N$,

$$|x_m| < X_m, \quad |(m-\beta)^{-1}| < H,$$

where the meaning is that the modulus of each individual element of the row or matrix on the left is less than the corresponding element on the right, the possibility of the inequalities $|\alpha_i| < A_i$, $|x_m| < X_m$ being a consequence of the assumed convergence for sufficiently small t of the series for the matrix v and the row x; then the equations, for $m \geqslant N$,

$$y_m = (m-\beta)^{-1}[\alpha_1 y_{m-1} + \ldots + \alpha_m y_0 + x_m]$$

lead to

$$|y_m| < H[A_1 Y_{m-1} + \ldots + A_m Y_0 + X_m];$$

or, if we put, for $m \geqslant N$,

$$Y_m = H[A_1 Y_{m-1} + \ldots + A_m Y_0 + X_m],$$

lead to

$$y_m| < Y_m.$$

It is therefore sufficient to show that the successive equations, for $m \geqslant N$,

$$Y_m = H[A_1 Y_{m-1} + \ldots + A_m Y_0 + X_m],$$

wherein $A_1, A_2, \ldots$ are any given matrices of real positive elements such that the series

$$A_1 t + A_2 t^2 + \ldots$$

converges when t is small enough; $Y_0, Y_1, \ldots, Y_{N-1}$ are any given rows of each real positive elements; $X_N, X_{N+1}, \ldots$ are any given rows of real positive elements such that the series

$$t^N X_N + t^{N+1} X_{N+1} + \ldots$$

converges when t is small enough, give rise to a series

$$t^N Y_N + t^{N+1} Y_{N+1} + \ldots,$$

converging when t is small enough.

For this, consider the equations expressed by

$$\begin{aligned} &Y - Y_0 - tY_1 - \ldots - t^{N-1} Y_{N-1} \\ &= H\{(tA_1 + t^2 A_2 + \ldots) Y - [tA_1 Y_0 + t^2 (A_1 Y_1 + A_2 Y_0) + \ldots \\ &\qquad \ldots + t^{N-1} (A_1 Y_{N-2} + \ldots + A_{N-1} Y_0)] + t^N X_N + t^{N+1} X_{N+1} + \ldots\}, \end{aligned}$$

which, when written at length, are n implicit equations for the n functions of t which are the n elements of the row Y, in fact of the form

$$\begin{aligned} C_{11}(y_1 - y_1^0) + \ldots + C_{1n}(y_n - y_n^0) &= t\phi_1(t, y_1, \ldots, y_n), \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ C_{n1}(y_1 - y_1^0) + \ldots + C_{nn}(y_n - y_n^0) &= t\phi_n(t, y_1, \ldots, y_n), \end{aligned}$$

where $\phi_1, \ldots, \phi_n$ are convergent series in t, linear in $y_1, \ldots, y_n$, and $C_{ij} = 0$, except $C_{ii} = 1$.

Such a set of equations is known to have convergent solutions; say in our case

$$Y = Y_0 + tZ_1 + \ldots + t^{N-1} Z_{N-1} + t^N Y_N + t^{N+1} Y_{N+1} + \ldots,$$

which on substitution gives

$$\begin{aligned} &t(Z_1 - Y_1) + \ldots + t^{N-1}(Z_{N-1} - Y_{N-1}) + t^N Y_N + t^{N+1} Y_{N+1} + \ldots \\ &\quad = H\{(tA_1 + t^2 A_2 + \ldots)(Y_0 + tZ_1 + \ldots + t^{N-1} Z_{N-1} + t^{N+1} Y_{N+1} + \ldots) \\ &\qquad - [tA_1 Y_0 + \ldots + t^{N-1}(A_1 Y_{N-2} + \ldots + A_{N-1} Y_0)] + t^N X_N + \ldots\}, \end{aligned}$$

and hence, so far as the terms in t^{N-1},

$$Z_1 - Y_1 = H\{A_1 Y_0 - A_1 Y_0\},$$
$$Z_2 - Y_2 = H\{A_1 Z_1 + A_2 Y_0 - (A_1 Y_1 + A_2 Y_0)\},$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$Z_{N-1} - Y_{N-1} = H\{A_1 Z_{N-2} + A_2 Z_{N-3} + \dots + A_{N-1} Y_0 - (A_1 Y_{N-2} + \dots + A_{N-1} Y_0)\},$$

leading to $Z_1 = Y_1, \quad Z_2 = Y_2, \quad \dots, \quad Z_{N-1} = Y_{N-1},$

while, for the term in t^m, for $m \geqslant N$,

$$Y_m = H\{A_1 Y_{m-1} + A_2 Y_{m-2} + \dots + A_m Y_0 + X_m\},$$

which is the series of real and positive elements occurring in the series whose convergence was previously shown sufficient for our theorem.

19. We have thus established that

$$\Omega\left(\frac{\alpha}{t} + v\right) = g\Omega(\phi) g_0^{-1} = g\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1}) g_0^{-1},$$

where g is a matrix of functions of t developable in a convergent ordinary power series about $t = 0$, the matrix g reducing to unity for $t = 0$, and g_0 is the value of g, or its continuation, at $t = t_0$. Thence, when, as before,

$$\mu^{-1} A \mu = \alpha, \quad \mu^{-1} V \mu = v,$$

we have $$\Omega\left(\frac{A}{t} + V\right) = G\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1})\, G_0^{-1},$$

where $$G = \mu g$$

reduces to μ for $t = 0$, and satisfies the equations

$$\frac{dG}{dt} = \left(\frac{A}{t} + V\right) G - G\phi.$$

The matrices γ, Θ, and ϕ have been explained in § 15.

In the subsidiary equations for the determination of the columns of g, the position of t_0 enters only in the combinations $c_{ij} t^{-(\theta_i - \theta_j)}$; if we put this $= e_{ij}$, these subsidiary equations will contain no reference to t_0. Thus the matrix g does not alter when t_0 is taken differently, nor therefore does the matrix G; but the matrix $\Omega(\phi)$ does alter, being

$$\Omega^{t,\, t_0}(\phi) = \Omega^{t,\, t_1}(\phi)\, \Omega^{t_1,\, t_0}(\phi).$$

The simplest case of our general formula is when no two of the roots of the determinantal equation $|A-\rho|=0$ are equal or differ by integers; then the matrix γ consists only of zeros and $\Omega(\gamma t^{-1})=1$.

Another case which may occur is that in which no two roots differ by an integer unless they are exactly equal. Then in the series which solve the subsidiary equation for the columns of the matrix ϕ no critical terms occur after the first terms, which may be critical owing to sequences of equal roots; in such case, as is seen on referring back to the work, every constant c_{ij} in which $j>i+1$ may be taken to be zero and the constants $c_{i,i+1}$ are those, $a_{i,i+1}$, arising at once from the given form of the differential equations, which occur in the canonical matrix a; thus

$$\phi=\frac{1}{t}\begin{Bmatrix}\theta_1 & c_{12}(t/t_0)^{\theta_1-\theta_2} & 0 & 0 & . \\ 0 & \theta_2 & c_{23}(t/t_0)^{\theta_2-\theta_3} & 0 & . \\ 0 & 0 & \theta_3 & c_{34}(t/t_0)^{\theta_3-\theta_4} & . \\ . & . & . & . & .\end{Bmatrix},$$

in which $c_{i,i+1}=0$, unless $\theta_i=\theta_{i+1}$, and $c_{i,i+1}=a_{i,i+1}$ when $\theta_i=\theta_{i+1}$. In other words, in this case, we have

$$\phi=\frac{a}{t},$$

and $$\Omega\left(\frac{A}{t}+V\right)=\mu g\,\Omega(at^{-1})\,g_0^{-1}\mu^{-1}=\mu g\mu^{-1}\Omega(At^{-1})(\mu g_0\mu^{-1})^{-1}$$

$$=h\Omega\left(\frac{A}{t}\right)h_0^{-1},$$

where $$h=\mu g\mu^{-1}$$

reduces to unity when $t=0$ and satisfies the equations

$$\frac{dh}{dt}=\left(\frac{A}{t}+V\right)h-h\frac{A}{t}.$$

In both these cases, it is to be noticed, the form of the matrix $\Omega(\phi)$ is determinable at once by inspection of the given differential equations from the matrix A alone.

20. In the case of a linear system derived from a single linear equation, as in § 9, the matrix A, there written at length, has the peculiarity that in $|A-\rho|$ the minor of the first element of the last row has a determinant equal to unity; thus, if θ be a root of multi-

plicity l, the first invariant factor corresponding thereto is $(\rho-\theta)^l$, and in the canonical form $\alpha = \mu^{-1} A \mu$ of the matrix A, the $l-1$ constants $a_{i,\,i+1}$ corresponding to the root have all the value unity, so that the equality of two roots necessarily involves a logarithm in the solution of the system. The matrix V is further special, in that all its first $n-1$ rows consist of zeros. In fact, the ordinary theory of a single linear equation leads us to expect that in this case all the constants c_{ij} of the matrix γ vanish in which j is not equal to $i+1$. We have not deduced this result in the present paper, the expression for the most general case given in § 15,

$$\nabla\left(\frac{\gamma_1}{t}\right) = 1+\gamma_1\lambda+\frac{\gamma_1^2}{2!}\lambda^2+\ldots,$$

appearing to be of sufficient simplicity. But that it is not possible in all cases to arrange to have all constants c_{ij} in which $j \neq i+1$ equal to zero appears to follow from such an example as the system

$$t\frac{dx_1}{dt} = \theta_1 x_1 + t(1-\theta_1+\theta_2)x_2 + \frac{t(1-\theta_1+\theta_3)+t^{\theta_1-\theta_3}}{1+t^2}x_3,$$

$$t\frac{dx_2}{dt} = \theta_2 x_2,$$

$$t\frac{dx_3}{dt} = \theta_3 x_3 + \frac{2t^2}{1+t^2}x_3,$$

where we may suppose $\theta_1-\theta_3 = 0$, or a positive integer; this has a form

$$\frac{dx}{dt} = \left(\frac{a}{t}+v\right)x,$$

where

$$a = \begin{Bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{Bmatrix},$$

$$v = \begin{Bmatrix} 0 & 1-\theta_1+\theta_2 & \dfrac{1-\theta_1+\theta_3+t^{\theta_1-\theta_3-1}}{1+t^2} \\ 0 & 0 & 0 \\ 0 & & \dfrac{2t}{1+t^2} \end{Bmatrix}.$$

This system is satisfied by the elements of each of the three columns

of the product of the matrices

$$\begin{Bmatrix} 1 & t & t \\ 0 & 1 & 0 \\ 0 & 0 & 1+t^2 \end{Bmatrix} \begin{Bmatrix} (t/t_0)^{\theta_1} & 0 & t^{\theta_1} t_0^{-\theta_3} \log t/t_0 \\ 0 & (t/t_0)^{\theta_2} & 0 \\ 0 & 0 & (t/t_0)^{\theta_3} \end{Bmatrix};$$

namely, in $\Omega\left(\frac{\alpha}{t}+v\right) = g\Omega(\phi)\, g_0^{-1} = g\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1})\, g_0^{-1}$,

we have

$$g = \begin{Bmatrix} 1 & t & t \\ 0 & 1 & 0 \\ 0 & 0 & 1+t^2 \end{Bmatrix},$$

$$\Omega(\Theta/t) = \begin{Bmatrix} (t/t_0)^{\theta_1} & 0 & 0 \\ 0 & (t/t_0)^{\theta_2} & 0 \\ 0 & 0 & (t/t_0)^{\theta_3} \end{Bmatrix},$$

$$\Omega(\gamma/t) = \begin{Bmatrix} 1 & 0 & t_0^{\theta_3 - \theta_2} \log(t/t_0) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix},$$

$$\gamma = \begin{Bmatrix} 0 & 0 & t_0^{\theta_3-\theta_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}, \qquad \phi = \frac{1}{t} \begin{Bmatrix} \theta_1 & 0 & t_0^{\theta_3-\theta_2}(t/t_0)^{\theta_1-\theta_2} \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{Bmatrix}.$$

21. As a simple actual example of the method for an assigned linear equation take the equation (Forsyth, *Linear Differential Equations*, p. 103, Ex. 8)

$$t^3(1+t)\,y''' - (2+4t)\,t^2y'' + (4+10t)\,ty' - (4+12t)\,y = 0,$$

which, as in § 9, leads to the system

$$\frac{dx}{dt} = \left[\frac{1}{t}\begin{Bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 4 & -4 & 4 \end{Bmatrix} + \frac{1}{t+1}\begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 8 & -6 & 2 \end{Bmatrix}\right] x = \left(\frac{A}{t} + \frac{B}{t+1}\right) x, \text{ say.}$$

The roots of $|A-\rho| = 0$ are $2, 2, 1$; with

$$\mu = \begin{Bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 0 \end{Bmatrix},$$

we find $\mu^{-1}A\mu = \begin{Bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{Bmatrix} = \alpha$, $\mu^{-1}B\mu = \begin{Bmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{Bmatrix}$,

$$\phi = \frac{1}{t}\begin{Bmatrix} 2 & c_{12} & c_{13}(t/t_0) \\ 0 & 2 & c_{23}(t/t_0) \\ 0 & 0 & 1 \end{Bmatrix},$$

and, with $u = \frac{\alpha}{t} + \frac{\beta}{t+1}$, the subsidiary equations

$$\frac{dg}{dt} = ug - g\phi$$

lead to $$\Omega\left(\frac{\alpha}{t} + \frac{\beta}{t+1}\right) = \begin{Bmatrix} 1 & 0 & -2t \\ 0 & 1 & 2t(1+t) \\ 0 & 0 & (1+t)^2 \end{Bmatrix} \Omega(\phi)\, g_0^{-1}$$

$$= g\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1})\, g_0^{-1},$$

with $\phi = \frac{1}{t}\begin{Bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2t \\ 2 & 0 & 1 \end{Bmatrix}$, $\Omega(\Theta t^{-1}) = \begin{Bmatrix} (t/t_0)^2 & 0 & 0 \\ 0 & (t/t_0)^2 & 0 \\ 0 & 0 & t/t_0 \end{Bmatrix}$,

$$\gamma = \begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2t_0 \\ 0 & 0 & 0 \end{Bmatrix}, \qquad \Omega(\gamma t^{-1}) = \begin{Bmatrix} 1 & \lambda & t_0\lambda^2 \\ 0 & 1 & 2t_0\lambda \\ 0 & 0 & 1 \end{Bmatrix},$$

where $\lambda = \log t/t_0$.

Part III.

22. A problem to which the previous investigation can be usefully applied is the elucidation of the connexion between the form of the linear system and the form of the linear substitutions which generate the monodromy group of the system. We consider only the case where the functions of t in the matrix u are single valued over the whole finite part of the plane. The star region in which the matrix $\Omega(u)$ is single valued and developable having been defined as previously explained, let the barrier joining one of its angular points to $t = \infty$ be removed, and let $\Omega_1(u)$ denote the value for the matrix obtained by integrating first from t_0 to t by the path by which $\Omega(u)$ was defined, and then from t once round the single corner now

isolated in the positive direction back to t. Then, as follows from the equation

$$\Omega^{t't_0}(u) = \Omega^{t't}(u)\,\Omega^{tt_0}(u),$$

we have

$$\Omega_1(u) = \Omega(u)\,\Phi,$$

where Φ is the value obtained by integrating from t_0 round the corner back to t_0. The group of linear substitutions formed by the combinations of the matrices Φ, one for each corner, is the group in question. More generally, if C be an arbitrary matrix of constants, a matrix whose columns are sets of solutions of the linear system is $\Omega(u)\,C$, and

$$\Omega_1(u)\,C = \Omega(u)\,C\,.\,C^{-1}\Phi C\,;$$

so that we may, instead of the group (Φ), consider the group generated by the substitutions $C^{-1}\Phi C$, which is said to be a translation or transformation of the other. In practice it is convenient to choose C so as to obtain the greatest possible simplicity.

23. Taking now the form we have investigated for linear systems of a certain type

$$\Omega(U) = G\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1})\,G_0^{-1},$$

the factor Φ for the matrix $\Omega(\Theta t^{-1})$, when t describes a circuit about $t=0$ from t_0 back to t_0, consists of a matrix having only diagonal elements of the form $e^{2\pi i\theta_1}, e^{2\pi i\theta_2}, \ldots$, which we denote by $\omega_1, \omega_2, \ldots$; corresponding to a sequence of roots from $\theta_1, \theta_2, \ldots$, each of which is less by an integer than the preceding, the corresponding quantities ω are equal; namely, in a notation previously employed (§ 14), the first s_1 quantities $\omega_1, \omega_2, \ldots$ are equal; then the following $s_2 - s_1$; and so on.

Denoting the quantity $2\pi i$ by ϵ, the factor Φ for the matrix (§ 15)

$$\Omega(\gamma t^{-1}) = \left\{ \begin{matrix} \nabla(\gamma_1/t) & & \\ & \nabla(\gamma_2/t) & \\ & & \ddots \end{matrix} \right\}$$

is a matrix

$$\left\{ \begin{matrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \ddots \end{matrix} \right\}$$

where Δ_1, of s_1 rows and columns, has the form

$$\Delta_1 = 1 + \gamma_1\epsilon + \frac{\gamma_1^2}{2!}\,\epsilon^2 + \ldots;$$

Δ_2, of $s_2 - s_1$ rows and columns, has a similar form in terms of γ_2; and so on.

If now such a matrix, with separate square matrices arranged about its diagonal, and other elements all zero, be called a diagonal matrix, and two such matrices be considered of the same kind when the numbers of rows and columns in the respective component matrices are the same for both, it is immediately obvious that the product of two such

$$\begin{Bmatrix} \Delta_1 & & \\ & \Delta_2 & \\ & & \ddots \end{Bmatrix} \begin{Bmatrix} \Delta_1' & & \\ & \Delta_2' & \\ & & \ddots \end{Bmatrix} = \begin{Bmatrix} \Delta_1\Delta_1' & & \\ & \Delta_2\Delta_2' & \\ & & \ddots \end{Bmatrix}$$

is independent of their order.

Thus it follows that the matrix Φ arising for the matrix $\Omega(\Theta t^{-1})$ is commutable with the matrix $\Omega(\gamma t^{-1})$, and we have

$$\Omega_1(U) = G\Omega(\Theta t^{-1})\,\Omega(\gamma t^{-1})\,\Phi G_0^{-1},$$

where
$$\Phi = \begin{Bmatrix} \omega\Delta_1 & & \\ & \omega'\Delta_2 & \\ & & \ddots \end{Bmatrix},$$

ω denoting the value of $e^{2\pi i\theta_1}$ for $\theta_1, \theta_2, \ldots, \theta_{s_1}$, and ω' the value for $\theta_{s_1+1}, \ldots, \theta_{s_2}$, and so on.

Thus
$$\Omega_1(U) = \Omega(U)\,G_0\Phi G_0^{-1},$$

and the monodromy group of the linear system is generated by linear substitutions of the form

$$G_0\Phi G_0^{-1} = G_0 \begin{Bmatrix} \omega\Delta_1 & & \\ & \omega'\Delta_2 & \\ & & \ddots \end{Bmatrix} G_0^{-1}.$$

24. Consider, for example, particular cases as in § 19.

(*a*) When all the roots $\theta_1, \theta_2, \ldots$ belonging to the corner $t = 0$ are different and no two differ by integers, each of $\Delta_1, \Delta_2, \ldots$ reduces to unity.

(*b*) when no two of the roots differ by integers unless they are exactly equal, the matrix Φ is the value of $\Omega\left(\frac{\alpha}{t}\right)$ taken round $t = 0$,

and equal to
$$1 + \alpha\epsilon + \frac{\alpha^2}{2!}\epsilon^2 + \ldots,$$

and determinable at once on inspection of the differential equations

without solution. If then t_0 can be taken so that the matrices G arising for the various corners have all the same value at t_0, the group of the system is particularly simple.

(*c*) For a corner at which all the roots belong to one sequence, differing by integers or zero, if all the constants c_{ij} of the matrix γ are zero, and no logarithms enter into the solutions about this corner, the matrix Φ reduces to a single constant ω, and the substitution of the group arising for this corner is independent of G_0, reducing to the matrix having only the quantity ω in each diagonal place.

25. Conversely, consider necessary conditions that the group should be finite. Then each substitution must be of finite order, and we must have equations for each corner of the form

$$1 = \Phi^m = \begin{Bmatrix} \omega^m \Delta_1^m & & \\ & \omega'^m \Delta_2^m & \\ & & \ddots \end{Bmatrix},$$

and hence $$\omega^m \Delta_1^m = 1, \quad \omega'^m \Delta_2^m = 1, \quad \ldots,$$

of which the first is

$$\omega^m \left(1 + m\gamma_1\epsilon + \frac{m^2\epsilon^2}{2!}\gamma_1^2 + \ldots\right) = 1,$$

where γ_1 is a matrix satisfying, for its equation of lowest order, an equation of the form $\gamma_1^{k_1} = 0$, in which k_1 is at most s_1, and the series in the bracket terminates with the term involving $\gamma_1^{k_1-1}$. This equation therefore involves $\omega^m = 1$ and $\gamma_1 = 0$.

A necessary condition is therefore that all the roots $\theta_1, \theta_2, \ldots$ should be rational numerical fractions and the matrix γ be zero, so that no logarithms enter into the solutions. In case no two of the roots differ by integers or zero, the matrix γ is zero of itself; on the other hand, if every two of the roots differ by integers, and still the matrix γ is zero, the particular substitution $G_0 \Phi G_0^{-1}$ is independent of G_0, and reduces to a numerical constant ω, which in the case supposed is a root of unity; for a system derived from a single linear equation we have seen that the invariant factors corresponding to a repeated root θ cannot be linear, and γ cannot be zero when there are repeated roots.

The condition is not generally sufficient. If a set of matrices N generate a finite group, it is known that a single matrix ϖ can be

found such that the set of matrices $M = \varpi N \varpi^{-1}$ satisfy the relation $\bar{M}_c M = 1$, where M_c denotes the matrix whose elements are the conjugate imaginaries of those of M, and $\bar{M}_c$ is the transposed of M_c. In our case, denoting Φ, which in the case supposed consists only of diagonal elements which are roots of unity, by Ω, so that $\bar{\Omega}_c \Omega = 1$, the matrices M are of the form

$$M = \varpi G_0 \Omega G_0^{-1} \varpi^{-1}.$$

Part IV.

26. In illustration of the previous theory consider the case of the system

$$\frac{dx}{dt} = \left[\begin{pmatrix} 0 & 0 \\ \alpha\beta & 0 \end{pmatrix} + \frac{1}{t(1-t)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] x = u(t)\, x,$$

in which $\beta = 1-\alpha$, derived from the single linear equation

$$t(1-t)\, y'' + (1-2t)\, y' - \alpha(1-\alpha)\, y = 0$$

by putting $x_1 = y$, $x_2 = t(1-t)\, y'$. For $t = 0$, $t = 1$ the system is already in canonical form; integrating from $t_0 = \frac{1}{2}$, we have, by the theory,

$$\Omega[u(t)] = g(t) \begin{pmatrix} 1 & \log 2t \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2}).$$

We find, however,

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -u(t) = -u(1-t);$$

and hence, if $s = 1-t$ and $\Omega[u(t)] = F(t)$, we have, by § 3,

$$F(s) = \Omega\left[u(s) \frac{ds}{dt} \right] = \Omega[-u(t)] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Omega[u(t)] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that

$$g(t) \begin{pmatrix} 1 & \log 2t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g(s) \begin{pmatrix} 1 & \log 2s \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g(\tfrac{1}{2}). \tag{A}$$

With $\varpi = t(1-t)$, the subsidiary equations for the determination of the columns of

$$g(t) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

are (§ 16)
$$\frac{dg_{11}}{dt} = \frac{1}{\varpi} g_{21}, \quad \frac{dg_{12}}{dt} = \frac{1}{\varpi} g_{22} - \frac{g_{11}}{t},$$
$$\frac{dg_{21}}{dt} = \alpha\beta g_{11}, \quad \frac{dg_{22}}{dt} = \alpha\beta g_{12} - \frac{g_{21}}{t},$$

leading to
$$\frac{d}{dt}(g_{11}g_{22} - g_{12}g_{21}) = 0;$$

so that, as at $t = 0$, so for all t,
$$g_{11}g_{22} - g_{12}g_{21} = 1,$$

the ordinary Abel relation in a disguised form.* The equation (A), with $h_{ij} = g_{ij}(1-t)$ and $\gamma_{ij} = g_{ij}(\frac{1}{2})$, is the same as

$$\begin{pmatrix} g_{11} & g_{11}\log 2t + g_{12} \\ g_{21} & g_{21}\log 2t + g_{22} \end{pmatrix}$$
$$= \begin{pmatrix} h_{11} & h_{12} \\ -h_{21} & -h_{22} \end{pmatrix} \begin{pmatrix} 1 & \log 2s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{22} & -\gamma_{12} \\ -\gamma_{21} & \gamma_{11} \end{pmatrix} \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ -\gamma_{21} & -\gamma_{22} \end{pmatrix}$$
$$= \begin{pmatrix} h_{11} & h_{11}\log 2s + h_{12} \\ -h_{21} & -h_{21}\log 2s - h_{22} \end{pmatrix} \begin{pmatrix} A & B \\ C & -A \end{pmatrix},$$

where
$$A = \gamma_{11}\gamma_{22} + \gamma_{12}\gamma_{21}, \quad B = 2\gamma_{12}\gamma_{22}, \quad C = -2\gamma_{11}\gamma_{21},$$
$$A^2 + BC = 1,$$

and leads to four relations effectively all reducible to $g_{11}g_{22} - g_{12}g_{21} = 1$ together with
$$h_{11} = Ag_{11} + C(g_{11}\log 2t + g_{12}). \qquad (B)$$

We know that g_{11}, g_{12} are power series in t reducing respectively to 1 and 0 for $t = 0$, and h_{11} is the same power series in s as is g_{11} in t; if
$$g_{11}(t) = 1 + \sum_{n=1}^{\infty} \kappa_n t^n,$$

the equation (B) gives, putting $t=0$, and assuming $\log t$ [and $\log(1-t)$] real and negative for $0 < t < 1$,
$$A + C\log 2 = \left[g_{11}(1-t) - C\log t \right]_{t=0}$$
$$= \left[1 + \sum_{n=1} \left(\kappa_n + \frac{C}{n} \right) (1-t)^n \right]_{t=0},$$

and hence
$$C = (-n\kappa_n)_{n=\infty}.$$

Thus, when the series $g_{11}(t)$ is known, C can be found, and hence A and B, as we shall show below; it concerns us, however, first to show how far the method of this paper enables us to go towards

* In general the determinant of g is the exponential of the integral, from 0 to t, of the sum of the diagonal elements of the matrix v.

determining the group of the equation without calculation of details; we prove, in fact, at once that this group can be transformed so as to be generated by the two substitutions

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2(\pi i C)^2 & 1 \end{pmatrix}.$$

For the solution $\Omega(u)$ we have seen that the two substitutions about $t=0$, $t=1$ are

$$g(\tfrac{1}{2}) \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2}), \quad \mu g(\tfrac{1}{2}) \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2}) \mu,$$

where

$$\mu = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now take a matrix p such that

$$pg(\tfrac{1}{2}) \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2}) p^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

which is known to be possible, since the two matrices

$$\begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

have the same roots and invariant factors; putting for the determination of p,

$$pg(\tfrac{1}{2}) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

we find P and Q arbitrary, $R=0$ and $S=\pi i P$; next, assuming P not zero, take a matrix

$$\sigma = p\mu p^{-1} = \begin{pmatrix} P & Q \\ 0 & \pi i P \end{pmatrix} g^{-1}(\tfrac{1}{2}) \mu g(\tfrac{1}{2}) \begin{pmatrix} P & Q \\ 0 & \pi i P \end{pmatrix}^{-1}$$

$$= \begin{Bmatrix} A + \dfrac{QC}{P} & \dfrac{1}{\pi i P}\left(PB - 2QA - \dfrac{Q^2 C}{P}\right) \\ \pi i C & -\dfrac{1}{P^2}\left(A + \dfrac{QC}{P}\right) \end{Bmatrix},$$

and then, assuming C not zero, take $Q/P = -A/C$, so that, in virtue of $A^2 + BC = 1$,

$$\sigma = \begin{pmatrix} 0 & \dfrac{1}{\pi i C} \\ \pi i C & 0 \end{pmatrix}.$$

Consider now the solution $\Omega(u) p^{-1}$ of the differential system;

about $t = 0$, $t = 1$ it has the respective factors

$$\Theta = pg(\tfrac{1}{2}) \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2})\, p^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

$$\Phi = p\mu g(\tfrac{1}{2}) \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2})\, \mu p^{-1},$$

which, by $p\mu = \sigma p$, is equal to

$$\sigma p g(\tfrac{1}{2}) \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} g^{-1}(\tfrac{1}{2}) p^{-1}\sigma^{-1} = \sigma \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \sigma^{-1},$$

which on calculation is, as stated,

$$\begin{pmatrix} 1 & 0 \\ 2(\pi i C)^2 & 1 \end{pmatrix}.$$

In fact it will be seen that $g_{11}(t)$ is the hypergeometric series $F(\alpha, \beta, 1, t)$, $\beta = 1-\alpha$, and hence

$$C = -\left[\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{(n-1)!\, n!}\right]_{n=\infty};$$

it is, however, an elementary property of Γ-functions that, for general values of α, β,

$$\left[n\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{n!\,(\alpha+\beta)(\alpha+\beta+1)\ldots(\alpha+\beta+n-1)}\right]_{n=\infty} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)};$$

thus here, with $\alpha+\beta = 1$,

$$C = -\frac{1}{\Gamma\alpha\,\Gamma(1-\alpha)} = -\frac{\sin\pi\alpha}{\pi},$$

and the group is generated by

$$\Theta = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 1 & 0 \\ -2\sin^2\pi\alpha & 1 \end{pmatrix},$$

For instance, for $\alpha = \frac{1}{2}$, $\alpha = \frac{1}{3}$, respectively,

$$\Phi = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{pmatrix}.$$

Returning now to the determination of the matrix g, the equations for the first column are, with $\beta = 1-\alpha$,

$$t(1-t)\frac{dg_{11}}{dt} = g_{21}, \qquad \frac{dg_{21}}{dt} = \alpha\beta g_{11},$$

which are satisfied by

$$g_{11} = F(\alpha, \beta, 1, t), \qquad g_{21} = t(1-t)\frac{dF}{dt},$$

reducing respectively to 1 and 0 for $t = 0$, if only the identity

$$\alpha\beta F = t(1-t)F'' + (1-2t)F'$$

is satisfied, as is known; these are then the values for g_{11} and g_{21}. The equations for the second column

$$\frac{dg_{12}}{dt} = \frac{g_{22}}{t(1-t)} - \frac{g_{11}}{t}, \quad \frac{dg_{22}}{dt} = \alpha\beta g_{12} - \frac{g_{21}}{t}$$

are to be satisfied by forms

$$g_{12} = \sum_{n=1}^{\infty} \lambda_n t^n, \quad g_{22} = t(1-t)\frac{dg_{12}}{dt} + (1-t)g_{11},$$

which on substitution in the second equation are found to give

$$g_{12} = \sum_{n=1} \kappa_n t^n \left[\frac{1}{\alpha} + \ldots + \frac{1}{\alpha+n-1} + \frac{1}{\beta} + \ldots \right.$$

$$\left. \ldots + \frac{1}{\beta+n-1} - 2\left(1 + \ldots + \frac{1}{n}\right) \right],$$

which can be shown to be the same as

$$g_{12} = -\log(1-t)F(\alpha, \beta, 1, t) - 2\sum_{n=1} \kappa_n t^n \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).$$

Thus C is as stated above, and hence from equation (B)

$$A = \frac{\sin \pi\alpha}{\pi} \log 2 + \left[F(\alpha, 1-\alpha, 1;\ 1-t) + \frac{\sin \pi\alpha}{\pi} \log t \right]_{t=0}.$$

To evaluate this we use the identity holding for general values of α, β (St. John's College, Cambridge, Examination Paper of June 4th, 1894, 9–12),

$$2\psi(1) - \psi(\alpha) - \psi(\beta) = \lim_{t=0} \left[\log t + \frac{\Gamma\alpha\Gamma\beta}{\Gamma(\alpha+\beta)} F(\alpha, \beta, \alpha+\beta, 1-t) \right],$$

where

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)};$$

by $\beta = 1-\alpha$, we thence at once find

$$A = \frac{\sin \pi\alpha}{\pi} \left[\log 2 + 2\psi(1) - \psi(\alpha) - \psi(1-\alpha) \right]$$

$$= \frac{\sin \pi\alpha}{\pi} \log 2 + \frac{2 \sin \pi\alpha}{\pi} \left[\psi(1) - \psi(\alpha) \right] - \cos \pi\alpha;$$

and hence, from $A^2 + BC = 1$,

$$B = \frac{\sin \pi\alpha}{\pi} \left\{ \left[\log 2 + 2\psi(1) - 2\psi(\alpha) - \pi \cot \pi\alpha \right]^2 - \pi^2 \operatorname{cosec}^2 \pi\alpha \right\},$$

whereby, in fact, the relation connecting the fundamental integrals of the original hypergeometric equation about $t = 0$ and $t = 1$, for a general value of α, is found in a manner which appears to the writer simpler than that employed* by Tannery for the particular case $\alpha = \frac{1}{2}$. It can be proved that

$$\psi(1) - \psi(\tfrac{1}{2}) = 2\log 2, \quad \psi(1) - \psi(\tfrac{1}{3}) = \tfrac{1}{2}(3\log 3 + \pi/\sqrt{3});$$

thus, for $\alpha = \frac{1}{2}$,

$$A = \frac{5}{\pi}\log 2, \qquad B = \frac{1}{\pi}\left[(5\log 2)^2 - \pi^2\right],$$

and, for $\alpha = \frac{1}{3}$,

$$A = \frac{\sqrt{3}}{2\pi}\log(54), \quad B = \frac{\sqrt{3}}{2\pi}\left[(\log 54)^2 - \tfrac{4}{3}\pi^2\right].$$

Another remark seems worth making. We have had the relations expressing the integrals about $t = 0$ in terms of those about $t = 1$; by elimination of the constants we obtain four functions of these integrals which are constant; putting

$$\xi = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}\begin{pmatrix} 1 & \log 2t \\ 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}\begin{pmatrix} 1 & \log 2s \\ 0 & 1 \end{pmatrix},$$

these relations are, in fact,

$$\overline{\xi}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\eta = \begin{pmatrix} -C & A \\ A & B \end{pmatrix},$$

and can be obtained in this form directly from the two expressions for $\Omega(u)$.

[*June 17th*, 1903.—The following deal with systems of linear equations:—Königsberger, *Lehrbuch der . . . Differentialgleichungen*, 1889, pp. 441–469. Sauvage, *Ann. de l'Ec. norm.*, 1886, 1888, 1889. Sauvage, *Toulouse Ann.*, Vol. VIII., 1894, pp. 1–24; Vol. IX., 1895, pp. 25–100 and pp. 1–75. Grünfeld, *Denkschr. der Wiener Akad., math.-naturw. Cl.*, Bd. LIV., 1888. Horn, *Math. Ann.*, Vol. XXXIX., 1891, pp. 391–408, and Vol. XL., 1892, pp. 527–550. Picard, *Traité d'Analyse*, Vol. III., 1896, p. 266. Dunkel, *American Akad.*, May 14th, 1902.]

* Repeated in Forsyth's *Linear Differential Equations* (1902), pp. 129–134. We remark in passing that the first four lines of p. 148 of that volume do not appear to be correctly printed.

On some cases of Matrices with Linear Invariant Factors. *By* H. F. Baker. Received and Read December 11th, 1902.

When an important theorem is capable of very brief proof it is desirable that the fact should be widely known. This more than their novelty may perhaps justify the following lines. We denote a matrix of n rows and columns by a single capital letter, say M; that obtained from it by changing rows into columns by M; that obtained from it by changing every element into its conjugate complex by M_0. And a row of n single quantities is denoted by a small letter, such as x or y; their conjugate complexes by x_0 or y_0. The symbol Mx is the row of n quantities obtained by combining the element of each row of M, in turn, with the elements of x; while Mxy is the single quantity obtained by combining the elements of Mx with those of y. We utilize the obvious identity $Mxy = Myx$, and assume the fundamental theorem as to invariant factors explained in detail in § 13 of the preceding paper "On the Integration of Linear Differential Equations."

1. For a matrix M such that $M^p = 1$, p being a positive integer, if θ be a root of the determinantal equation $|M-\rho| = 0$, we can find x so that $Mx = \theta x$, and hence $M^p x = \theta^p x$; so that $\theta^p = 1$, or θ is a p-th root of unity. We cannot, however, find y, other than zero and linearly independent of x, such that $My = \theta y + x$, or else we should have in succession

$$M^2y = \theta(\theta y + x) + \theta x = \theta^2 y + 2\theta x, \quad M^3y = \theta^2(\theta y + x) + 2\theta^2 x = \theta^3 y + 3\theta^2 x,$$

and finally $$M^p y = \theta^p y + p\theta^{p-1} x,$$

leading to $$p\theta^{p-1} x = 0,$$

which is absurd, provided $p \neq 0$ and $\theta \neq 0$, of which the latter follows from $M^p = 1$, giving $|M| \neq 0$.

Thus the invariant factors of the matrix are linear, and a matrix μ can be found such that $\mu^{-1} M \mu$ consists only of diagonal elements, the roots of $|M-\rho| = 0$. It is known (E. H. Moore, *Math. Ann.*, Vol. L., 1898, p. 214) that, if $M_1, M_2, \ldots$ be the generating matrices of a finite group, a matrix m can be found such that the matrices $N_1 = m^{-1} M_1 m$, $N_2 = m^{-1} M_2 m$, ... satisfy the relation $\overline{N}_0 N = 1$. Thus the result is a particular case of that proved under (3) below.

2. If a matrix M satisfy the relation $\overline{M} = M_0$ so that, if it consist of real elements, it is a symmetrical matrix, and if θ be a root of $|M-\rho| = 0$, we have for a suitable row x the sequences

$$Mx = \theta x, \quad Mxx_0 = \theta xx_0;$$

$$M_0x_0 = \theta_0 x_0, \quad M_0x_0x = \theta_0 x_0 x, \quad \overline{M}_0 xx_0 = \theta_0 xx_0, \quad Mxx_0 = \theta_0 xx_0,$$

and can infer $$(\theta-\theta_0)\,xx_0 = 0, \quad \theta = \theta_0,$$

so that the roots θ are necessarily real; but we cannot have an equation $(M-\theta)\,y = x$, or else the real quantity

$$xx_0 = (M-\theta)\,yx_0 = (M_0-\theta)\,y_0x = (\overline{M}_0-\theta)\,xy_0 = (M-\theta)\,xy_0 = 0$$

would be zero.

Thus the invariant factors of the matrix M are necessarily linear, and the roots are real.

If A, B be two matrices of the n-th order such that $\overline{A} = A_0$, $\overline{B} = B_0$, and Bxx_0 only vanishes if each element of x is zero, so that $|B| \neq 0$, and θ be a root of $|A-\rho B| = 0$, we can, from $(B^{-1}A-\theta)\,x = 0$, infer

$$Axx_0 = \theta Bxx_0, \quad \overline{A}x_0x = \theta \overline{B}x_0x, \quad A_0x_0x = \theta B_0x_0x, \quad Axx_0 = \theta_0 Bxx_0,$$

and hence $(\theta-\theta_0)\,Bxx_0 = 0$, showing that θ is real; but we cannot have $(B^{-1}A-\theta)\,y = x$, or else the real quantity

$$\overline{B}_0xx_0 = B_0x_0x = \overline{B}x_0x = Bxx_0 = (A-\theta B)\,yx_0 = (A_0-\theta B_0)\,y_0x$$
$$= (\overline{A}_0-\theta\overline{B}_0)\,xy_0 = (A-\theta B)\,xy_0 = 0;$$

so that the invariant factors of $B^{-1}A$ are linear.*

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* If θ, θ' be two different roots of $|B^{-1}A-\rho| = 0$, equations

$$Ax = \theta Bx, \quad Ax' = \theta' Bx'$$

give $$\theta Bxx_0' = Axx_0' = \overline{A}x_0'x = A_0x_0'x = \theta' B_0x_0'x = \theta'\overline{B}_0xx_0' = \theta' Bxx_0',$$

and hence $(\theta-\theta')\,Bxx_0' = 0$, so that $Bxx_0' = 0$. While, if, for a repeated root, we have $Ax = \theta Bx$, $Ay = \theta By$, $Az = \theta Bz$, ... and put $\eta = y+\lambda x$, where λ is a single quantity, we have $B\eta\eta_0 = Bxy_0 + \lambda_0 Bxx_0$: thus we may suppose $Bxy_0 = 0$, giving $Byx_0 = 0$; then, putting $\zeta = z+\mu y+\nu x$, when μ, ν are single quantities, we have $B\zeta\zeta_0 = Bzz_0 + \nu_0 Bxx_0$, $By\zeta_0 = Byz_0 + \mu_0 Byy_0$; so we may suppose $Bxz_0 = 0$, $Byz_0 = 0$, and so on; and this leaves the sets $x, y, z, \ldots$ independent. We may then suppose, if μ be the matrix of which any column consists of one of the n independent sets satisfying, for the various values of ρ, the equations $(B^{-1}A-\rho)\,x = 0$, that, when $i \neq j$,

$$0 = \sum_{r,\,s} B_{rs}\mu_{si}(\mu_{rj})_0 = \sum_{r} (B\mu)_{ri}(\mu_{rj})_0 = (\overline{\mu}_0 B\mu)_{ji},$$

and denote the real quantity $(\overline{\mu}_0 B\mu)_{ii}$ by ϕ_i, and the diagonal matrix $\overline{\mu}_0 B\mu$ by Φ. Then it is a well known consequence of the definition of μ that the matrix $\mu^{-1}B^{-1}A\mu$

3. If S be a matrix such that the bilinear form Sxx_0 does not vanish unless every element of x is zero, and M be a matrix such that $\overline{M}_0 SM = S$, a particular case being when $S = 1$, and thereof a particular case being the ordinary orthogonal transformation when M consists of real elements; then, if $|M-\theta| = 0$ and $Mx = \theta x$, $t = Mx$, $t_0 = M_0 x_0$, we have

$$Sxx_0 = \overline{M}_0 SMxx_0 = SMxt_0 = Stt_0 = \theta\theta_0 Sxx_0,$$

so that $\theta\theta_0 = 1$ and the roots θ are of modulus unity. But we cannot have $My = \theta y + x$, or else, noticing that the determinant of S, and therefore of M, and therefore θ, is not zero, we could infer the sequences

$$y = \theta M^{-1}y + M^{-1}x, \quad Sy = \theta SM^{-1}y + SM^{-1}x = \theta \overline{M}_0 Sy + \overline{M}_0 Sx,$$

$$\overline{M}_0 Sy = \theta_0 Sy - \theta_0 \overline{M}_0 Sx,$$

$$\overline{S}M_0 y_0 y = \overline{M}_0 Syy_0 = (\theta_0 Sy - \theta_0 \overline{M}_0 Sx)\, y_0 = \theta_0 Syy_0 - \theta_0 \overline{M}_0 Sxy_0;$$

$$\overline{S}M_0 y_0 y = \overline{S}(\theta_0 y_0 + x_0)\, y = (\theta_0 \overline{S}y_0 y + \overline{S}x_0 y = \theta_0 Syy_0 + Sx_0 y;$$

and hence deduce $\qquad H = \theta_0 \overline{M}_0 Sxy_0 + \overline{S}x_0 y = 0,$

of which, however, the left side H is the same as

$$\begin{aligned}
\theta_0 \overline{S}M_0 y_0 x + \overline{S}x_0 y &= \theta_0 \overline{S}(\theta_0 y_0 + x_0)\, x + \overline{S}x_0 y \\
&= \theta_0^2 \overline{S}y_0 x + \overline{S}x_0 y + \theta_0 \overline{S}x_0 x \\
&= \theta_0^2 Sxy_0 + \overline{S}x_0 y + \theta_0 Sxx_0 \\
&= \theta_0^2 \overline{M}_0 SMxy_0 + \overline{S}x_0 y + \theta_0 Sxx_0 \\
&= \theta_0^2 \theta \overline{M}_0 Sxy_0 + \overline{S}x_0 y + \theta_0 Sxx_0 \\
&= H + \theta_0 Sxx_0,
\end{aligned}$$

giving the impossible result $Sxx_0 = 0$.

---

is a diagonal matrix, say Θ, having only the roots of $|A-\rho B| = 0$ in its diagonal, each to its own multiplicity; thus

$$\mu^{-1}B^{-1}(A-\rho B)\,\mu = \Theta - \rho, \quad \overline{\mu}_0 (A-\rho B)\,\mu = \Phi\Theta - \rho\Phi,$$

wherein, as the equations $(A-\theta B)\,x = 0$ only determine the ratios of the elements of x, the real quantities ϕ_i in Φ are arbitrary. Putting $\Phi\Theta = \Psi$, $\tau = \mu t$, this is equivalent to

$$(A-\rho B)\,\tau\tau_0 = (\Psi - \rho\Phi)\, tt_0,$$

whereby the bilinear forms $A\tau\tau_0$, $B\tau\tau_0$, are simultaneously transformed, each to contain only n terms, of the form $\psi_i t_i (t_i)_0$. In particular when A, B are real symmetric matrices and the quadratic form Bx^2 does not vanish unless $x = 0$, the equations $(A-\theta B)\,x = 0$ give only real elements for μ, and the real quadratic forms $A\tau^2$, $B\tau^2$ are hereby transformed simultaneously into sums of real positive or negative squares.

Thus the invariant factors of $|M-\rho|=0$ are linear. (*Cf.* Loewy, *Nova Acta Kais. Leop. Carol. Deut. Ak. der Naturf.*, LXXI., Halle, 1898, where another proof, involving somewhat more detailed considerations, is given.)

4. One simple example of the results in (2) and (3) deserves notice from its connexion with the theory of algebraically integrable linear differential equations. If a be a matrix as in (2) for which $\bar{a}=a_0$ and $\Omega(at^{-1})$ denote the matrix

$$\Omega\left(\frac{a}{t}\right)=1+aQ(t^{-1})+a^2Q(t^{-1}(Qt^{-1}))+a^3Q(t^{-1}(Qt^{-1}(Qt^{-1})))+\ldots$$

where Q denotes integration from an arbitrary position $t=t_0$, and the integration be extended completely round $t=0$, back to t_0, it is easy to see that $\Omega\left(-\frac{\bar{a}}{t}\right)$ is the conjugate complex of $\Omega\left(\frac{a}{t}\right)$. For an incomplete circuit the inverse of the matrix $\Omega\left(\frac{a}{t}\right)$ can be shown to be the transposed of $\Omega\left(-\frac{\bar{a}}{t}\right)$. Thus, if M denote the complete value of $\Omega\left(\frac{a}{t}\right)$, we have*

$$\bar{M}_0M=1;$$

which is simply $\sum_0^\infty \frac{(2\pi i a)^m}{m!}$, and it can be shown that the correspondence of the roots θ, $\phi=e^{2\pi i\theta}$ of the equations $|a-\theta|=0$, $|M-\phi|=0$ is complete.

5. If E denote the matrix occurring in the theory of continuous groups for which the general element is

$$E_{\rho\sigma}=\sum_{\tau=1}^{r} c_{\sigma\tau\rho}\,e_\tau \quad (\rho,\sigma=1\ldots r),$$

where the constants $c_{\sigma\tau\rho}$ are such that we have identically (*Proc. Lond. Math. Soc.*, Vol. XXXIV., p. 93)

$$Ee'+E'e=0,\quad EE'e''+E'E''e+E''Ee'=0,$$

there is a result of importance (Killing, *Math. Ann.*, Vol. XXXIII., 1889, p. 5) which we can prove briefly by means of a result proved in a previous note (*Proc. Lond. Math. Soc.*, Vol. XXXIV., p. 351). We have identically $Ee=0$, and, of the determinantal equation $|E+\rho|=0$,

* [*June 17th*, 1903.—And conversely any matrix M satisfying this equation can be so written as $\Omega(at^{-1})$; in particular for a real orthogonal transformation $a=i\beta$, where β is a skew symmetrical real matrix.]

zero is a root of at least multiplicity unity. If, for general values of $e_1, \ldots, e_r$, we form the descending series of positive integers which are the exponents of the invariant factors corresponding to this root zero, the theorem is that the last of them is necessarily unity; so that the last invariant factor is linear. To prove this it is sufficient to show that a set e' of quantities independent of e cannot be found such that, beside $Ee = 0$, we have $Ee' = e$. First, when the root zero of the equation $|E+\rho| = 0$ is of multiplicity r for all values of $e_1, \ldots, e_r$, the equation $Ee' = e$ would give $(E'+1)e = 0$, and so the impossible equation $|E'+1| = 0$. Next, when the root zero is at least of multiplicity k $(0 < k < r)$ for all values of $e_1, \ldots, e_r$, but is of no higher multiplicity for at least one particular set of values $e_1, \ldots, e_r$, if then e' can be found such that $e = Ee' = -E'e$, and if the series $1 + E + \frac{E^2}{2!} + \ldots$ be denoted by Δ_e, we have with arbitrary λ, since $E'^2 e = -E'e = e$, $E'^3 e = E'e = -e$, ..., putting $f = \lambda e'$,

$$\begin{aligned}\Delta_f e &= e + \lambda E'e + \frac{\lambda^2 E'^2 e}{2!} + \frac{\lambda^3 E'^3}{3!} + \ldots \\ &= e - \lambda + \frac{\lambda^2 e}{2!} - \frac{\lambda^3 e}{3!} + \ldots \\ &= \mu e,\end{aligned}$$

where $\mu = \exp(-\lambda)$. It has been proved, however, that, if $e' = \Delta_f e$,

then $$E' = \Delta_f E \Delta_f^{-1};$$

thus here we have $$\mu E = E_{\mu e} = \Delta_f E \Delta_f^{-1},$$

and so $$|E+\rho| = |\mu E + \rho|.$$

The coefficients in the equation $|E+\rho| = 0$ are therefore unaltered when $e_1, \ldots, e_r$ are replaced respectively by $\mu e_1, \ldots, \mu e_r$. As $\mu = \exp(-\lambda)$ and λ is arbitrary, this can only be so if the equation reduces to $\theta^r = 0$, which by hypothesis is not the case.

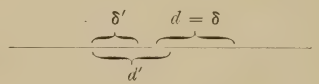
6. The result in (5) is for general values of $e_1, \ldots, e_r$. Consider an integrable group, and let $e_1, \ldots, e_r$ be a particular set for which $P = e_1 X_1 + \ldots + e_r X_r$ is the infinitesimal transformation such that when P' is any other the combinant (P', P) is zero or a constant multiple of P, so that we have $Ee' = \lambda e$; if then $Ee'' = \mu e$, and therefore $E(\mu e' - \lambda e'') = 0$, and neither of λ, μ be zero, we can replace one of e', e'' by $\mu e' - \lambda e''$. In other words, there is at most only one independent transformation P' for which the combinant (P', P) is not

zero. But the proof in (5) avails when $Ee' = \lambda e$ is not zero to show that for this special $e_1, \ldots, e_r$ the equation $| E + \rho | = 0$ has no root but zero; thus, besides the equation $Ee' = \lambda e$, we have $(r-2)$ equations $Ef' = 0$, together with $Ee = 0$. In other words, the first invariant factor of $| E + \rho | = 0$ is of exponent 2, and the remaining ones are linear, and the matrix E satisfies the equation $E^2 = 0$. For a group known to be necessarily integrable, such that $| E + \rho | = 0$ reduces to $\rho^r = 0$ for all values of $e_1, \ldots, e_r$, it can be shown that a set $e_1, \ldots, e_r$ exists, not identically zero, for which $E = 0$.

Overlapping Intervals. *By* W. H. Young. Received November 30th, 1902. Read December 11th, 1902. Revised March 10th, 1903.

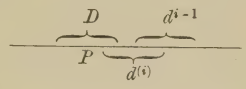
1. Given any set of overlapping intervals, we will show how to determine a countable set from among them which by themselves determine the most important properties of the given set.

Take, first, any one of the intervals, and let us denote it by d or δ. Then either there is no interval of the given set which abuts or overlaps with d on the left, or else there is such an interval. In the latter case we denote by δ' the part of this interval which extends beyond d to the left, and by d' the interval itself, which coincides with δ' if d' abuts with d, and which otherwise contains δ' as a part, and has its left-hand end point coincident with that of δ'.



Proceeding in this way towards the left, we must ultimately either come to an interval of the given set having no interval abutting or overlapping with it on the left, or else the parts of intervals $\delta', \delta'', \delta''', \ldots$ must get smaller and smaller without limit, and define a limiting point P external to all of them, and therefore external to the intervals $d', d'', d''', \ldots$ of the given set. Such a point P may, however, be internal to some other interval of the given set; in this case we choose out any one of the intervals containing P, say D. There will only be a finite number of the intervals $d, d', d'', \ldots$

which do not overlap with D. Let $d^{(i)}$ be the first which overlaps with D; then we select the intervals $d, d', d'', \ldots, d^{(i)}, D$, and omit from consideration all the intervals $d^{i+1}, d^{i+2}, \ldots$. Proceeding on these lines we can only be stopped (1) by coming to an interval of the given set having no interval of the given set abutting or overlapping with it on the left, (whose left-hand end point P is therefore neither internal to any interval of the given set nor a right-hand end point of any one of the intervals), or (2) by the parts of intervals $\delta', \delta'', \delta''', \ldots$ becoming smaller and smaller and defining a limiting point Q on the left of all of them, such that no interval of the given set contains Q as internal point. Q would be external to every one of the given intervals, unless it were a right-hand end point of one or more of them. Such a point, P or Q, may be properly called *an external* or *semi-external point of the given set*, though it should be remarked that a semi-external point of a set of overlapping intervals is not necessarily, as in the case of non-overlapping intervals, an end point of only *one* of the intervals.



Having proceeded in like manner on the right of d, we start afresh in each of the one or two segments left over, and take again any one of the intervals and treat it as we did d. Continuing this process, we get a set of *non-overlapping* intervals $\delta, \delta', \delta'', \ldots$, &c., which, by the theory of sets of non-overlapping intervals, can be arranged in *countable* order $\delta_1, \delta_2, \delta_3, \ldots$, and, corresponding to these, a *countable* set of the given intervals, $d_1, d_2, d_3, \ldots$, such that δ_i coincides with the whole or a part of d_i, and has at least one end point common with it. Any other of the given set of intervals lies entirely within one of the d_i's or else lies entirely within a set of the d_i's which overlap or abut all along.

Such a countable set $d_1, d_2, \ldots$ chosen in the manner indicated from among the intervals of the given set serves to classify the points of the straight line with reference to the given set in a manner analogous to that used in the theory of non-overlapping intervals. Any internal point of $d_1, d_2, \ldots$ is internal to the given set (of course), and the converse is also true, except that an internal point of the given set *may* be an end point of two intervals d_i, d_j which abut. Any external or semi-external point of $d_1, d_2, \ldots$ is external or semi-external to the given set, and *vice versa*.

2. We notice that the set $\delta_1, \delta_2 \ldots$ is not unique; but, since the ex-

ternal and semi-external points of the set then determined are the same as before, all such sets have the same derived set $\delta_1', \delta_2', \delta_3', \ldots$, and we may very properly call this *the derived set of the given set*, since it is obtained by amalgamating all abutting and *overlapping* intervals of the given set.

3. This gives us a direct proof of the Heine-Borel theorem, and shows us how to determine a finite number of the intervals such as are asserted to exist in the theorem. The enunciation of the theorem is as follows:—*Given a set of intervals such that every point of a given segment* (A, B) *is internal to some interval of the set, we can choose out a finite number of the intervals having the same property.*

For in this case there is no point of the segment external or semi-external to the intervals δ_i; hence they are finite in number, and therefore the same is true of the intervals d_i. Let these be $d_1, d_2, \ldots, d_k$, and let the end points of these intervals inside (A, B) be $P_1, P_2, \ldots$. Then we can choose out $(k-1)$ or less of the given intervals containing as internal points those of the points $P_1, P_2, \ldots$ which are not already internal to $d_1, d_2, \ldots, d_k$, and get in this way at most $(2k-1)$ of the intervals having the desired property.

4. Content.—It is convenient to *define* the content of the set as being the content I_δ of the derived set. This agrees with the definition already given in the case where the intervals do not overlap, and enables us to extend the theorems given in the first part of this paper concerning external and semi-external points and their connexion with the content to sets of intervals of the most general character.

5. The intervals of the derived set are such that every internal point of the given intervals is internal to some interval of the derived set, and no external or semi-external point of the given intervals is internal to the derived set; but it is not necessarily true that every internal point of the derived set is internal to some interval of the given set. If, however, any point P, internal to an interval δ', is not internal to any interval of the given set, then, however, we construct $d_1, d_2 \ldots$. P must be an end point of two abutting intervals δ_i and δ_j; and, since it is not internal to either of the corresponding d_i or d_j, it must also be a common end point of d_i and d_j, and these must therefore abut. Thus, if there be such a point P, there will be two or more intervals abutting at P,

and no interval containing P as internal point. Hence the number of these points P is at most countably infinite, and they are completely determined by the given intervals. We can call these points properly *isolated end points.* If we plot them down in the derived intervals, *we get a new set of intervals*, (non-overlapping), $D_1, D_2, \ldots$, *uniquely determined by the given set,* and such that every internal point of the given intervals is internal to this set, and* vice versa.

6. From this theorem we can, if we please, deduce as a special case the so-called counterpart of the Heine-Borel theorem. The enunciation as given in Schoenflies's *Bericht*, p. 109, is as follows :—

"If to every point x of a set of points $X = (x)$, dense everywhere in a segment (A, B), there corresponds an interval ϵ containing x as internal point, and if every point of (A, B) is not internal to one of these intervals, then the intervals ϵ determine a finite or a closed set of points Q such that no point of Q is internal to the intervals ϵ, while this is the case for every point of the set complementary to Q."

It will be observed that the apparent restriction as to the relation of the set of points X to the intervals ϵ and to the continuum is superfluous, and only tends to complicate the issue. If we remove this restriction, the theorem is synonymous with the theorem stated at the end of § 5.

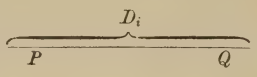
7. We can now prove the following *extension of the Heine-Borel theorem :—Given any closed set of points on a straight line and a set of intervals so that every point of the closed set of points is an internal point of at least one of the intervals, then there exist a finite number of the given intervals having the same property.*

For let us construct the equivalent set of non-overlapping intervals $D_1, D_2, \ldots$ (§ 5).

Then, by the fundamental property of this set, each point of the closed set of points is internal to one of the intervals D_i. If there be not a finite number of the intervals D_i having the same property, there must be a limiting point P of those intervals D_i, each of which contains at least one point of the closed set. P could not be a point of the closed set, since it is not internal to any one of these intervals D_i; but it is clearly a limiting point of the given set of points, and therefore belongs to the set, since it is a closed set.

* But not necessarily contained in the given set.

This is a contradiction, and therefore there can only be a finite number of the intervals D_i which contain points of the given set. Let these be $D_1, D_2, \ldots, D_k$. In any one of these, say D_i, the points of the closed set form a closed set lying entirely internal to D_i. Hence we can assign two points P and Q internal to D_i such that the points of the closed set which lie in D_i form a closed set lying betwen P and Q (not inclusive).



Now between P and Q, (both inclusive), there are only internal points of the given set of intervals (§ 5); hence, by the Heine-Borel theorem, we can assign a *finite* number of the given intervals so as to entirely cover up the segment (P, Q). In this finite number of the given intervals every point of the closed set of points in D_i must lie. Performing this for all values of i from 1 to k, we get a finite number of the original intervals having the required property.

The Continuations of certain Fundamental Power Series. *By* M. J. M. Hill, M.A., Sc.D., F.R.S., Astor Professor of Mathematics at University College, London. Received, in revised form, February 6th, 1903.*

1. The theory of the continuation of power series has not, so far as I can ascertain, been hitherto illustrated by applications to simple cases where the work is unartificial.

By means of some well known formulæ, proved rigidly for the first time by Abel, in his famous memoir on *The Binomial Series*, I have succeeded in working out the continuations of the binomial series, the logarithmic series, and the series for $\arctan x$ along arbitrary circuits. Using conjugate functions I have also succeeded in finding the continuation of the series for $\arcsin x$.

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* The paper has been condensed from two papers communicated at the meetings of November 13th, 1902, and December 11th, 1902; the latter dealt exclusively with the case of $\arcsin x$.

The continuation of the logarithmic series is obtained by Harkness and Morley in their treatise on *The Theory of Functions*, pp. 123–4, by considering a circuit in the form of a regular polygon containing an infinite number of sides.

The observation that the result must be the same for an arbitrary circuit led me to the consideration of a circuit of that kind, and finally to the results in this paper.

The continuation of the binomial series will be found at the end of Art. 3 and Art. 4, of the logarithmic series in Art. 7, of the series for arc tan x in Art. 8, of the series for arc sin x at the end of Art. 17.

2. *First Illustration.—Continuation of the Binomial Series.*

The work depends on a formula which can be deduced from those proved in Abel's memoir on *The Binomial Theorem.*

If $|x| < 1$, and if k_n be an abbreviation for $k(k-1)\ldots(k-n+1)$, then

$$1+k_1x+\frac{k_2x^2}{2!}+\ldots+\frac{k_nx^n}{n!}+\ldots$$
$$=|1+x|^k\left[\cos k\,\mathrm{amp}(1+x)+i\sin k\,\mathrm{amp}(1+x)\right],$$

where amp $(1+x)$ is the principal amplitude of $1+x$. This principal amplitude lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, because $|x| < 1$.

3. Let
$$Px = 1+k_1x+k_2\frac{x^2}{2!}+\ldots+k_n\frac{x^n}{n!}+\ldots, \qquad \text{(I.)}$$

where $|x| < 1$.

Put
$$x = x_1+(x-x_1).$$

Then
$$P(x\,|\,x_1) = P(x_1)+(x-x_1)P'(x_1)+\frac{(x-x_1)^2}{2!}P''(x_1)+\ldots$$
$$\ldots+\frac{(x-x_1)^n}{n!}P^n(x_1)+\ldots. \qquad \text{(II.)}$$

The condition for convergency will presently be shown to be
$$|x-x_1| < |1+x_1|.$$

In order to calculate the right-hand side of (II.) observe that
$$P(cx) = 1+k_1(cx)+\frac{k_2(cx)^2}{2!}+\ldots+\frac{k_n(cx)^n}{n!}+\ldots,$$

where
$$|cx| < 1. \qquad \text{(III.)}$$

Then by actual differentiation, and substitution in the equation

$$(1+cx)\frac{d^rP(cx)}{dx^r} = c(k-r+1)\frac{d^{r-1}P(cx)}{dx^{r-1}}, \qquad \text{(IV.)}$$

the validity of (IV.) can be established. From this it follows that

$$\frac{d^rP(cx)}{dx^r} = k_r\left(\frac{c}{1+cx}\right)^r P(cx). \qquad \text{(V.)}$$

Using the formula (V.) in (II.) it follows that

$$P(x\,|\,x_1) = P(x_1)\left[1+k_1\frac{x-x_1}{1+x_1}+\frac{k_2}{2!}\left(\frac{x-x_1}{1+x_1}\right)^2+\ldots+\frac{k_n}{n!}\left(\frac{x-x_1}{1+x_1}\right)^n+\ldots\right];$$

therefore

$$P(x\,|\,x_1) = P(x_1)\,P\left(\frac{x-x_1}{1+x_1}\right), \qquad \text{(VI.)}$$

provided that

$$|\,x-x_1\,| < |\,1+x_1\,|.$$

Put next

$$x-x_1 = (x_2-x_1)+(x-x_2).$$

Then

$$P\left(\frac{x-x_1}{1+x_1}\right) = P\left(\frac{x_2-x_1}{1+x_1}\right)+(x-x_2)\left[\frac{d}{dx}P\left(\frac{x-x_1}{1+x_1}\right)\right]_{x=x_2}$$
$$+\frac{(x-x_2)^2}{2!}\left[\frac{d^2}{dx^2}P\left(\frac{x-x_1}{1+x_1}\right)\right]_{x=x_2}+\ldots$$
$$+\frac{(x-x_2)^n}{n!}\left[\frac{d^n}{dx^n}P\left(\frac{x-x_1}{1+x_1}\right)\right]_{x=x_2}+\ldots.$$

Now, by (V.),

$$\frac{d^nP\left(\frac{x-x_1}{1+x_1}\right)}{dx^n} = k_n\left\{\frac{\left(\frac{1}{1+x_1}\right)}{1+\frac{x-x_1}{1+x_1}}\right\}^n P\left(\frac{x-x_1}{1+x_1}\right);$$

therefore

$$\left[\frac{d^nP\left(\frac{x-x_1}{1+x_1}\right)}{dx^n}\right]_{x=x_2} = \frac{k_n}{(1+x_2)^n}P\left(\frac{x_2-x_1}{1+x_1}\right);$$

therefore

$$P\left(\frac{x-x_1}{1+x_1}\right) = P\left(\frac{x_2-x_1}{1+x_1}\right)\left[1+k_1\frac{x-x_2}{1+x_2}+\frac{k_2}{2!}\left(\frac{x-x_2}{1+x_2}\right)^2+\ldots\right.$$
$$\left.\ldots+\frac{k_n}{n!}\left(\frac{1-x_2}{1+x_2}\right)^n+\ldots\right]$$
$$= P\left(\frac{x_2-x_1}{1+x_1}\right)P\left(\frac{x-x_2}{1+x_2}\right),$$

provided

$$|\,x-x_2\,| < |\,1+x_2\,|.$$

Hence (VI.) becomes

$$P(x \mid x_1 \mid x_2) = P(x_1)\, P\left(\frac{x_2 - x_1}{1 + x_1}\right) P\left(\frac{x - x_2}{1 + x_2}\right), \qquad \text{(VII.)}$$

and so on;

$$P(x \mid x_1 \mid x_2 \mid \ldots \mid x_s)$$

$$= P(x_1)\, P\left(\frac{x_2 - x_1}{1 + x_1}\right) P\left(\frac{x_3 - x_2}{1 + x_2}\right) \ldots P\left(\frac{x_s - x_{s-1}}{1 + x_{s-1}}\right) P\left(\frac{x - x_s}{1 + x_s}\right), \qquad \text{(VIII.)}$$

provided $$| x - x_s | < | 1 + x_s | .$$

4. Now let a series of points be taken which form a complete circuit. We can suppose x_s to be the same point as x_1.

Put $x_s = x_1$ in (VIII.), and then using (VI.) we have

$$\frac{P(x \mid x_1 \mid x_2 \mid \ldots x_{s-1} \mid x_1)}{P(x \mid x_1)} = P\left(\frac{x_2 - x_1}{1 + x_1}\right) P\left(\frac{x_3 - x_2}{1 + x_2}\right) \ldots P\left(\frac{x_1 - x_{s-1}}{1 + x_{s-1}}\right). \qquad \text{(IX.)}$$

To calculate the value of the expression last written, consider the expression

$$P\left(\frac{x_{r+1} - x_r}{1 + x_r}\right).$$

The point x_{r+1} is inside the circle of convergence whose centre is x_r: therefore

$$\left| \frac{x_{r+1} - x_r}{1 + x_r} \right| < 1;$$

therefore the amplitude of $1 + \dfrac{x_{r+1} - x_r}{1 + x_r}$ lies between $-\dfrac{\pi}{2}$ and $+\dfrac{\pi}{2}$.

This amplitude is the amplitude of $\dfrac{1 + x_{r+1}}{1 + x_r}$, and it is therefore the acute angle, positive or negative, through which the vector $1 + x_r$ must be turned to fall on the vector $1 + x_{r+1}$.

Hence by the result due to Abel quoted above

$$P\left(\frac{x_{r+1} - x_r}{1 + x_r}\right)$$

$$= \left| \frac{1 + x_{r+1}}{1 + x_r} \right|^k \left[\cos k \operatorname{amp} \left(\frac{1 + x_{r+1}}{1 + x_r}\right) + i \sin k \operatorname{amp} \left(\frac{1 + x_{r+1}}{1 + r_r}\right) \right];$$

therefore

$$P\left(\frac{x_2 - x_1}{1 + x_1}\right) P\left(\frac{x_3 - x_2}{1 + x_2}\right) \ldots P\left(\frac{x_1 - x_{s-1}}{1 + x_{s-1}}\right)$$

$$= \left\{ \begin{array}{l} \cos k \left[\operatorname{amp} \left(\dfrac{1 + x_2}{1 + x_1}\right) + \operatorname{amp} \left(\dfrac{1 + x_3}{1 + x_2}\right) + \ldots + \operatorname{amp} \left(\dfrac{1 + x_1}{1 + x_{s-1}}\right) \right] \\ + i \sin k \left[\operatorname{amp} \left(\dfrac{1 + x_2}{1 + x_1}\right) + \operatorname{amp} \left(\dfrac{1 + x_3}{1 + x_2}\right) + \ldots + \operatorname{amp} \left(\dfrac{1 + x_1}{1 + x_{s-1}}\right) \right] \end{array} \right\}.$$

Now $\left[\text{amp}\left(\frac{1+x_2}{1+x_1}\right)+\text{amp}\left(\frac{1+x_3}{1+x_2}\right)+\ldots+\text{amp}\left(\frac{1+x_1}{1+x_{s-1}}\right)\right]$

is the angle through which the radius vector joining x to the point (-1) turns round as x describes the circuit $x_1x_2 \ldots x_{s-1}x_1$.

If the circuit do not include the point (-1), then the angle is zero, and

$$P(x \mid x_1 \mid \ldots x_{s-1} \mid x_1) = P(x \mid x_1).$$

If the circuit go once round the point (-1) in the positive direction, then the angle is 2π, and

$$P(x \mid x_1 \mid \ldots x_{s-1} \mid x_1) = P(x \mid x_1)\,[\cos 2k\pi + i \sin 2k\pi],$$

so that, if k be a positive or negative integer, then

$$P(x \mid x_1 \mid \ldots x_{s-1} \mid x_1) = P(x \mid x_1).$$

But, if k be equal to $\frac{1}{n}$, where n is a positive integer,

$$P(x \mid x_1 \mid \ldots x_{s-1} \mid x_1) = P(x \mid x_1)\left(\cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}\right).$$

Hence the point x must go n times round the circuit including the singular point in the positive direction before the continuation can return to the original value $P(x \mid x_1)$.

5. A particular case of the preceding results will be required for the continuation of the series for arc sin x.

Let $$Q(x) = \sum_{n=0}^{\infty} \frac{1.3 \ldots (2n-1)}{2.4 \ldots (2n)} x^n,$$

which is the power series for $(1-x)^{-\frac{1}{2}}$. Let

$$R(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1.3 \ldots (2n-1)}{2.4 \ldots (2n)} x^n,$$

which is the power series for $(1+x)^{-\frac{1}{2}}$. Let

$$P(x) = \sum_{n=0}^{\infty} \frac{1.3 \ldots (2n-1)}{2.4 \ldots (2n)} x^{2n},$$

which is the power series for $(1-x^2)^{-\frac{1}{2}}$. These are all convergent, and

$$P(x) = Q(x)\,R(x),$$

if $|x| < 1$. Also

$$Q(x \mid x_1 \mid \ldots x_{s-1} \mid x_s) = Q(x_1)\,Q\left(\frac{x_2-x_1}{1-x_1}\right) \ldots Q\left(\frac{x_s-x_{s-1}}{1-x_{s-1}}\right) Q\left(\frac{x-x_s}{1-x_s}\right)$$

and

$$R(x \mid x_1 \mid \ldots x_{s-1} \mid x_s) = R(x_1)\,R\left(\frac{x_2-x_1}{1+x_1}\right) \ldots R\left(\frac{x_s-x_{s-1}}{1+x_{s-1}}\right) R\left(\frac{x-x_s}{1+x_s}\right).$$

Consequently $P(x \mid x_1 \mid \ldots x_{s-1} \mid x_s)$ is equal to

$$Q(x_1)\, Q\left(\frac{x_2-x_1}{1-x_1}\right) \ldots Q\left(\frac{x_s-x_{s-1}}{1-x_{s-1}}\right) \qquad \text{(A)}$$

multiplied by

$$R(x_1)\, R\left(\frac{x_2-x_1}{1+x_1}\right) \ldots R\left(\frac{x_s-x_{s-1}}{1+x_{s-1}}\right) \qquad \text{(B)}$$

multiplied by

$$Q\left(\frac{x-x_s}{1-x_s}\right) R\left(\frac{x-x_s}{1+x_s}\right). \qquad \text{(C)}$$

The product of the two power series in the line marked (C) is a convergent power series in $(x-x_s)$ if $|x-x_s|$ is less than the smaller of the two quantities $|1-x_s|$ and $|1+x_s|$.

If now $x_1 x_2 \ldots x_{s-1} x_s$ form a circuit including $x=1$, but not including $x=-1$, and x_s be the same point as x_1, then, using the results of Art. 4, and putting therein $k=-\frac{1}{2}$, we have

$$Q\left(\frac{x_2-x_1}{1-x_1}\right) \ldots Q\left(\frac{x_1-x_{s-1}}{1-x_{s-1}}\right) = \cos(-\tfrac{1}{2})2\pi + i\sin(-\tfrac{1}{2})2\pi = -1,$$

$$R\left(\frac{x_2-x_1}{1+x_1}\right) \ldots R\left(\frac{x_1-x_{s-1}}{1+x_{s-1}}\right) = \cos(-\tfrac{1}{2}) \quad 0 + i\sin(-\tfrac{1}{2}) \quad 0 = +1$$

therefore

$$\begin{aligned} P(x \mid x_1 \mid \ldots x_{s-1} \mid x_1) &= -\,Q(x_1)\, R(x_1)\, Q\left(\frac{x-x_1}{1-x_1}\right) R\left(\frac{x-x_1}{1+x_1}\right) \\ &= -\,P(x \mid x_1). \end{aligned}$$

In like manner, if the circuit include $x=-1$, but does not include $x=+1$, the same result will be obtained. But a circuit round both $x=+1$ and $x=-1$ brings the function back to its original value.

6. To apply the results to the series for arc $\sin x$, it is necessary to transform the preceding results by the aid of conjugate functions.

If $$x = \sin y,$$

and $$x = \xi + i\eta, \quad y = u + iv,$$

then $$\xi = \sin u \cosh v, \qquad \text{(X.)}$$

$$\eta = \cos u \sinh v\,; \qquad \text{(XI.)}$$

therefore $$(\xi \operatorname{cosec} u)^2 - (\eta \sec u)^2 = 1, \qquad \text{(XII.)}$$

$$(\xi \operatorname{sech} v)^2 + (\eta \operatorname{cosech} v)^2 = 1. \qquad \text{(XIII.)}$$

The equations (XII.) and (XIII.) are not equivalent to (X.) and

(XI.), because it has been necessary to square (X.) and (XI.) to find (XII.) and (XIII.).

It is convenient, however, to have (XII.) and (XIII.), and we will call (XII.) the equation of the hyperbola u, and (XIII.) the equation of the ellipse v.

Let the Riemann surface for the x-plane be constructed. Let the sheet corresponding to the principal branch be placed immediately below that corresponding to the $(+1)$-th branch. Then, if a point in the sheet corresponding to the principal branch be denoted by any letter K, the point immediately above it in the sheet corresponding to the $(+1)$-th branch will be denoted by K_1.

The x-plane will be slit along the ξ-axis from $+1$ to $+\infty$, and from -1 to $-\infty$; and there will be a junction from each edge of the slit ($+1$ to $+\infty$) for points to pass from the principal branch to the $(+1)$-th branch, and conversely.

Let corresponding points in the x and y planes be denoted by the same letter, and let us consider the circuit in the x-plane which corresponds to the following circuit in the y-plane. Let the circuit commence at the point $M(u = a,\ v = b)$, and pass along the line $u = a$ to the point $N(u = a,\ v = -b)$, then pass along the line $v = -b$ to the point $M_1(u = \pi - a,\ v = -b)$, then pass along the line $u = \pi - a$ to the point $N_1(u = \pi - a,\ v = +b)$, then pass along the line $v = +b$ to the starting point $M(u = a,\ v = b)$.

THE y-PLANE.

$y = u + iv$.

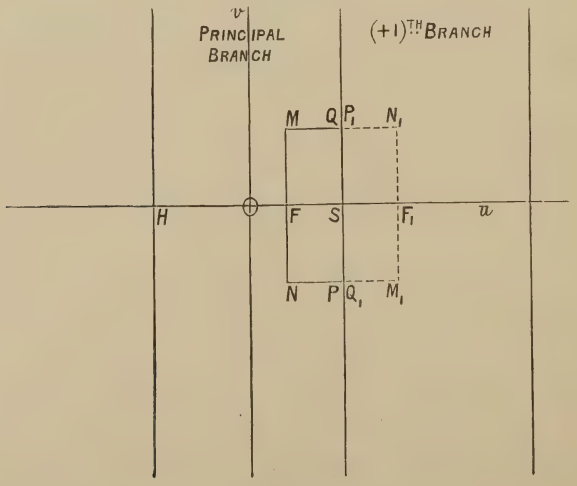

FIG. 1.

THE x-PLANE.

$x = \xi + i\eta.$

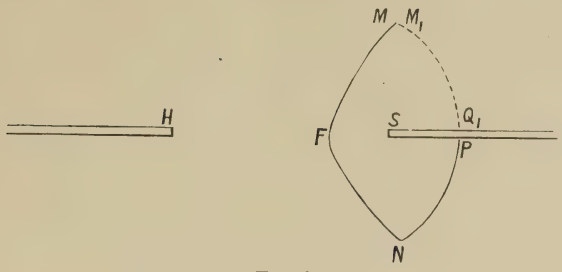

FIG. 2.

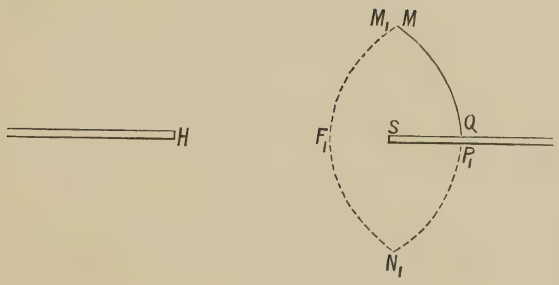

FIG. 3.

The point M, $u = a$, $v = b$, in the y-plane corresponds to

$$\xi = \sin a \cosh b, \quad \eta = \cos a \sinh b,$$

in the x-plane.

To the path MFN along $u = a$ in the y-plane corresponds the path MFN along the hyperbola $u = a$.

To the point N, $u = a$, $v = -b$ corresponds

$$\xi = \sin a \cosh b, \quad \eta = -\cos a \sinh b.$$

To the path NP along $v = -b$ corresponds the path NP along the ellipse $v = -b$.

At the point P the path crosses out of the sheet of the Riemann's surface representing the principal branch into the sheet representing the $(+1)$-th branch. It still remains, however, on the ellipse $v = -b$. When $u = \pi - a$, $v = -b$,

$$\xi = \sin a \cosh b, \quad \eta = \cos a \sinh b\,;$$

so that the point corresponding to M_1 is in the $(+1)$-th branch vertically above the point corresponding to M.

Hence in the x-plane a complete circuit has been made, corresponding to the path MNM_1 in the y-plane.

The portion $MFNP$ of the path is in the sheet corresponding to the principal branch.

The portion Q_1M_1 of the path, where M_1 is exactly above M, is in the sheet corresponding to the $(+1)$-th branch.

It remains to consider the path in the x-plane, which corresponds to $M_1F_1N_1P_1QM$ in the y-plane. $M_1F_1N_1$ is an arc of the hyperbola $u = \pi - a$.

This is in the sheet corresponding to the $(+1)$-th branch, and is immediately above the hyperbola $u = a$, which is in the sheet corresponding to the principal branch.

From N_1 to P_1 the point is moving along the ellipse $v = +b$.

At P_1 it crosses from the sheet representing the $(+1)$-th branch into the sheet representing the principal branch, and goes from Q to M, the original starting point along the ellipse $v = +b$. So that a second circuit in the x-plane has now been completed.

It is to be noticed that the ellipse $v = +b$ has the* upper half in the sheet corresponding to the principal branch, and the lower half in the sheet corresponding to the $(+1)$-th branch; whilst the ellipse $v = -b$ has the upper half in the sheet corresponding to the $(+1)$-th branch, and the lower half in the sheet corresponding to the principal branch.

Since $$x = \sin u \cosh v + i \cos u \sinh v,$$

therefore $$x_s = \sin u_s \cosh v_s + i \cos u_s \sinh v_s;$$

therefore

$$Q(x_1)\, Q\left(\frac{x_2 - x_1}{1 - x_1}\right) \ldots Q\left(\frac{x_s - x_{s-1}}{1 - x_{s-1}}\right) = |1 - x_s|^{-\frac{1}{2}} \left[\cos\left(-\tfrac{1}{2}\right)\sigma_s + i \sin\left(-\tfrac{1}{2}\right)\sigma_s\right],$$

where σ_s is the angle through which the vector $1 - x$, starting from a position in the positive direction of the real axis, must turn to fall on the vector $1 - x_s$ when the point x goes round the circuit. Similarly,

$$R(x_1)\, R\left(\frac{x_2 - x_1}{1 + x_1}\right) \ldots R\left(\frac{x_s - x_{s-1}}{1 + x_{s-1}}\right) = |1 + x_s|^{-\frac{1}{2}} \left[\cos\left(-\tfrac{1}{2}\right)\tau_s + i \sin\left(-\tfrac{1}{2}\right)\tau_s\right],$$

---

* For the upper half $\eta > 0$, for the lower half $\eta < 0$.

where τ_s is the angle through which the vector $1+x$, starting from a position in the positive direction of the real axis, must turn to fall on the vector $1+x_s$ when the point x goes round the circuit.

We have also the following expressions:—

$$|1-x_s| = \cosh v_s - \sin u_s,$$

$$\cos \operatorname{amp}(1-x_s) = \frac{1-\sin u_s \cosh v_s}{\cosh v_s - \sin u_s},$$

$$\sin \operatorname{amp}(1-x_s) = \frac{-\cos u_s \sinh v_s}{\cosh v_s - \sin u_s},$$

$$\tan \tfrac{1}{2} \operatorname{amp}(1-x_s) = -\tanh \tfrac{1}{2} v_s \cot\left(\frac{\pi}{4} - \tfrac{1}{2} u_s\right),$$

$$\cos \tfrac{1}{2} \operatorname{amp}(1-x_s) = \pm \frac{\cosh \tfrac{1}{2} v_s \sin\left(\frac{\pi}{4} - \tfrac{1}{2} u_s\right)}{\sqrt{\sin^2\left(\frac{\pi}{4} - \tfrac{1}{2} u_s\right) + \sinh^2 \tfrac{1}{2} v_s}},$$

$$\sin \tfrac{1}{2} \operatorname{amp}(1-x_s) = \mp \frac{\sinh \tfrac{1}{2} v_s \cos\left(\frac{\pi}{4} - \tfrac{1}{2} u_s\right)}{\sqrt{\sin^2\left(\frac{\pi}{4} - \tfrac{1}{2} u_s\right) + \sinh^2 \tfrac{1}{2} v_s}};$$

$$|1+x_s| = \cosh v_s + \sin u_s,$$

$$\cos \operatorname{amp}(1+x_s) = \frac{1+\sin u_s \cosh v_s}{\cosh v_s + \sin u_s},$$

$$\sin \operatorname{amp}(1+x_s) = \frac{\cos u_s \sinh v_s}{\cosh v_s + \sin u_s},$$

$$\tan \tfrac{1}{2} \operatorname{amp}(1+x_s) = \tanh \tfrac{1}{2} v_s \tan\left(\frac{\pi}{4} - \tfrac{1}{2} u_s\right),$$

$$\cos \tfrac{1}{2} \operatorname{amp}(1+x_s) = \pm \frac{\cosh \tfrac{1}{2} v_s \cos\left(\frac{\pi}{4} - \tfrac{1}{2} u_s\right)}{\sqrt{\cos^2\left(\frac{\pi}{4} - \tfrac{1}{2} u_s\right) + \sinh^2 \tfrac{1}{2} v_s}},$$

$$\sin \tfrac{1}{2} \operatorname{amp}(1+x_s) = \pm \frac{\sinh \tfrac{1}{2} v_s \sin\left(\frac{\pi}{4} - \tfrac{1}{2} u_s\right)}{\sqrt{\cos^2\left(\frac{\pi}{4} - \tfrac{1}{2} u_s\right) + \sinh^2 \tfrac{1}{2} v_s}}.$$

The ambiguous signs will now be determined. To do this consider a

portion $x_1 x_2 x_3 \ldots x_s$ of a circuit round $x = 1$, but not round $x = -1$, as in the figure.

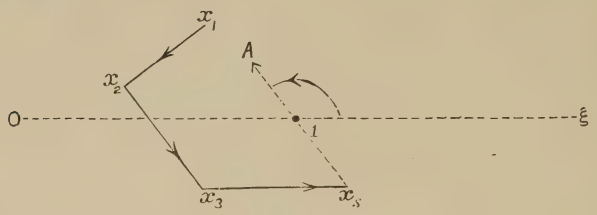

FIG. 4.

In this figure $\quad \text{amp}\,(1-x_s) = \angle\,\xi 1A,$

and also $\quad \sigma_s = \angle\,\xi 1A.$

We have also $\quad O < u_s < \tfrac{1}{2}\pi\,;$

therefore $\quad \cos u_s > 0.$

If, as in this figure, $\quad \sigma_s < \pi, \quad \cos \tfrac{1}{2}\sigma_s > 0,$

and $\quad \sigma_s = \text{amp}\,(1-x_s),$

therefore

$$\cos \tfrac{1}{2}\sigma_s = \frac{\cosh \tfrac{1}{2}v_s \left| \sin\left(\frac{\pi}{4} - \tfrac{1}{2}u_s\right)\right|}{\sqrt{\sin^2\left(\frac{\pi}{4} - \tfrac{1}{2}u_s\right) + \sinh^2 \tfrac{1}{2}v_s}},$$

$$\sin \tfrac{1}{2}\sigma_s = \frac{-\sinh \tfrac{1}{2}v_s \cot\left(\frac{\pi}{4} - \tfrac{1}{2}u_s\right)\left|\sin\left(\frac{\pi}{4} - \tfrac{1}{2}u\right)\right|}{\sqrt{\sin^2\left(\frac{\pi}{4} - \tfrac{1}{2}u_s\right) + \sinh^2 \tfrac{1}{2}v_s}}.$$

Also in this figure $\quad -\frac{\pi}{2} < \text{amp}\,(1+x_s) < +\frac{\pi}{2},$

and $\quad \tau_s = \text{amp}\,(1+x_s)\,;$

therefore $\quad \cos \tfrac{1}{2}\tau_s > 0\,;$

therefore

$$\cos\tfrac{1}{2}\tau_s = \frac{\cosh\tfrac{1}{2}v_s\left|\cos\left(\frac{\pi}{4}-\tfrac{1}{2}u_s\right)\right|}{\sqrt{\cos^2\left(\frac{\pi}{4}-\tfrac{1}{2}u_s\right)+\sinh^2\tfrac{1}{2}v_s}},$$

$$\sin\tfrac{1}{2}\tau_s = \frac{\sinh\tfrac{1}{2}v_s\tan\left(\frac{\pi}{4}-\tfrac{1}{2}u_s\right)\left|\cos\left(\frac{\pi}{4}-\tfrac{1}{2}u_s\right)\right|}{\sqrt{\cos^2\left(\frac{\pi}{4}-\tfrac{1}{2}u_s\right)+\sinh^2\tfrac{1}{2}v_s}};$$

therefore $$\cos\tfrac{1}{2}(\sigma_s+\tau_s) = \frac{|\cos u_s|\cosh v_s}{\sqrt{\cos^2 u_s+\sinh^2 v_s}},$$

$$\sin\tfrac{1}{2}(\sigma_s+\tau_s) = \frac{-\sinh v_s\sin u_s\dfrac{|\cos u_s|}{\cos u_s}}{\sqrt{\cos^2 u_s+\sinh^2 v_s}};$$

therefore

$$\cos\tfrac{1}{2}(\sigma_s+\tau_s)+i\sin\tfrac{1}{2}(\sigma_s+\tau_s) = \frac{|\cos u_s|}{\cos u_s}\,\frac{\cos u_s\cosh v_s - i\sin u_s\sinh v_s}{\sqrt{\cos^2 u_s+\sinh^2 v_s}}$$

$$= \frac{\cos u_s\cosh v_s - i\sin u_s\sinh v_s}{\sqrt{\cos^2 u_s+\sinh^2 v_s}};$$

because $\cos u_s > 0$. Also

$$|1-x_s^2| = \cos^2 u_s+\sinh^2 v_s;$$

therefore

$$|1-x_s^2|^{-\frac{1}{2}}\left[\cos\left(-\tfrac{1}{2}\right)(\sigma_s+\tau_s)+i\sin\left(-\tfrac{1}{2}\right)(\sigma_s+\tau_s)\right]$$

$$= \frac{1}{\cos u_s\cosh v_s - i\sin u_s\sinh v_s} = \sec(u_s+iv_s).$$

The next case to be considered is that in which x_s is on the real axis between $+1$ and $+\infty$.

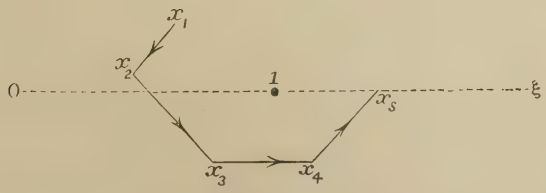

FIG. 5.

Here
$$\text{amp}\,(1-x_s) = \sigma_s = \pi,$$
$$u_s = \frac{\pi}{2},$$
$$\tau_s = 0,$$
$$|\,1+x_s\,| = \cosh v_s + 1,$$
$$|\,1-x_s\,| = \cosh v_s - 1,$$
$$|\,1-x_s^2\,| = \sinh^2 v_s,$$
$$|\,1-x_s^2\,|^{-\frac{1}{2}} = -\,\text{cosech}\, v_s,$$

the negative sign being taken, because here v_s is negative. [Compare this with Fig. 2. For the part of the path NP, v is negative.]

Therefore $\cos\frac{1}{2}(\sigma_s+\tau_s) = 0$, $\sin\frac{1}{2}(\sigma_s+\tau_s) = 1$;

therefore $|\,1-x_s^2\,|^{-\frac{1}{2}}\left[\cos(-\frac{1}{2})(\sigma_s+\tau_s)+i\sin(-\frac{1}{2})(\sigma_s+\tau_s)\right]$
$$= -\,\text{cosech}\, v_s(-i) = i\,\text{cosech}\, v_s.$$

[This is included in the previous result, as is seen by putting $u_s = \frac{\pi}{2}$ in it.]

There remains only the case in which $\pi < \sigma_s < 2\pi$.

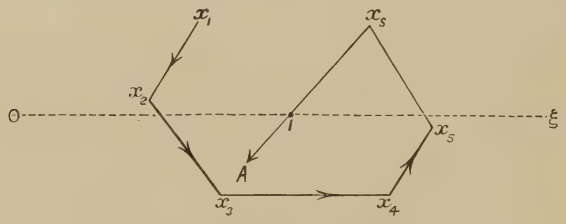

Fig. 6.

Here
$$\sigma_s = \pi + OIA,$$
$$\text{amp}\,(1-x_s) = -\,(\pi - OIA);$$

therefore
$$\sigma_s = \text{amp}\,(1-x_s) + 2\pi;$$
$$\tfrac{1}{2}\sigma_s = \pi + \tfrac{1}{2}\,\text{amp}\,(1-x_s).$$

Hence the expressions for $\cos\frac{1}{2}\sigma_s$ and $\sin\frac{1}{2}\sigma_s$ have the opposite signs to those obtained in the first case.

But $\cos\frac{1}{2}\tau_s$ and $\sin\frac{1}{2}\tau_s$ are unchanged.

Hence $\cos\frac{1}{2}(\sigma_s+\tau_s)$ and $\sin\frac{1}{2}(\sigma_s+\tau_s)$ have the opposite signs to those obtained in the first case.

Therefore

$$\cos\tfrac{1}{2}(\sigma_s+\tau_s)+i\sin\tfrac{1}{2}(\sigma_s+\tau_s)$$
$$=-\frac{|\cos u_s|}{\cos u_s}\,\frac{\cos u_s\cosh v_s-i\sin u_s\sinh v_s}{\sqrt{\cos^2 u_s+\sinh^2 v_s}}.$$

Now, in this case,

$$\frac{\pi}{2}<u_s<\pi,\quad \cos u_s<0;$$

therefore

$$\cos\tfrac{1}{2}(\sigma_s+\tau_s)+i\sin\tfrac{1}{2}(\sigma_s+\tau_s)=\frac{\cos u_s\cosh v_s-i\sin u_s\sinh v_s}{\sqrt{\cos^2 u_s+\sinh^2 v_s}},$$

as in the first case.

Further, because $|1-x_s^2|=\cos^2 u_s+\sinh^2 v_s$,

therefore

$$|1-x_s^2|^{-\frac{1}{2}}\left[\cos(-\tfrac{1}{2})(\sigma_s+\tau_s)+i\sin(-\tfrac{1}{2})(\sigma_s+\tau_s)\right]$$
$$=\frac{1}{\cos u_s\cosh v_s-i\sin u_s\sinh v_s}=\sec(u_s+iv_s),$$

as in the first case.

Hence it is proved in all cases that

$$Q(x_1)\,Q\left(\frac{x_2-x_1}{1-x_1}\right)\ldots Q\left(\frac{x_s-x_{s-1}}{1-x_{s-1}}\right)\times R(x_1)\left(\frac{x_2-x_1}{1+x_1}\right)\ldots R\left(\frac{x_s-x_{s-1}}{1+x_{s-1}}\right)$$
$$=\sec(u_s+iv_s).$$

Hence the continuation of the power series for $(1-x^2)^{-\frac{1}{2}}$, when $x=\sin(u+iv)$, has everywhere the value $\sec(u+iv)$. In this form it is easy to trace the changes in the value of the power series.

When $u=a,\ v=b$,

$$\sec(u+iv)=\quad\sec(a+ib).$$

When $u=a,\ v=-b$,

$$\sec(u+iv)=\quad\sec(a-ib).$$

When $u=\pi-a,\ v=-b$,

$$\sec(u+iv)=-\sec(a+ib).$$

Hence $u+iv$ has changed from $a+ib$ to $\pi-(a+ib)$, and the value of the power series has merely changed its sign.

It is possible to deduce from the above the corresponding results for the power series for $(1+x^2)^{-\frac{1}{2}}$ [which is not in reality a distinct series from the preceding].

The result is that the power series for $(1+x^2)^{-\frac{1}{2}}$, where

$x = \sinh(u+iv)$, and its continuations are everywhere represented by the expression $\operatorname{sech}(u+iv)$.

7. *Second Illustration.—The Continuation of the Logarithmic Series.*

The results only are given.

The work depends on the theorem due to Abel,

$$x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots = \log|1+x| + i\,\text{amp}\,(1+x),$$

where $|x| < 1$, and amp $(1+x)$ is the principal amplitude of $(1+x)$. Since $|x| < 1$, this principal amplitude lies between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$.

Let $$P(x) = x - \frac{x^2}{2} + \ldots + (-1)^{r-1}\frac{x^r}{r} + \ldots \quad (|x| < 1).$$

Then $$P(x \mid x_1) = P(x_1) + P\left(\frac{x-x_1}{1+x_1}\right),$$

$$P(x \mid x_1 \mid \ldots \mid x_{s-1} \mid x_s) = P(x_1) + P\left(\frac{x_2-x_1}{1+x_1}\right) + \ldots + P\left(\frac{x_s-x_{s-1}}{1+x_{s-1}}\right) + P\left(\frac{x-x_s}{1+x_s}\right).$$

If therefore x_s be the same point as x_1,

$$P(x \mid x_1 \mid \ldots \mid x_{s-1} \mid x_1) - P(x \mid x_1) = P\left(\frac{x_2-x_1}{1+x_1}\right) + \ldots + P\left(\frac{x_1-x_{s-1}}{1+x_{s-1}}\right).$$

Now $$P\left(\frac{x_{r+1}-x_r}{1+x_r}\right) = \log\left|1+\frac{x_{r+1}-x_r}{1+x_r}\right| + i\,\text{amp}\left(1+\frac{x_{r+1}-x_r}{1+x_r}\right)$$

$$= \log\left|\frac{1+x_{r+1}}{1+x_r}\right| + i\,\text{amp}\left(\frac{1+x_{r+1}}{1+x_r}\right);$$

therefore $$P(x \mid x_1 \mid \ldots \mid x_{s-1} \mid x_1) - P(x \mid x_1)$$

$$= \log\left\{\left|\frac{1+x_2}{1+x_1}\right|\left|\frac{1+x_3}{1+x_2}\right|\ldots\left|\frac{1+x_1}{1+x_{s-1}}\right|\right\} + i\left[\text{amp}\,\frac{1+x_2}{1+x_1} + \text{amp}\,\frac{1+x_3}{1+x_2} + \ldots + \text{amp}\,\frac{1+x_1}{1+x_{s-1}}\right].$$

The logarithm vanishes.

In consequence of the hypothesis regarding the vertices of the polygon, viz., that each is in the circle of convergence whose centre is the preceding vertex, the value of the coefficient of i is equal to the angle which the radius vector joining x to the point (-1) describes as x describes the polygon. It is therefore zero if the polygon do not contain the point (-1), but equal to 2π if the point (-1) lie in the interior of the polygon, and the describing point has

gone once round the origin in the positive direction. Therefore

$$\left[P(x \mid x_1 \mid x_2 \ldots \mid x_{s-1} \mid x_1)\right] - P(x \mid x_1) = 2i\pi$$

if the polygon contain the point (-1). Also

$$\left[P(x \mid x_1 \mid x_2 \ldots \mid x_{s-1} \mid x_1)\right] = P(x \mid x_1)$$

if the polygon do not contain the point (-1).

8. *Third Illustration.—Continuation of the Series for* arc tan x.

The results only are given.

Let $$P(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots,$$

where $$|x| < 1.$$

Let $$Q(x) = x - \tfrac{1}{2}x^2 + \ldots + \frac{(-1)^{n-1}}{n}x^n + \ldots.$$

Then $$P(x \mid x_1) = P(x_1) + \frac{1}{2i}\left[Q\left(\frac{x-x_1}{x_1-i}\right) - Q\left(\frac{x-x_1}{x_1+i}\right)\right];$$

therefore

$$P(x \mid x_1 \ldots \mid x_{s-1} \mid x_s)$$

$$= P(x_1) + \frac{1}{2i}\left\{\begin{array}{l} Q\left(\frac{x_2-x_1}{x_1-i}\right) + \ldots + Q\left(\frac{x_s-x_{s-1}}{x_{s-1}-i}\right) + Q\left(\frac{x-x_s}{x_s-i}\right) \\ -Q\left(\frac{x_2-x_1}{x_1+i}\right) - \ldots - Q\left(\frac{x_s-x_{s-1}}{x_{s-1}+i}\right) - Q\left(\frac{x-x_s}{x_s+i}\right) \end{array}\right\}.$$

If x_s be the same point as x_1, then

$$P(x \mid x_1 \mid \ldots x_{s-1} \mid x_1) - P(x \mid x_1)$$

$$= \frac{1}{2i}\left\{\begin{array}{l} Q\left(\frac{x_2-x_1}{x_1-i}\right) + \ldots + Q\left(\frac{x_1-x_{s-1}}{x_{s-1}-i}\right) \\ -Q\left(\frac{x_2-x_1}{x_1+i}\right) - \ldots - Q\left(\frac{x_1-x_{s-1}}{x_{s-1}+i}\right) \end{array}\right\}$$

$$= \tfrac{1}{2}\left\{\begin{array}{l} \operatorname{amp}\left(\frac{x_2-i}{x_1-i}\right) + \operatorname{amp}\left(\frac{x_3-i}{x_2-i}\right) + \ldots + \operatorname{amp}\left(\frac{x_1-i}{x_{s-1}-i}\right) \\ -\operatorname{amp}\left(\frac{x_2+i}{x_1+i}\right) - \operatorname{amp}\left(\frac{x_3+i}{x_2+i}\right) - \ldots - \operatorname{amp}\left(\frac{x_1-i}{x_{s-1}+i}\right) \end{array}\right\}$$

$= \pi$ for a circuit round $x = i$ in the positive direction,

but equal to $-\pi$,, ,, $x = -i$,, ,, ,

and equal to 0 for a circuit including *neither* $x = i$ nor $x = -i$ or for a circuit including *both* $x = i$ and $x = -i$.

9. *Fourth Illustration.—Continuation of the Power Series for* $\arcsin x$.

General Formulæ for the Continuation of Power Series.—

If $f(x)$ be a power series, then

$$f(x \mid x_1) = f(x_1) + \sum_{n=1}^{\infty} \frac{(x-x_1)^n}{n!} f^n(x_1);$$

therefore $$\left[f(x \mid x_1)\right]_{x=x_2} = f(x_1) + \sum_{n=1}^{\infty} \frac{(x_2-x_1)^n}{n!} f^n(x_1).$$

Putting $$x - x_1 = (x_2 - x_1) + (x - x_2),$$

it follows that

$$\begin{aligned} f(x \mid x_1 \mid x_2) &= \left[f(x_1) + \sum_{n=1}^{\infty} \frac{(x_2-x_1)^n}{n!} f^n(x_1)\right] + (x-x_2)\left[f'(x_1) + \sum_{n=1}^{\infty} \frac{(x_2-x_1)^n}{n!} f^{n+1}(x_1)\right] \\ &\quad + \frac{(x-x_2)^2}{2!}\left[f''(x_1) + \sum_{n=1}^{\infty} \frac{(x_2-x_1)^n}{n!} f^{n+2}(x_1)\right] \\ &\quad + \ldots \\ &= \left[f(x \mid x_1)\right]_{x=x_2} + (x-x_2)\left[\frac{df(x \mid x_1)}{dx}\right]_{x=x_2} + \frac{(x-x_2)^2}{2!}\left[\frac{d^2 f(x \mid x_1)}{dx^2}\right]_{x=x_2} + \ldots. \end{aligned}$$

Putting $$x - x_2 = (x_3 - x_2) + (x - x_3),$$

it follows in like manner that

$$\begin{aligned} f(x \mid x_1 \mid x_2 \mid x_3) &= \left[f(x \mid x_1 \mid x_2)\right]_{x=x_3} + (x-x_3)\left[\frac{df(x \mid x_1 \mid x_2)}{dx}\right]_{x=x_3} \\ &\quad + \frac{(x-x_3)^2}{2!}\left[\frac{d^2 f(x \mid x_1 \mid x_2)}{dx^2}\right]_{x=x_3} + \ldots, \end{aligned}$$

and so on; therefore

$$\begin{aligned} f(x \mid x_1 \mid x_2 \mid x_3) &= f(x_1) + \left\{(x_2-x_1)\left[\frac{df(x)}{dx}\right]_{x=x_1} + \frac{(x_2-x_1)^2}{2!}\left[\frac{d^2 f(x)}{dx^2}\right]_{x=x_1} + \ldots\right\} \\ &\quad + \left\{(x_3-x_2)\left[\frac{df(x \mid x_1)}{dx}\right]_{x=x_2} + \frac{(x_3-x_2)^2}{2!}\left[\frac{d^2 f(x \mid x_1)}{dx^2}\right]_{x=x_2} + \ldots\right\} \\ &\quad + \left\{(x-x_3)\left[\frac{df(x \mid x_1 \mid x_2)}{dx}\right]_{x=x_3} + \frac{(x-x_3)^2}{2!}\left[\frac{d^2 f(x \mid x_1 \mid x_2)}{dx^2}\right]_{x=x_3} + \ldots\right\}, \end{aligned}$$

and so on.

Denoting therefore the series

$$\sum_{n=0}^{\infty} \frac{(x-x_r)^{n+1}}{(n+1)!} \left[\frac{d^{n+1} f(x \mid x_1 \mid \ldots \mid x_{r-1})}{dx^{n+1}}\right]_{x=x_r}$$

by $\chi(x, x_r)$, it follows that

$$f(x \mid x_1 \mid \ldots \mid x_r) = f(x_1) + \chi(x_2, x_1) + \chi(x_3, x_2) + \ldots + \chi(x_{r-1}, x_r) + \chi(x, x_r).$$

10. The quantity χ satisfies the following relations :—

$$\frac{\partial \chi(x, x_r)}{\partial x} = \sum_{n=0}^{\infty} \frac{(x-x_r)^n}{n!} \left[\frac{d^{n+1} f(x \mid x_1 \mid \ldots \mid x_{r-1})}{dx^{n+1}}\right]_{x=x_r}$$

$$= \frac{df(x \mid x_1 \mid \ldots \mid x_{r-1} \mid x_r)}{dx},$$

$$\frac{\partial \chi(x, x_r)}{\partial x_r} = -\left[\frac{df(x \mid x_1 \mid \ldots \mid x_{r-1})}{dx}\right]_{x=x_r}.$$

The quantity $\chi(x, x_r)$ does not involve $x_1, x_2, \ldots, x_{r-1}$, except in so far as the choice of these points affects the branch of the function $f(x)$ which is represented by $f(x \mid x_1 \mid \ldots \mid x_r)$.

Its value will be determined later on in the case where $f(x)$ is the power series for arc sin x.

11. *Calculation of the Differential Coefficients of the Continuations of the Power Series for* arc sin x.

We will now suppose that $f(x)$ is the power series

$$\sum_{n=0}^{\infty} \frac{1.3 \ldots (2n-1)}{2.4 \ldots (2n)} \frac{x^{2n+1}}{2n+1},$$

which is convergent if $|x| \leqslant 1$. And we choose x_1 so that $|x_1| < 1$.

$$\frac{df(x)}{dx} = \sum_{n=0}^{\infty} \frac{1.3 \ldots (2n-1)}{2.4 \ldots (2n)} x^{2n} = \text{power series for } \frac{1}{\sqrt{1-x^2}};$$

therefore, using the notation of Art. 5,

$$\frac{df(x \mid x_1 \mid \ldots \mid x_r)}{dx} = P(x \mid x_1 \mid \ldots \mid x_r)$$

$$= Q(x_1) Q\left(\frac{x_2-x_1}{1-x_1}\right) \ldots Q\left(\frac{x_r-x_{r-1}}{1-x_{r-1}}\right) Q\left(\frac{x-x_r}{1-x_r}\right)$$

$$\times R(x_1) R\left(\frac{x_2-x_1}{1+x_1}\right) \ldots R\left(\frac{x_r-x_{r-1}}{1+x_{r-1}}\right) R\left(\frac{x-x_r}{1+x_r}\right);$$

therefore

$$\frac{d^{n+1}f(x|x_1|\dots|x_r)}{dx^{n+1}} = Q(x_1)\,Q\left(\frac{x_2-x_1}{1-x_1}\right)\dots Q\left(\frac{x_r-x_{r-1}}{1-x_{r-1}}\right)$$
$$\times R(x_1)\,R\left(\frac{x_2-x_1}{1+x_1}\right)\dots R\left(\frac{x_r-x_{r-1}}{1+x_{r-1}}\right)$$
$$\times \frac{d^n}{dx^n}\left[Q\left(\frac{x-x_r}{1-x_r}\right)R\left(\frac{x-x_r}{1+x_r}\right)\right].$$

Now $$\frac{d^n}{dx^n}\left[Q\left(\frac{x-x_r}{1-x_r}\right)R\left(\frac{x-x_r}{1+x_r}\right)\right]$$
$$= \sum_{s=0}^{s=n} {}_nC_s\left[\frac{d^{n-s}}{dx^{n-s}}Q\left(\frac{x-x_r}{1-x_r}\right)\right]\left[\frac{d^s}{dx^s}R\left(\frac{x-x_r}{1+x_r}\right)\right].$$

Now $$Q\left(\frac{x-\xi}{1-\xi}\right) = \sum_{t=0}^{t=\infty}\frac{1.3\dots(2t-1)}{2.4\dots(2t)}\left(\frac{x-\xi}{1-\xi}\right)^t;$$

therefore

$$\frac{d^v}{dx^v}Q\left(\frac{x-\xi}{1-\xi}\right) = \sum_{t=v}^{t=\infty}\frac{1.3\dots(2t-1)}{2.4\dots(2t)}\,\frac{t(t-1)\dots(t-v+1)}{(1-\xi)^v}\left(\frac{x-\xi}{1-\xi}\right)^{t-v}$$
$$= \sum_{w=0}^{w=\infty}\frac{1.3\dots(2v+2w-1)}{2.4\dots(2v+2w)}\,\frac{(v+w)(v+w-1)\dots(w+1)}{(1-\xi)^v}\left(\frac{x-\xi}{1-\xi}\right)^w$$
$$= \sum_{w=0}^{w=\infty}\frac{1.3\dots(2v+2w-1)}{2^{v+w}\,w!}\,\frac{1}{(1-\xi)^v}\left(\frac{x-\xi}{1-\xi}\right)^w.$$

I will prove that this is equal to

$$(-\tfrac{1}{2})_v\,\frac{Q\left(\frac{x-\xi}{1-\xi}\right)}{(x-1)^v}$$

if $$\left|\frac{x-\xi}{1-\xi}\right| < 1.$$

Now $$(-\tfrac{1}{2})_v\,\frac{Q\left(\frac{x-\xi}{1-\xi}\right)}{(x-1)^v} = (-\tfrac{1}{2})_v\left(-\frac{1}{1-\xi}\right)^v\frac{Q\left(\frac{x-\xi}{1-\xi}\right)}{\left[1-\frac{x-\xi}{1-\xi}\right]^v},$$

i.e., to

$$(-\tfrac{1}{2})_v\left(-\frac{1}{1-\xi}\right)^v\sum_{w=0}^{w=\infty}\frac{1.3\dots(2w-1)}{2.4\dots(2w)}\left(\frac{x-\xi}{1-\xi}\right)^w$$
$$\times\left[1+v\left(\frac{x-\xi}{1-\xi}\right)+\frac{v(v+1)}{2!}\left(\frac{x-\xi}{1-\xi}\right)^2+\dots\right.$$
$$\left.\dots+\frac{v(v+1)\dots(v+w-1)}{w!}\left(\frac{x-\xi}{1-\xi}\right)^w+\dots\right].$$

The coefficient of $\left(\frac{x-\xi}{1-\xi}\right)^w$ in this product is

$$(-\tfrac{1}{2})_v\left(-\frac{1}{1-\xi}\right)^v \times\Big[\frac{v(v+1)\dots(v+w-1)}{w!} + \tfrac{1}{2}\frac{v(v+1)\dots(v+w-2)}{(w-1)!} + \frac{1.3}{2.4}\frac{v(v+1)\dots(v+w-3)}{(w-2)!}+\dots \dots+\frac{1.3\dots(2w-3)}{2.4\dots(2w-2)}v+\frac{1.3\dots(2w-1)}{2.4\dots(2w)}\Big].$$

The bracket is the coefficient of z^w in the product of the power series for $(1-z)^{-v}$ multiplied by the power series for $(1-z)^{-\frac{1}{2}}$.

Hence it is the coefficient of z^w in $(1-z)^{-v-\frac{1}{2}}$. It is therefore

$$\frac{(v+\frac{1}{2})(v+\frac{3}{2})\dots\left(v+\frac{2w-1}{2}\right)}{w!}.$$

The coefficient of $\left(\frac{x-\xi}{1-\xi}\right)^w$ is therefore

$$\left(\frac{1}{1-\xi}\right)^v\frac{\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2v-1}{2}\right)\left(\frac{2v+1}{2}\right)\left(\frac{2v+3}{2}\right)\cdots\left(\frac{2v+2w-1}{2}\right)}{w!} = \frac{1}{(1-\xi)^v}\,\frac{1.3\dots(2v+2w-1)}{2^{v+w}w!};$$

therefore $$\frac{d^v}{dx^v}Q\left(\frac{x-\xi}{1-\xi}\right) = (-\tfrac{1}{2})_v\frac{Q\left(\frac{x-\xi}{1-\xi}\right)}{(x-1)^v},$$

provided $|x-\xi|<|1-\xi|$. Similarly,

$$\frac{d^v}{dx^v}R\left(\frac{x-\xi}{1+\xi}\right) = (-\tfrac{1}{2})_v\frac{R\left(\frac{x-\xi}{1+\xi}\right)}{(x+1)^v};$$

therefore

$$\frac{d^n}{dx^n}\left[Q\left(\frac{x-x_r}{1-x_r}\right)R\left(\frac{x-x_r}{1+x_r}\right)\right] = \sum_{s=0}^{s=n}\left[{}_nC_s(-\tfrac{1}{2})_{n-s}(-\tfrac{1}{2})_sQ\left(\frac{x-x_r}{1-x_r}\right)R\left(\frac{x-x_r}{1+x_r}\right)\frac{1}{(x-1)^{n-s}}\frac{1}{(x+1)^s}\right]$$

$$= Q\left(\frac{x-x_r}{1-x_r}\right)R\left(\frac{x-x_r}{1+x_r}\right)\sum_{s=0}^{s=n}\left[{}_nC_s(-\tfrac{1}{2})_{n-s}(-\tfrac{1}{2})_s\frac{1}{(x-1)^{n-s}}\frac{1}{(x+1)^s}\right];$$

therefore

$$\frac{1}{(n+1)!}\frac{d^{n+1}f(x\,|\,x_1|\,\dots\,|\,x_r)}{dx^{n+1}}$$

$$= Q(x_1)\,Q\left(\frac{x_2-x_1}{1-x_1}\right)\dots Q\left(\frac{x_r-x_{r-1}}{1-x_{r-1}}\right)Q\left(\frac{x-x_r}{1-x_r}\right)$$

$$\times R(x_1)\,R\left(\frac{x_2-x_1}{1+x_1}\right)\dots R\left(\frac{x_r-x_{r-1}}{1+x_{r-1}}\right)R\left(\frac{x-x_r}{1+x_r}\right)$$

$$\times\frac{1}{n+1}\sum_{s=0}^{s=n}\left[\frac{(-\tfrac{1}{2})_{n-s}}{(n-s)!}\,\frac{(-\tfrac{1}{2})_s}{s!}\,\frac{1}{(x-1)^{n-s}}\,\frac{1}{(x+1)^s}\right].$$

Hence $\dfrac{(x-x_r)^{n+1}}{(n+1)!}\left[\dfrac{d^{n+1}f(x\,|\,x_1|\,\dots\,|\,x_{r-1})}{dx^{n+1}}\right]_{x=x_r}$

$$= Q(x_1)\,Q\left(\frac{x_2-x_1}{1-x_1}\right)\dots Q\left(\frac{x_r-x_{r-1}}{1-x_{r-1}}\right)$$

$$\times R(x_1)\,R\left(\frac{x_2-x_1}{1+x_1}\right)\dots R\left(\frac{x_r-x_{r-1}}{1+x_{r-1}}\right)$$

$$\times\frac{(x-x_r)^{n+1}}{n+1}\sum_{s=0}^{s=n}\left[\frac{(-\tfrac{1}{2})_{n-s}}{(n-s)!}\,\frac{(-\tfrac{1}{2})_s}{s!}\,\frac{1}{(x_r-1)^{n-s}}\,\frac{1}{(x_r+1)^s}\right].$$

12. *Determination of the Quantities* $\chi(x, x_r)$, $\chi(x_{s+1}, x_s)$.

Hence

$$\chi(x, x_r) = Q(x_1)\,Q\left(\frac{x_2-x_1}{1-x_1}\right)\dots Q\left(\frac{x_r-x_{r-1}}{1-x_{r-1}}\right)$$

$$\times R(x_1)\,R\left(\frac{x_2-x_1}{1+x_1}\right)\dots R\left(\frac{x_r-x_{r-1}}{1+x_{r-1}}\right)$$

$$\times\sum_{n=0}^{\infty}\left\{\frac{(x-x_r)^{n+1}}{n+1}\sum_{s=0}^{s=n}\left[\frac{(-\tfrac{1}{2})_{n-s}}{(n-s)!}\,\frac{(-\tfrac{1}{2})_s}{s!}\,\frac{1}{(x_r-1)^{n-s}}\,\frac{1}{(x_r+1)^s}\right]\right\}.$$

Now take a circuit round $x=1$, such that $|x_r-1|$ is always less than $|x_r+1|$; then it will be proved that the above series is convergent if

$$|x-x_r| < |x_r-1|.$$

Since $\left|\dfrac{(-\frac{1}{2})_{n-s}}{(n-s)!}\right|$ and $\left|\dfrac{(-\frac{1}{2})_s}{s!}\right|$ are each less than unity, therefore the modulus of the general term of the series is less than

$$\frac{|x-x_r|^{n+1}}{n+1}\sum_{s=0}^{s=n}\frac{1}{|x_r-1|^{n-s}}\,\frac{1}{|x_r+1|^s} < \frac{|x-x_r|^{n+1}}{|x_r-1|^n}\,\frac{1}{n+1}\sum_{s=0}^{s=n}\left|\frac{x_r-1}{x_r+1}\right|^s;$$

and, since $$\left| \frac{x_r - 1}{x_r + 1} \right| < 1,$$

therefore $$\frac{1}{n+1} \sum_{s=0}^{s=n} \left| \frac{x_r - 1}{x_r + 1} \right|^s < 1.$$

Hence the modulus of the general term of the series is less than

$$|x - x_r| \left| \frac{x - x_r}{x_r - 1} \right|^n.$$

Hence the series is convergent if

$$|x - x_r| < |x_r - 1|.$$

In like manner, if a circuit round $x = -1$ be taken, the series will be convergent if

$$|x - x_r| < |x_r + 1|.$$

Hence the series will be convergent if x be so taken that $|x - x_r|$ is always less than the smaller of the two moduli $|x - x_r|$ and $|x_r + 1|$.

Hence $\chi(x, x_r)$ and $\chi(x_{s+1}, x_s)$ will be convergent if the vertices $x_1, x_2, \ldots, x_r$ are all chosen so that each of them is in the circle of convergence whose centre is at the preceding vertex.

In order to evaluate $\chi(x_{s+1}, x_s)$ we will suppose that x_s and x_{s+1} both lie either on one of the hyperbolas for which $u =$ constant, or on one of the ellipses for which $v =$ constant. This is not in reality a restriction, for, if x_s and x_{s+1} are not so placed, it is possible to draw the ellipse (or hyperbola) through x_s, and find where it meets the hyperbola (or ellipse) through x_{s+1} and take their point of intersection as a vertex of the polygon intermediate between x_s and x_{s+1}.

Now it was proved in Art. 10 that

$$\frac{\partial \chi(x, x_r)}{\partial x} = \frac{df(x \mid x_1 \mid \ldots \mid x_r)}{dx}$$

and $$\frac{\partial \chi(x, x_r)}{\partial x_r} = -\left[\frac{df(x \mid x_1 \mid \ldots \mid x_{r-1})}{dx} \right]_{x = x_r};$$

therefore $$\frac{\partial \chi(x_{s+1}, x_s)}{\partial x_{s+1}} = \left[\frac{df(x \mid x_1 \mid \ldots \mid x_s)}{dx} \right]_{x = x_{s+1}},$$

$$\frac{\partial \chi(x_{s+1}, x_s)}{\partial x_s} = -\left[\frac{df(x \mid x_1 \mid \ldots \mid x_{s-1})}{dx} \right]_{x = x_s}.$$

Therefore, by Art. 11,

$$\frac{\partial\chi(x_{s+1},\,x_s)}{\partial x_{s+1}} = Q(x_1)\,Q\left(\frac{x_2-x_1}{1-x}\right)\ldots Q\left(\frac{x_{s+1}-x_s}{1-x_s}\right)$$

$$\times R(x_1)\,R\left(\frac{x_2-x_1}{1+x_1}\right)\ldots R\left(\frac{x_{s+1}-x_s}{1+x_s}\right)$$

$$= \frac{1}{\cos u_{s+1}\cosh v_{s+1} - i\sin u_{s+1}\sinh v_{s+1}} \quad \text{(by Art. 6)},$$

$$\frac{\partial\chi(x_{s+1},\,x_s)}{\partial x_s} = -Q(x_1)\,Q\left(\frac{x_2-x_1}{1-x_1}\right)\ldots Q\left(\frac{x_s-x_{s-1}}{1-x_s}\right)$$

$$\times R(x_1)\,R\left(\frac{x_2-x_1}{1+x_1}\right)\ldots R\left(\frac{x_s-x_{s-1}}{1+x_s}\right)$$

$$= \frac{-1}{\cos u_s\cosh v_s - i\sin u_s\sinh v_s} \quad \text{(by Art. 6)}.$$

Now suppose that x_s and x_{s+1} both lie on the ellipse for which $v = b$. Then

$$x_s = \sin u_s\cosh b + i\cos u_s\sinh b,$$

$$x_{s+1} = \sin u_{s+1}\cosh b + i\cos u_{s+1}\sinh b\,;$$

therefore

$$\frac{D\chi(x_{s+1},\,x_s)}{Db}$$

$$= \frac{\partial\chi(x_{s+1},\,x_s)}{\partial x_s}\,\frac{Dx_s}{Db} + \frac{\partial\chi(x_{s+1},\,x_s)}{\partial x_{s+1}}\,\frac{Dx_{s+1}}{Db}$$

$$= \frac{-1}{\cos u_s\cosh b - i\sin u_s\sinh b}(\sin u_s\sinh b + i\cos u_s\cosh b)$$

$$+ \frac{1}{\cos u_{s+1}\cosh b - i\sin u_{s+1}\sinh b}(\sin u_{s+1}\sinh b + i\cos u_{s+1}\cosh b)$$

$$= -i + i = 0.$$

In this case then we may find the value of $\chi(x_{s+1},\,x_s)$ by putting $b = 0$; then x_s becomes $\sin u_s$ and x_{s+1} becomes $\sin u_{s+1}$. Here u_s and u_{s+1} are the parameters* of the hyperbolas through the vertices x_s, x_{s+1} of the polygon.

Again, if x_s and x_{s+1} both lie on the hyperbola for which $u = a$, then

$$x_s = \sin a\cosh v_s + i\cos a\sinh v_s,$$

$$x_{s+1} = \sin a\cosh v_{s+1} + i\cos a\sinh v_{s+1},$$

* The values of the parameters depend on the path of the variable x.

$$\frac{D\chi(x_{s+1}, x_s)}{Da}$$

$$= \frac{\partial\chi(x_{s+1}, x_s)}{\partial x_s}\frac{Dx_s}{Da} + \frac{\partial\chi(x_{s+1}, x_s)}{\partial x_{s+1}}\frac{Dx_{s+1}}{Da}$$

$$= \frac{-1}{\cos a\cosh v_s - i\sin a\sinh v_s}(\cos a\cosh v_s - i\sin a\sinh v_s)$$

$$+\frac{1}{\cos a\cosh v_{s+1} - i\sin a\sinh v_{s+1}}(\cos a\cosh v_{s+1} - i\sin a\sinh v_{s+1})$$

$$= -1+1 = 0.$$

In this case we may find the value of $\chi(x_{s+1}, x_s)$ by putting $a = 0$; then x_s becomes $i\sinh v_s$, x_{s+1} becomes $i\sinh v_{s+1}$. Here v_s and v_{s+1} are the parameters* of the ellipses which pass through the vertices x_s and x_{s+1} of the polygon.

13. Now consider $\chi(x_{s+1}, x_s)$ in the case in which x_s and x_{s+1} both lie on the ellipse whose parameter $= b$. Its value is, by Art. 12,

$$Q(x_1)\,Q\left(\frac{x_2-x_1}{1-x_1}\right)\ldots Q\left(\frac{x_s-x_{s-1}}{1-x_{s-1}}\right)$$

$$\times R(x_1)\,R\left(\frac{x_2-x_1}{1+x_1}\right)\ldots R\left(\frac{x_s-x_{s-1}}{1+x_{s-1}}\right)$$

$$\times \sum_{n=0}^{n=\infty}\left[\frac{(x_{s+1}-x_s)^{n+1}}{(n+1)}\sum_{t=0}^{t=n}\frac{(-\frac{1}{2})_{n-t}}{(n-t)!}\frac{(-\frac{1}{2})_t}{t!}\frac{1}{(x_s-1)^{n-t}}\frac{1}{(x_s+1)^t}\right]$$

$$= \frac{1}{\cos u_s\cosh b - i\sin u_s\sinh b}$$

$$\times \sum_{n=0}^{n=\infty}\left[\frac{(x_{s+1}-x_s)^{n+1}}{(n+1)}\sum_{t=0}^{t=n}\frac{(-\frac{1}{2})_{n-t}}{(n-t)!}\frac{(-\frac{1}{2})_t}{t!}\frac{1}{(x_s-1)^{n-t}}\frac{1}{(x_s+1)^t}\right].$$

Now we have seen that the value of this does not depend on the value of b; therefore, putting $b = 0$, and consequently $x_s = \sin u_s$, $x_{s+1} = \sin u_{s+1}$, $\chi(x_{s+1}, x_s)$ becomes

$$\frac{1}{\cos u_s}\sum_{n=0}^{n=\infty}\left[\frac{(\sin u_{s+1}-\sin u_s)^{n+1}}{(n+1)}\sum_{t=0}^{t=n}\frac{(-\frac{1}{2})_{n-t}}{(n-t)!}\frac{(-\frac{1}{2})_t}{t!}\right.$$

$$\left.\times\frac{1}{(\sin u_s-1)^{n-t}}\frac{1}{(\sin u_s+1)^t}\right].$$

* The values of the parameters depend on the path of the variable x.

Now, by Taylor's theorem,

the acute angle whose sine is x_2 = the acute angle whose sine is x_1

$$+\frac{1}{\sqrt{1-x_1^2}}\sum_{n=0}^{n=\infty}\left[\frac{(x_2-x_1)^{n+1}}{(n+1)}\sum_{t=0}^{t=n}\frac{(-\frac{1}{2})_{n-t}}{(n-t)!}\frac{(-\frac{1}{2})_t}{t!}\frac{1}{(x_1-1)^{n-t}}\frac{1}{(x_1+1)^t}\right],$$

where the square root is to be taken with the positive sign.

Now, if u_s and u_{s+1} are both acute angles, we can put

$$x_1 = \sin u_s, \quad x_2 = \sin u_{s+1},$$

$$\sqrt{1-x_1^2} = \cos u_s.$$

Also the acute angles whose sines are x_1 and x_2 are u_s, u_{s+1} respectively.

Hence $$u_{s+1} = u_s + \chi(x_{s+1}, x_s);$$

therefore $$\chi(x_{s+1}, x_s) = u_{s+1} - u_s.$$

The next case that comes up for consideration in forming a circuit is that in which u_s is an acute angle, but u_{s+1} an obtuse angle. There is in this case a difficulty. As b approaches zero the ellipse shrinks up into the line joining the points -1 and $+1$. But it always has two sides, and we have to work out the continuation of the series from the point $\sin u_s$ on one side up to the point $+1$, and round on the other side to the point $\sin u_{s+1}$. It is therefore better to regard it as consisting of two parts, viz., of the part from $\sin u_s$ to $\sin\frac{\pi}{2}$, which belongs to the preceding case; and of the part from $\sin\frac{\pi}{2}$ to $\sin u_{s+1}$, which belongs to the following case.

Consider therefore the case where both u_s and u_{s+1} are obtuse.

We put here, in applying Taylor's theorem,

$$x_1 = \sin(\pi - u_s), \quad x_2 = \sin(\pi - u_{s-1}),$$

$$\sqrt{1-x_1^2} = \cos(\pi - u_s).$$

In this case the acute angles whose sines are x_1 and x_2 are $\pi - u_s$ and $\pi - u_{s+1}$ respectively; therefore

$$\pi - u_{s+1} = \pi - u_s + \frac{1}{\cos(\pi - u_s)}\sum_{n=0}^{n=\infty}\left[\frac{(\sin u_{s+1} - \sin u_s)^{n+1}}{n+1}\right.$$

$$\left.\times\sum_{t=0}^{t=n}\frac{(-\frac{1}{2})_{n-t}}{(n-t)!}\frac{(-\frac{1}{2})_t}{t}\frac{1}{(\sin u_s - 1)^{n-t}}\frac{1}{(\sin u_s + 1)^t}\right];$$

therefore $$\pi - u_{s+1} = \pi - u_s - \chi(x_{s+1}, x_s);$$

therefore $$\chi(x_{s+1}, x_s) = u_{s+1} - u_s,$$

as before.

We are now in a position to take the case where u_s is an acute angle and u_{s+1} is obtuse. Let us call the point $+1$, $x_{s'}$. We take first u from u_s to $\frac{\pi}{2}$, which gives by the first case

$$\chi(x_{s'}, x_s) = \frac{\pi}{2} - u_s.$$

Then we take u from $\frac{\pi}{2}$ to u_{s+1}, which gives by the last case

$$\chi(x_{s+1}, x_{s'}) = u_{s+1} - \frac{\pi}{2},$$

Hence $$\chi(x_{s'}, x_s) + \chi(x_{s+1}, x_{s'}) = u_{s+1} - u_s.$$

Hence in all cases we have

$$\chi(x_{s+1}, x_s) = u_{s+1} - u_s.$$

14. We have now to proceed to the case where x_{s+1} and x_s both lie on the hyperbola for which the parameter is equal to a. Here

$$\chi(x_{s+1}, x_s) = \frac{1}{\cos a \cosh v_s - i \sin a \sinh v_s} \times \sum_{n=0}^{n=\infty} \left[\frac{(x_{s+1} - x_s)^{n+1}}{(n+1)} \sum_{t=0}^{t=n} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \frac{1}{(x_s - 1)^{n-t}} \frac{1}{(x_s + 1)^t} \right].$$

The value of this is independent of the value of a. So we can put $a = 0$. Then x_s, x_{s+1} become $i \sinh v_s$, $i \sinh v_{s+1}$ respectively. And $\chi(x_{s+1}, x_s)$ becomes

$$\frac{1}{\cosh v_s} \left\{ \sum_{n=0}^{n=\infty} \frac{(i \sinh v_{s+1} - i \sinh v_s)^{n+1}}{n+1} \times \left[\sum_{t=0}^{t=n} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \frac{1}{(i \sinh v_s - 1)^{n-t}} \frac{1}{(i \sinh v_s + 1)^t} \right] \right\}$$

$$= \frac{i}{\cosh v_s} \left\{ \sum_{n=0}^{n=\infty} \frac{(\sinh v_{s+1} - \sinh v_s)^{n+1}}{n+1} \times \left[\sum_{t=0}^{t=n} \frac{(-\frac{1}{2})_{n-t}}{(n-t)!} \frac{(-\frac{1}{2})_t}{t!} \frac{1}{(\sinh v_s + i)^{n-t}} \frac{1}{(\sinh v_s - i)^t} \right] \right\}.$$

To evaluate this we consider the real logarithm of $z+\sqrt{1+z^2}$ where z is real. Call this real logarithm

$$\log_e(z+\sqrt{1+z^2}).$$

Then $$\frac{d\log_e(z+\sqrt{1+z^2})}{dz} = \frac{1}{\sqrt{1+z^2}} = (z+i)^{-\frac{1}{2}}(z-i)^{-\frac{1}{2}};$$

therefore

$$\frac{d^{n+1}\log_e(z+\sqrt{1+z^2})}{dz^{n+1}} = \sum_{t=0}^{t=n} {}_nC_t\left(-\tfrac{1}{2}\right)_{n-t}(z+i)^{-n+t-\frac{1}{2}}\left(-\tfrac{1}{2}\right)_t(z-i)^{-t-\frac{1}{2}};$$

therefore

$$\frac{1}{(n+1)!}\frac{d^{n+1}\log_e(z+\sqrt{1+z^2})}{dz^{n+1}}$$

$$= \frac{1}{n+1}\frac{1}{\sqrt{1+z^2}}\sum_{t=0}^{t=n}\frac{\left(-\frac{1}{2}\right)_{n-t}}{(n-t)!}\frac{\left(-\frac{1}{2}\right)_t}{t!}\frac{1}{(z+i)^{n-t}}\frac{1}{(z-i)^t},$$

where the square root is to be taken with the positive sign.

Put now $$z = \sinh v_s;$$

therefore

$$\left[\frac{1}{(n+1)!}\frac{d^{n+1}\log_e(z+\sqrt{1+z^2})}{dz^{n+1}}\right]_{z=\sinh v_s}$$

$$= \frac{1}{n+1}\frac{1}{\cosh v_s}\sum_{t=0}^{t=n}\frac{\left(-\frac{1}{2}\right)_{n-t}}{(n-t)!}\frac{\left(-\frac{1}{2}\right)_t}{t!}\frac{1}{(\sinh v_s+i)^{n-t}}\frac{1}{(\sinh v_s-i)^t}.$$

Hence the value of $\chi(x_{s+1}, x_s)$ to be obtained is

$$= i\sum_{n=0}^{\infty}\frac{(\sinh v_{s+1}-\sinh v_s)^{n+1}}{(n+1)!}\left[\frac{d^{n+1}\log_e(z+\sqrt{1+z^2})}{dz^{n+1}}\right]_{z=\sinh v_s}$$

$$= i\left[\log_e(z+\sqrt{1+z^2})_{z=\sinh v_{s+1}} - \log_e(z+\sqrt{1+z^2})_{z=\sinh v_s}\right]$$

$$= i(v_{s+1}-v_s).$$

The series is convergent if $|\sinh v_{s+1}-\sinh v_s|$ is less than the smaller of the two moduli $|\sinh v_s - i|$ and $|\sinh v_s + i|$.

15. Hence, by Art. 13, if the vertices x_s, x_{s+1} be taken along an arc of one of the ellipses $v =$ constant,

$$\chi(x_{s+1}, x_s) = \text{value of } u \text{ at } x_{s+1} - \text{value of } u \text{ at } x_s$$
$$= \{\text{value of } (u+iv) \text{ at } x_{s+1} - \text{value of } (u+iv) \text{ at } x_s\},$$

since the values of v at x_{s+1} and x_s are the same.

If, however, x_s, x_{s+1} be taken along an arc of one of the hyperbolas $u =$ constant, then, by Art. 14,

$$\chi(x_{s+1}, x_s) = i(\text{value of } v \text{ at } x_{s+1} - \text{value of } v \text{ at } x_s)$$
$$= \text{value of } (u+iv) \text{ at } x_{s+1} - \text{value of } (u+iv) \text{ at } x_s,$$

since the values of u at x_{s+1}, x_s are the same. Putting the two results together, when x_s and x_{s+1} are not situated as above, we may, by interposing one or more intermediary points, prove that

$$\chi(x_{s+1}, x_s) = \text{value of } (u+iv) \text{ at } x_{s+1} - \text{value of } (u+iv) \text{ at } x_s,$$

where the values of u, v will depend on the path of the variable x round the circuit.

16. Let us now examine

$$\chi(x, x_r) = (x-x_r)\left[\frac{df(x \mid x_1 \mid \ldots \mid x_{r-1})}{dx}\right]_{x=x_r}$$
$$+ \frac{(x-x_r)^2}{2!}\left[\frac{d^2f(x \mid x_1 \mid \ldots \mid x_r)}{dx^2}\right]_{x=x_r}$$
$$+ \ldots .$$

Suppose that the circuit has been taken round $x = 1$, but not round $x = -1$; and that x_r coincides with x_1.

Then $\chi(x, x_r)$, where x_r coincides with x_1, is, by Art. 12,

$$Q(x_1)\, Q\left(\frac{x_2-x_1}{1-x_1}\right) \ldots Q\left(\frac{x_1-x_{r-1}}{1-x_{r-1}}\right)$$
$$\times R(x_1)\, R\left(\frac{x_2-x_1}{1-x_1}\right) \ldots R\left(\frac{x_1-x_{r-1}}{1-x_{r-1}}\right)$$
$$\times \sum_{n=0}^{\infty}\left\{\frac{(x-x_1)^{n+1}}{n+1} \sum_{s=0}^{s=n} \frac{(-\frac{1}{2})_{n-s}}{(n-s)!} \frac{(-\frac{1}{2})_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s}\right\}$$

$$= -Q(x_1)\, R(x_1) \sum_{n=0}^{\infty}\left\{\frac{(x-x_1)^{n+1}}{n+1} \sum_{s=0}^{s=n} \frac{(-\frac{1}{2})_{n-s}}{(n-s)!} \frac{(-\frac{1}{2})_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s}\right\},$$

by Art. 5.

17. *Continuation of the Power Series for* $\arcsin x$ *round a Circuit enclosing* $x = +1$, *but not* $x = -1$.

$$f(x\,|\,x_1\,|\,\dots\,|\,x_{r-1}\,|\,x_1)$$

$$= f(x_1) + \chi(x_2, x_1) + \chi(x_3, x_2) + \dots + \chi(x_{r-1}, x_r) + \chi(x, x_r)$$

[where $x_r = x_1$]

$$= (u_1 + iv_1) + (u_2 + iv_2 - u_1 - iv_1) + \dots + (u_r + iv_r - u_{r-1} - iv_{r-1})$$

$$- Q(x_1)\,R(x_1) \sum_{n=0}^{\infty} \left\{ \frac{(x-x_1)^{n+1}}{n+1} \sum_{s=0}^{s=n} \frac{(-\frac{1}{2})_{n-s}}{(n-s)!} \frac{(-\frac{1}{2})_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s} \right\}.$$

But $$u_r = \pi - u_1 \quad \text{and} \quad v_r = -v_1;$$

therefore

$$f(x\,|\,x_1\,|\,\dots\,|\,x_{r-1}\,|\,x_1)$$

$$= \pi - u_1 - iv_1$$

$$- Q(x_1)\,R(x_1) \sum_{n=0}^{\infty} \frac{(x-x_1)^{n+1}}{n+1} \sum_{s=0}^{s=n} \frac{(-\frac{1}{2})_{n-s}}{(n-s)!} \frac{(-\frac{1}{2})_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s}$$

$$= \pi - f(x_1)$$

$$- Q(x_1)\,R(x_1) \sum_{n=0}^{\infty} \frac{(x-x_1)^{n+1}}{n+1} \sum_{s=0}^{s=n} \frac{(-\frac{1}{2})_{n-s}}{(n-s)!} \frac{(-\frac{1}{2})_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s}.$$

But $f(x\,|\,x_1) = f(x_1)$

$$+ Q(x_1)\,R(x_1) \sum_{n=0}^{\infty} \frac{(x-x_1)^{n+1}}{n+1} \sum_{s=0}^{s=n} \frac{(-\frac{1}{2})_{n-s}}{(n-s)!} \frac{(-\frac{1}{2})_s}{s!} \frac{1}{(x_1-1)^{n-s}} \frac{1}{(x_1+1)^s};$$

therefore $$f(x\,|\,x_1\,|\,\dots\,|\,x_{r-1}\,|\,x_1) = \pi - f(x\,|\,x_1)$$

for a circuit in the positive direction round $x = +1$, which does not include $x = -1$.

18. In like manner for a circuit in the positive direction round $x = -1$, which does not include $x = +1$, the result would be $-\pi - f(x\,|\,x_1)$.

For a circuit in the positive direction round both $x = +1$ and $x = -1$ the result would be $-2\pi + f(x\,|\,x_1)$, provided that the point x_1 is situated in such a manner that the circuit is equivalent to a circuit round $x = -1$, followed by a circuit round $x = 1$, both these circuits being described in the positive direction.

Other circuits may be treated in a similar way by resolving them into circuits round $x = +1$ and $x = -1$.

On the Functions associated with the Parabolic Cylinder in Harmonic Analysis. *By* E. T. Whittaker. Received and Read December 11th, 1902.

1. *Introduction.*

The following paper is a study of the functions defined by the differential equation

$$\frac{d^2y}{dz^2} + Zy = 0,$$

where Z is a quadratic function of z. To these functions the name *functions of the parabolic cylinder* may be given, as it was shown by Weber (*Math Ann.*, Vol. I.) that the harmonic functions applicable to the parabolic cylinder satisfy this equation; some power series which represent these harmonic functions and satisfy the differential equation have been found by Baer (*Diss., Cästrin*) and Haentzschel (*Zeitschrift für Math.*, Vol. XXXIII.)

The differential equation, however, has an importance quite apart from its occurrence in harmonic analysis; for, owing to its simplicity, it naturally is presented to us as the first object of study in the general scheme of defining new transcendental functions by means of differential equations; and it will be evident from the results found in the present paper that many definite integrals, which have been suggested at various times by different writers as suitable definitions for new transcendental functions, are really special cases of definite integrals which satisfy the differential equation of the parabolic cylinder.

2. *Description of Paper.*

In the present paper the differential equation associated with the parabolic cylinder is first (§ 3) solved by means of a family of definite integrals. It is then (§ 4) shown that one of the integrals of this family possesses properties which fit it to be taken as a standard solution, and this integral is denoted by $D_n(z)$. It is shown that $D_n(z)$ possesses an asymptotic expansion which is well suited for the purpose of calculating its numerical values, and that in certain cases it degenerates into an elementary function. It is then (§ 5) shown that the general solution of the equation does not introduce any

function other than $D_n(z)$, and a second family of definite integrals is found which satisfies the differential equation.

It is then shown (§ 6) that the function $D_n(z)$ possesses two sets of recurrence formulæ, and that in the case in which n is an integer (§ 7) this function can be expressed in a simple form, and (§ 8) that it possesses integral properties and can be used to form expansions of arbitrary functions. A third family of definite integral solutions of the differential equation is found in § 9, and a fourth family in § 10, and the members of these families which represent $D_n(z)$ are determined.

3. *Determination of a Family of Definite Integrals which satisfy the Differential Equation.*

As has already been said, the differential equation found by Weber for the functions of the parabolic cylinder is

$$\frac{d^2y}{dz^2} + Zy = 0,$$

where Z is a quadratic function of z. There is clearly no real loss of generality if we write this in the form

$$\frac{d^2y}{dz^2} + (n+\tfrac{1}{2}-\tfrac{1}{4}z^2)\,y = 0,$$

where n is a constant. We shall accordingly adopt this as our standard form of the equation.* It is evident that two independent solutions of this equation exist of the forms

$$1 - \frac{2n+1}{4}z^2 + \frac{4n^2+4n+3}{96}z^4 + \ldots$$

and

$$z - \frac{2n+1}{12}z^3 + \frac{4n^2+4n+7}{480}z^5 + \ldots$$

respectively. These series are convergent over the entire z-plane, and may be taken as the two fundamental solutions. As, however, series-solutions of this kind have been discussed by the two previous writers, and, as we shall show that it is advantageous to choose the two fundamental solutions otherwise, these series need not further be mentioned.

In order to obtain a pair of fundamental solutions suitable to be taken as standards, we shall first show that a family of definite

* This differs from the standard forms previously adopted; the reason for it will appear subsequently.

integrals exists, by means of which the solutions can be represented; for in the differential equation

$$\frac{d^2y}{dz^2} + (n+\tfrac{1}{2}-\tfrac{1}{4}z^2)\,y = 0$$

let a new independent variable be introduced by the relation

$$y = e^{-\frac{1}{4}z^2} z^n u.$$

The equation becomes

$$z^2\frac{d^2u}{dz^2} + (2nz-z^3)\frac{du}{dz} + n(n-1)\,u = 0.$$

Now, if v denote the integral

$$\int_\gamma e^{-t-\frac{1}{2}(t^2/z^2)}\,t^{-n-1}\,dt,$$

where the path of integration γ is for the present unspecified, we find that

$$z^2\frac{d^2v}{dz^2} + (2nz-z^3)\frac{dv}{dz} + n(n-1)\,v$$
$$= -\frac{1}{z^2}\int_\gamma \frac{d}{dt}\left[\left\{t^2-tz^2+(n-1)\,z^2\right\}t^{-n}e^{-t-\frac{1}{2}(t^2/z^2)}\right]dt.$$

It follows from this that *any integral of the type*

$$e^{-\frac{1}{4}z^2}z^n\int_\gamma e^{-t-\frac{1}{2}(t^2/z^2)}\,t^{-n-1}\,dt$$

is a solution of the differential equation of the parabolic cylinder, provided the path of integration γ is such that the quantity

$$\left\{t^2-tz^2-(n-1)\,z^2\right\}t^{-n}e^{-t-\frac{1}{2}(t^2/z^2)}$$

resumes its initial value after describing the path.

This result furnishes a large number of definite integrals which satisfy the equation; thus, the quantity

$$\left\{t^2-tz^2-(n-1)\,z^2\right\}t^{-n}e^{-t-\frac{1}{2}(t^2/z^2)}$$

is zero at the points

$$t = \tfrac{1}{2}z^2+\tfrac{1}{2}z\left\{z^2-4(n-1)\right\}^{\frac{1}{2}},$$
$$t = \tfrac{1}{2}z^2-\tfrac{1}{2}z\left\{z^2-4(n-1)\right\}^{\frac{1}{2}},$$
$$t = 0 \quad \text{(when } n \text{ is negative)},$$
$$t = \infty \quad \left(\text{provided the direction at infinity is such that } \frac{t}{z} \text{ is real}\right);$$

and therefore, if γ be a path joining any of these points, the expression

$$e^{-\frac{1}{4}z^2}z^n\int_\gamma e^{-t-\frac{1}{2}(t^2/z^2)}t^{-n-1}dt$$

will be a solution of the differential equation.

4. *Asymptotic Expansion of one of the Integrals found in* § 3. *Determination of a Standard Solution.*

Among these solutions there is one which (as will appear later) is especially suited to be taken as a standard solution. This is the integral taken along a path which encircles the origin in the t-plane and begins and ends at infinity in that direction which makes $\frac{t}{z}$ a positive real quantity. This path will be denoted by δ. We shall now show that this solution

$$e^{-\frac{1}{4}z^2}z^n\int_\delta e^{-t-\frac{1}{2}(t^2/z^2)}t^{-n-1}dt$$

possesses an asymptotic expansion when z is real and positive.

For let z be any real positive quantity greater than some fixed quantity k^2. Then the path of integration can be divided into two parts, namely, (1) a part δ_1 consisting of the portion of δ which is contained within a circle of radius k and centre the origin, and (2) a part δ_2 consisting of the two portions of δ which are outside this circle. Thus

$$\begin{aligned}
&e^{-\frac{1}{4}z^2}z^n\int_\delta e^{-t-\frac{1}{2}(t^2/z^2)}t^{-n-1}dt\\
&=\ e^{-\frac{1}{4}z^2}z^n\int_\delta e^{-t}t^{-n-1}\left\{1-\frac{t^2}{2z^2}+\frac{1}{2!}\left(\frac{t^2}{2z^2}\right)^2-\ldots+\frac{1}{(r-1)!}\left(\frac{-t^2}{2z^2}\right)^{r-1}\right\}dt\\
&\quad+e^{-\frac{1}{4}z^2}z^n\int_{\delta_1}e^{-t}\left\{e^{-\frac{1}{2}(t^2/z^2)}-1+\frac{t^2}{2z^2}-\ldots-\frac{1}{(r-1)!}\left(\frac{-t^2}{2z^2}\right)^{r-1}\right\}t^{-n-1}dt\\
&\quad+e^{-\frac{1}{4}z^2}z^n\int_{\delta_2}e^{-t}\left\{e^{-\frac{1}{2}(t^2/z^2)}-1+\frac{t^2}{2z^2}-\ldots-\frac{1}{(r-1)!}\left(\frac{-t^2}{2z^2}\right)^{r-1}\right\}t^{-n-1}dt.
\end{aligned}$$

The first of these integrals can be evaluated by help of the relation

$$\frac{1}{\Gamma(k+1)}=\frac{i}{2\pi}\int_\delta e^{-t}(-t)^{-k-1}dt,$$

and thus furnishes the terminating series

$$\frac{2\pi(-1)^{n+1}}{i\Gamma(n+1)}e^{-\frac{1}{4}z^2}z^n\left\{1-\frac{n(n-1)}{2z^2}+\ldots\right.$$
$$\left.\ldots+\frac{(-1)^{r-1}n(n-1)\ldots(n-r+1)}{(r-1)!}\frac{1}{(2z^2)^{r-1}}\right\}.$$

Considering next the second integral, we observe that at all points of its path of integration δ_1 the quantity

$$e^{-\frac{1}{2}(t^2/z^2)}-1+\frac{t^2}{2z^2}-\ldots-\frac{1}{(r-1)!}\left(\frac{-t^2}{2z^2}\right)^{r-1}$$

can be expanded as an absolutely convergent series in the form

$$\frac{1}{r!}\left(\frac{-t^2}{2z^2}\right)^r+\frac{1}{(r+1)!}\left(\frac{-t^2}{2z^2}\right)^{r+1}+\ldots,$$

from which it follows that, when z is large (r being supposed predetermined independently of z), the second integral can be written

$$e^{-\frac{1}{4}z^2}\cdot z^n\cdot\frac{A}{z^{2r}},$$

where A remains finite when z is indefinitely increased.

Considering, lastly, the third integral, we observe that $|t|$ is greater than k at all points in the range of integration, and therefore (r being supposed predetermined independently of z) the occurrence of the factor e^{-t} in the integrand shows that the integral will at most be comparable with e^{-k} when k is large, *i.e.*, it will for large values of z be small compared with the second integral.

Summing up, we see that, if r be a predetermined integer, then the solution

$$e^{-\frac{1}{4}z^2}\int_\delta e^{-t-\frac{1}{2}(t^2/z^2)}t^{-n-1}\,dt$$

can for large positive values of z be expressed in the form

$$\frac{2\pi(-1)^{n+1}}{i\Gamma(n+1)}e^{-\frac{1}{4}z^2}z^n\left\{1-\frac{n(n-1)}{2z^2}+\ldots\right.$$
$$\left.\ldots+\frac{(-1)^{r-1}n(n-1)\ldots(n-r+1)}{(r-1)!}\frac{1}{(2z^2)^{r-1}}+\frac{A}{z^{2r}}\right\},$$

where A remains finite when z increases indefinitely; that is to say, *the divergent series*

$$\frac{2\pi(-1)^{n+1}}{i\Gamma(n+1)}e^{-\frac{1}{4}z^2}z^n\left\{1-\frac{n(n-1)}{2z^2}+\ldots\right.$$
$$\left.\ldots+\frac{(-1)^r n(n-1)\ldots(n-r)}{2^r r!}\frac{1}{z^{2r}}+\ldots\right\}$$

is an asymptotic expansion of this solution for large real positive values of z.

We can now define our standard solution in the following way :—

The integral $$\frac{i\Gamma(n+1)}{2\pi} e^{-\frac{1}{4}z^2} z^n \int_\delta e^{-t-\frac{1}{2}(t^2/z^2)} (-t)^{-n-1} dt$$

will be denoted by $D_n(z)$,

and will be taken as a standard solution of the differential equation of the parabolic cylinder.

The function $D_n(z)$ *is a one-valued analytic function, regular for all finite values of* z *; and, for large real positive values of* z, *it possesses the asymptotic expansion*

$$D_n(z) = e^{-\frac{1}{4}z^2} z^n \left\{ 1 - \frac{n(n-1)}{2z^2} + \frac{n(n-1)(n-2)(n-3)}{2.4.z^4} - \dots \right\}.$$

It is obvious that when n is a positive integer this last series terminates; in this case the function $D_n(z)$ therefore degenerates into the product of an exponential function and a polynomial. This case will be deferred for the present; but it may be observed that this circumstance is a justification for the choice we have made of a standard solution.

When the real part of n is negative, by deforming the path of integration, we find that *the function* $D_n(z)$ *can be expressed by the real integral*

$$D_n(z) = \frac{1}{\Gamma(-n)} e^{-\frac{1}{4}z^2} z^n \int_0^\infty e^{-t-\frac{1}{2}(t^2/z^2)} t^{-n-1} dt.$$

5. *Expression of the Second Solution of the Equation in terms of* $D_n(z)$. *A Second Family of Definite Integrals which satisfy the Equation.*

Since the differential equation

$$\frac{d^2y}{dz^2} + (n+\tfrac{1}{2}-\tfrac{1}{4}z^2)y = 0$$

is unaffected if we simultaneously replace n by $-n-1$ and z by $-iz$, and since from the asymptotic expansion it is evident that $D_n(z)$ and $D_{-n-1}(iz)$ cannot be mere multiples of each other, we see that *the general solution of the differential equation of the parabolic cylinder is expressible in terms of the D-function in the form*

$$aD_n(z) + bD_{-n-1}(iz),$$

where a and b are arbitrary constants.

This transformation shows that *integrals of the type*

$$e^{\frac{1}{4}z^2} z^{-n-1} \int_\gamma e^{-t+\frac{1}{2}(t^2/z^2)} t^n \, dt,$$

where γ is any path such that the quantity

$$\{t^2 + tz^2 - (n+2)\, z^2\}\, t^{n+1} e^{-t+\frac{1}{2}(t^2/z^2)}$$

resumes its initial value after t has described γ, constitute a second family of definite integrals satisfying the equation.

6. *Recurrence Formulæ for the Function $D_n(z)$.*

For all real or complex values of n and of z the function $D_n(z)$ satisfies certain recurrence formulæ. For, on integrating by parts the right-hand member of the equation

$$-\frac{2\pi i}{\Gamma(n+1)} e^{\frac{1}{4}z^2} z^n D_n(z) = \int_\delta e^{-t-\frac{1}{2}(t^2/z^2)} (-t)^{-n-1} \, dt,$$

we obtain the result

$$D_n(z) - z D_{n-1}(z) + (n-1)\, D_{n-2}(z) = 0,$$

and in the same way we can obtain a second recurrence formula, namely,

$$\frac{dD_n(z)}{dz} + \tfrac{1}{2} z D_n(z) - n D_{n-1}(z) = 0.$$

7. *The Functions $D_n(z)$ in the case in which n is an Integer.*

We have already seen (§ 4) that when n is an integer the function $D_n(z)$ ceases to be a new transcendental function and becomes merely the product of an exponential function and a polynomial. Its explicit expression may be found as follows :—We have

$$D_n(z) = \frac{i\,\Gamma(n+1)}{2\pi} e^{-\frac{1}{4}z^2} z^n \int_\delta e^{-t-\frac{1}{2}(t^2/z^2)} (-t)^{-n-1} \, dt.$$

If we take a new variable v defined by the relation $t = z(v-z)$, the resulting integral can, when n is a positive integer, be at once evaluated by Cauchy's formula for the n-th derivate of a function. *We thus obtain the formula*

$$D_n(z) = (-1)^n e^{\frac{1}{4}z^2} \frac{d^n}{dz^n} (e^{-\frac{1}{2}z^2})$$

which represents $D_n(z)$ when n is a positive integer.

Applying the transformation which changes z to iz and n to $-n-1$, we see that, when n is a negative integer, the expression

$$e^{-\frac{1}{2}z^2}\frac{d^{-n-1}}{dz^{-n-1}}(e^{\frac{1}{2}z^2})$$

is a solution of the differential equation. This gives a second set of solutions expressible as the product of an exponential and a polynomial.

8. *Integral Properties of the Functions $D_n(z)$ when n is an Integer. Expansion of an Arbitrary Function.*

We shall now show that the functions $D_n(z)$ possess properties analogous to those expressed in the theory of Legendre functions by the well known equations

$$\int_{-1}^{1} P_m(z)\,P_n(z) = 0, \quad \int_{-1}^{1} P_n^2(z)\,dz = \frac{2}{2n+1}.$$

For, from the differential equation, we have

$$D_n(z)\frac{d^2D_n(z)}{dz^2} - D_m(z)\frac{d^2D_n(z)}{dz^2} + (m-n)\,D_m(z)\,D_n(z) = 0,$$

and, integrating this relation between $z=-\infty$ and $z=\infty$, *we have the integral relation*

$$\int_{-\infty}^{\infty} D_m(z)\,D_n(z)\,dz = 0,$$

for all unequal integer values of m and n.

Moreover, we have by the recurrence-formulæ

$$\int_{-\infty}^{\infty} D_n^2(z)\,dz = \int_{-\infty}^{\infty} D_n(z)\left\{-2\frac{dD_n(z)}{dz} + (n-1)\,D_{n-1}(z)\right\}dz.$$

Transforming the right-hand member by use of the theorem just established, and by integration by parts, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} D_n^2(z)\,dz &= n\int_{-\infty}^{\infty} D_{n-1}^2(z)\,dz \\ &= n!\int_{-\infty}^{\infty}\{D_0(z)\}^2\,dz \\ &= n!\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}dz,\end{aligned}$$

whence, *for all integer values of n, we have the integral relation*

$$\int_{-\infty}^{\infty} \{D_n(z)\}^2 dz = (2\pi)^{\frac{1}{2}} n!\,.$$

From these integral relations it is evident that, *if any function* $f(z)$ *be expanded in a series of the form*

$$f(z) = a_0 D_0(z) + a_1 D_1(z) + a_2 D_2(z) + \ldots,$$

then the coefficients are given by the relations

$$a_n = \frac{1}{(2\pi)^{\frac{1}{2}} n!} \int_{-\infty}^{\infty} D_n(t) f(t)\, dt.$$

9. *A Third Family of Definite Integrals which satisfy the Differential Equation: Expression of* $D_n(z)$ *as an Integral of this Type.*

We shall now find another family of definite integrals which satisfy the differential equation.

For let $$y = \int_\gamma e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}n-1} (1-t)^{\frac{1}{2}(n-1)}\, dt,$$

where the path of integration is for the present left undetermined. Then

$$\frac{d^2y}{dz^2} + (n+\tfrac{1}{2}-\tfrac{1}{4}z^2)\, y$$

$$= \int_\gamma e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}n-1} (1-t)^{\frac{1}{2}(n-1)} \left\{\tfrac{1}{2}t + \tfrac{1}{4}z^2 t^2 + n + \tfrac{1}{2} - \tfrac{1}{4}z^2\right\} dt$$

$$= -\int_\gamma \frac{d}{dt} \left\{e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}n} (1-t)^{\frac{1}{2}(n+1)}\right\} dt.$$

It follows that *the differential equation of the parabolic cylinder is satisfied by any integral of the type*

$$\int_\gamma e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}n-1} (1-t)^{\frac{1}{2}(n-1)} dt,$$

provided the path of integration γ *is such that the quantity*

$$e^{\frac{1}{4}z^2 t} (1+t)^{-\frac{1}{2}n} (1-t)^{\frac{1}{2}(n-1)}$$

resumes its initial value after t has described γ.

A large number of paths γ can be found to satisfy this condition; among these we shall select the one which gives our standard solution $D_n(z)$.

Let ϵ denote a path of integration which encircles the point $t = -1$,

and begins and ends at infinity in that direction which makes z^2t negative. Then, if we form the integral corresponding to this path, we easily find that it possesses an asymptotic expansion of the form $z^n e^{-\frac{1}{4}z^2} \times$ a power series in $\frac{1}{z}$; and, on evaluating the first coefficient in this series and comparing with the asymptotic expansion of $D_n(z)$, *we have the formula*

$$D_n(z) = -\frac{i\Gamma\left(\frac{n}{2}+1\right)}{2^{-\frac{1}{2}(n-1)}\pi}\int_\epsilon e^{\frac{1}{4}z^2t}(1+t)^{-\frac{1}{2}n-1}(1-t)^{\frac{1}{2}(n-1)}dt,$$

which expresses the function $D_n(z)$ as a member of this family of integrals.

When the real part of n is negative, this gives, on deformation of the path, the result

$$D_n(z) = \frac{1}{\pi}\sin\frac{n\pi}{2}\Gamma\left(\frac{n}{2}+1\right)e^{-\frac{1}{4}z^2}z^n\int_0^\infty e^{-s}s^{-\frac{1}{2}n-1}\left(1+\frac{2s}{z^2}\right)^{\frac{1}{2}(n-1)}ds.$$

10. *A Fourth Family of Definite Integrals which satisfy the Differential Equation. Expression of $D_n(z)$ as an Integral of this type.*

By a similar process to that employed in § 3 and § 9, it can be shown that *the differential equation of the parabolic cylinder is satisfied by any integral of the type*

$$z\int_\gamma e^{-\frac{1}{4}z^2t}(t-1)^{-\frac{1}{2}n-\frac{1}{2}}(t+1)^{\frac{1}{2}n}dt,$$

provided the path of integration γ is such that the quantity

$$e^{-\frac{1}{4}z^2t}(t-1)^{-\frac{1}{2}n+\frac{1}{2}}(t+1)^{\frac{1}{2}n+1}$$

resumes its initial value after t has described γ.

A large number of paths γ can be found to satisfy this condition, and these furnish integrals of the equation. Among these we shall now find one which represents the standard function $D_n(z)$.

Let η denote a path of integration which begins and ends at infinity in the direction which makes z^2t positive and which encircles the point $t = 1$.

Let y_1 denote the integral taken along this path, so that

$$y_1 = z\int_\eta e^{-\frac{1}{4}z^2t}(t-1)^{-\frac{1}{2}n-\frac{1}{2}}(t+1)^{\frac{1}{2}n}dt.$$

Then, if s be a new variable defined by the relation

$$t = \frac{4s}{z^2} + 1,$$

we have $$y_1 = 2^{-\frac{1}{2}n+1} e^{-\frac{1}{4}z^2} z^n \int e^{-s} s^{-\frac{1}{2}n-\frac{1}{2}} \left(1+\frac{2s}{z^2}\right)^{\frac{1}{2}n} ds,$$

where the integration is now taken round a contour which begins and ends at infinity, and encloses the point $s = 0$.

This quantity clearly possesses an asymptotic expansion of the form

$$e^{-\frac{1}{4}z^2} z^n \times \text{a power series in } \frac{1}{z},$$

and is therefore a multiple of $D_n(z)$. To find the constant multiplier, we observe that the first term in the expansion is

$$2^{-\frac{1}{2}n+1} e^{-\frac{1}{4}z^2} z^n \int e^{-s} s^{-\frac{1}{2}n-\frac{1}{2}} ds$$

or

$$\frac{2^{-\frac{1}{2}n+2} \pi i^n}{\Gamma\left(\frac{n+1}{2}\right)} e^{-\frac{1}{4}z^2} z^n.$$

It follows that $$y_1 = \frac{2^{-\frac{1}{2}n+2} \pi i^n}{\Gamma\left(\frac{n+1}{2}\right)} D_n(z),$$

and therefore that $D_n(z)$ *is expressed as a member of this family of integrals by the formula*

$$D_n(z) = \frac{2^{\frac{1}{2}n-2} \Gamma\left(\frac{n+1}{2}\right)}{\pi i^n} z \int_\eta e^{-\frac{1}{4}z^2 t} (t-1)^{-\frac{1}{2}n-\frac{1}{2}} (t+1)^{\frac{1}{2}n} dt.$$

When the real part of n is less than unity, it is easily seen by deformation of the path that *this can be replaced by the real integral*

$$D_n(z) = \frac{1}{\Gamma\left(\frac{1-n}{2}\right)} e^{-\frac{1}{4}z^2} z^n \int_0^\infty e^{-s} s^{-\frac{1}{2}n-\frac{1}{2}} \left(1+\frac{2s}{z^2}\right)^{\frac{1}{2}n} ds.$$

Some Applications of Fourier's Theorem. *By* H. M. Macdonald.
Received and Read 11th December, 1902.

Fourier's theorem may be written in the form

$$f(x) = \frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} \epsilon^{xy}\, dy \int_b^a c^{-zy} f(z)\, dz,$$

where a, b are two real quantities, the path of integration with respect to z is the part of the real axis in the z plane lying between $z = b$ and $z = a$, c is a real positive quantity, and the path with respect to y lies in the first and fourth quadrants of the y plane, being terminated as indicated. From this it follows that, if

$$\int_b^a e^{-zy} f(z)\, dz = \phi(y),$$

then

$$\frac{1}{2\pi\iota}\int e^{xy}\phi(y)\, dy = f(x), \qquad (A)$$

where the path of integration is any curve into which the path $(c-\infty\iota,\ c+\infty\iota)$ can be deformed without passing over a singularity of $\phi(y)$.

This suggests that there may be other paths of integration such that

$$f(x) = A\int e^{xy}\int e^{yz} f(z)\, dy\, dz,$$

and then corresponding relations between integrals would follow. For example, Cauchy's relation

$$f(x) = \frac{1}{2\pi\iota}\int \frac{f(z)\, dz}{z-x}$$

may be written

$$f(x) = \frac{1}{2\pi\iota}\int_0^\infty e^{-xy}\, dy \int_{\infty e^{\beta\iota}}^{\infty e^{\alpha\iota}} e^{yz} f(z)\, dz,$$

where α and β are such that $\cos\alpha$ and $\cos\beta$ are negative. If the value of the integral

$$\frac{1}{2\pi\iota}\int_{\infty e^{\beta\iota}}^{\infty e^{\alpha\iota}} e^{yz} f(z)\, dz = \phi(y)$$

is known, the value of the integral

$$\int_0^\infty e^{-xy}\phi(y)\, dy \qquad (B)$$

is $f(x)$.

In what follows these two relations will be applied to obtain the values of certain integrals.

It is known that the integral

$$\int_0^\infty e^{-yz} J_n(z)\, z^{-m}\, dz$$

can be expressed as a hypergeometric function; it therefore follows from the above that $J^n(x)\, x^{-m}$ can be expressed as an integral involving a hypergeometric function. Writing

$$\int_0^\infty e^{-yz} J_n(z)\, z^{-m}\, dz = \varphi(y),$$

then $\varphi(y) = \sum_{s=0}^{\infty} \frac{(-)^s}{2^{n+2s}\,\Pi(n+s)\,\Pi(s)} \int_0^\infty e^{-yz} z^{n+2s-m}\, dz,$

that is, $$\varphi(y) = \sum_0^\infty \frac{(-)^s\,\Pi(n-m+2s)}{2^{n+2s}\,\Pi(n+s)\,\Pi(s)} \frac{1}{y^{n-m+2s+1}};$$

therefore $$\varphi(y) = \sum_0^\infty \frac{(-)^s\,\Pi\left(\frac{n-m}{2}+s\right)\Pi\left(\frac{n-m-1}{2}+s\right)}{2^m\,\Pi(n+s)\,\Pi(s)\,\Pi(-\frac{1}{2})} \frac{1}{y^{n-m+2s+1}};$$

whence $$\varphi(y) = \frac{1}{2^m y^{n-m+1}} \frac{\Pi\left(\frac{n-m}{2}\right)\Pi\left(\frac{n-m-1}{2}\right)}{\Pi(-\frac{1}{2})\,\Pi(n)} \times F\left(\frac{n-m+1}{2},\ \frac{n-m+2}{2},\ n+1,\ -\frac{1}{y^2}\right),$$

using the usual notation for a hypergeometric function. The relation holds if the real part of $n-m$ is greater than -1 and the series converges. The hypergeometric function that occurs in the above expression can be expressed by means of a spherical harmonic, and the relation can be written

$$\int_0^\infty e^{-yz} J_n(z)\, z^{-m}\, dz = \sqrt{\frac{2}{\pi}}\, \iota^{n-m+1}\, e^{(m-\frac{1}{2})\pi\iota} (-y^2-1)^{\frac{1}{4}(2m-1)}\, Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\iota y). \qquad (1)$$

Hence, from the above,

$$J_n(x)\, x^{-m} = \frac{1}{2\pi\iota}\sqrt{\frac{2}{\pi}}\, \iota^{n-m+1} e^{(m-\frac{1}{2})\pi\iota} \int_{c-\infty\iota}^{c+\infty\iota} e^{xy} (-y^2-1)^{\frac{1}{4}(2m-1)}\, Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\iota y)\, dy,$$

that is,

$$J_n(x)\, x^{-m} = \sqrt{\frac{2}{\pi}}\, \iota^{n-m-1} e^{(m-\frac{1}{2})\pi\iota} \int_{\iota c+\infty}^{\iota c-\infty} e^{-\iota x\mu} (\mu^2-1)^{\frac{1}{4}(2m-1)}\, Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)\, d\mu.$$

The path of integration may be deformed so that it becomes a curve, beginning at $-\infty - \iota c$, ending at $-\infty + \iota c$, and crossing the real axis to the right of the point $\mu = 1$. Therefore, provided a quantity α, less than unity, can be found such that

$$(\mu^2-1)^{\frac{1}{4}(2m-1)+\alpha} Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)$$

does not tend to become infinite as μ approaches unity, the above relation can be written

$$J_n(x)x^{-m} = \sqrt{\frac{2}{\pi}}\, e^{(m-\frac{1}{2})\pi\iota} \frac{\iota^{n-m-1}}{2\pi} \int_{-\infty}^{1} e^{-\iota x\mu} \Big[\big\{(\mu-o\iota)^2-1\big\}^{\frac{1}{4}(2m-1)} Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu-o\iota) - \big\{(\mu+o\iota)^2-1\big\}^{\frac{1}{4}(2m-1)} Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu+o\iota)\Big]\, d\mu,$$

that is,

$$J_n(x)x^{-m} = \sqrt{\frac{2}{\pi}}\, e^{(m-\frac{1}{2})\pi\iota} \frac{\iota^{n-m-1}}{2\pi} \Big[\int_{-\infty}^{-1} e^{-\iota x\mu} (\mu^2-1)^{\frac{1}{4}(2m-1)} \big\{ e^{-(m-\frac{1}{2})\pi\iota} Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu-o\iota) - e^{(m-\frac{1}{2})\pi\iota} Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu+o\iota)\big\}\, d\mu + \int_{-1}^{1} e^{-\iota x\mu} (\mu^2-1)^{\frac{1}{4}(2m-1)} \big\{ e^{-\frac{1}{2}(m-\frac{1}{2})\pi\iota} Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu-o\iota) - e^{\frac{1}{2}(m-\frac{1}{2})\pi\iota} Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu+o\iota)\big\}\, d\mu \Big];$$

whence

$$J_n(x)x^{-m} = \sqrt{\frac{2}{\pi}}\, \iota^{n-m-1} e^{(m-\frac{1}{2})\pi\iota} \frac{1}{2\pi} \Big[\int_{-\infty}^{-1} e^{-\iota x\mu} (\mu^2-1)^{\frac{1}{4}(2m-1)} \big\{ e^{(n-m)\pi\iota} + e^{-(n-m)\pi\iota} \big\} Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\, d\mu + \int_{-1}^{1} e^{-\iota x\mu} (1-\mu^2)^{\frac{1}{4}(2m-1)} \iota\pi e^{(\frac{1}{2}-m)\pi\iota} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\, d\mu \Big],$$

where the harmonics are now harmonics of a real quantity. This becomes

$$J_n(x)x^{-m} = \sqrt{\frac{2}{\pi}}\, \iota^{n-m} e^{(m-\frac{1}{2})\pi\iota} \frac{1}{2\pi} \Big[-\int_{-\infty}^{-1} e^{-\iota x\mu} (\mu^2-1)^{\frac{1}{4}(2m-1)} Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\, d\mu \sin(n-m)\pi + \int_{-1}^{1} e^{-\iota x\mu} (1-\mu^2)^{\frac{1}{4}(2m-1)} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\, d\mu\, \pi e^{(\frac{1}{2}-m)\pi\iota} \Big],$$

that is,

$$J_n(x)x^{-m} = \frac{\iota^{n-m}}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\iota x\mu} (1-\mu^2)^{\frac{1}{4}(2m-1)} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\, d\mu - \sqrt{\frac{2}{\pi}}\, \iota^{n-m} \frac{\sin(n-m)\pi}{\pi} \int_{1}^{\infty} e^{(m-\frac{1}{2})\pi\iota} e^{\iota x\mu} (\mu^2-1)^{\frac{1}{4}(2m-1)} Q_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\, d\mu. \tag{2}$$

When $n-m$ is an integer the relation becomes

$$J_n(x)x^{-m} = \frac{\iota^{n-m}}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\iota x\mu}(1-\mu^2)^{\frac{1}{4}(2m-1)} P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu)\, d\mu.$$

Various known results can be obtained from the above as particular cases. Putting $m = n$, and remembering that

$$P_{n-\frac{1}{2}}^{\frac{1}{2}-n}(\mu) = \frac{1}{2^{n-\frac{1}{2}}\Pi(n-\frac{1}{2})}(1-\mu^2)^{\frac{1}{4}(2n-1)},$$

it becomes

$$J_n(x)x^{-n} = \frac{1}{2^n\sqrt{\pi}\,\Pi(n-\frac{1}{2})} \int_{-1}^{1} e^{-\iota x\mu}(1-\mu^2)^{n-\frac{1}{2}} d\mu, \qquad (3)$$

where the real part of n is greater than $-\frac{1}{2}$.

Putting $m = 0$, and using the relations

$$P_{n-\frac{1}{2}}^{\frac{1}{2}}(\cos\theta) = \sqrt{\frac{2}{\pi\sin\theta}}\cos n\theta,$$

$$Q_{n-\frac{1}{2}}^{\frac{1}{2}}(\cos\psi) = \sqrt{\frac{\pi}{2\sin h\psi}}\,\iota e^{-n\psi},$$

the equation (2) becomes

$$J_n(x) = \frac{\iota^n}{\pi}\left\{\int_0^{\pi} e^{-\iota x\cos\theta}\cos n\theta\, d\theta - \sin n\pi \int_0^{\infty} e^{\iota x\cosh\psi - n\psi}\, d\psi\right\}, \qquad (4)$$

which holds if the imaginary part of x is greater than zero. The corresponding relation, when the imaginary part of x is less than zero, is

$$J_n(x) = \frac{(-\iota)^n}{\pi}\left\{\int_0^{\pi} e^{\iota x\cos\theta}\cos n\theta\, d\theta - \sin n\pi \int_0^{\infty} e^{-\iota x\cosh\psi - n\psi}\, d\psi\right\}.^{*}$$

When $n-m$ is an integer, it may be shown that

$$P_{n-\frac{1}{2}}^{\frac{1}{2}-m}(\mu) = \frac{2^{m-\frac{1}{2}}\Pi(m-1)\,\Pi(n-m)}{\Pi(n+m-1)\,\Pi(-\frac{1}{2})}(1-\mu^2)^{\frac{1}{4}(2m-1)} C_{n-m}^{m}(\mu),$$

where $C_{n-m}^{m}(\mu)$ is the coefficient of x^{n-m} in the expansion of $(1-2\mu x+x^2)^{-m}$ in powers of x.

Hence, when $n-m$ is an integer,

$$J_n(x)x^{-m} = \frac{2^{m-1}\Pi(m-1)\,\Pi(n-m)}{\Pi(n+m-1)}\,\frac{\iota^{n-m}}{\pi}\int_{-1}^{1} e^{-\iota x\mu}(1-\mu^2)^{m-\frac{1}{2}} C_{n-m}^{m}(\mu)\, d\mu;^{*}$$

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* *Cf.* Sonine, *Math. Ann.*, Vol. xvi.

when $m = -n$,

$$Q^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu) = \frac{\Pi(2n)\,\Pi(-\frac{1}{2})}{2^{n+\frac{1}{2}}\,\Pi(n)} \frac{e^{(n+\frac{1}{2})\pi\iota}}{(\mu^2-1)^{\frac{1}{4}(2n+1)}},$$

and relation (1) becomes

$$\int_0^\infty e^{-yz} J_n(z)\, z^n\, dz = \sqrt{\frac{2}{\pi}} \frac{\Pi(2n)\,(\Pi-\frac{1}{2})}{2^{n+\frac{1}{2}}\,\Pi(n)} \frac{e^{(n+\frac{1}{2})\pi\iota}}{(-y^2-1)^{n+\frac{1}{2}}},$$

that is,

$$\int_0^\infty e^{-yz} J_n(z)\, z^n\, dz = \frac{2^n}{\sqrt{\pi}} \frac{\Pi(n-\frac{1}{2})}{(1+y^2)^{n+\frac{1}{2}}},* \qquad (5)$$

where the real part of n is greater than $-\frac{1}{2}$.

The relation (2) becomes in this case

$$J_n(z)\, z^n = \frac{2^n \Pi(n-\frac{1}{2})}{\sqrt{\pi}} \frac{1}{2\pi\iota} \int_{c-\infty\iota}^{c+\infty\iota} \frac{e^{xy}\, dy}{(1+y^2)^{n+\frac{1}{2}}},$$

that is,

$$J_n(z)\, z^n = \frac{2^{n-\frac{1}{2}} \Pi(n-\frac{1}{2})}{\Pi\sqrt{\pi}} \int_{-\infty-\iota c}^{\infty-\iota c} \frac{e^{xy}\, dy}{(1-y^2)^{n+\frac{1}{2}}}, \qquad (6)$$

where the real part of n is greater than $-\frac{1}{2}$.

This integral can be transformed into one along the real axis when the real part of n is less than $\frac{1}{2}$, and then

$$J^n(z)\, z^n = \frac{2^{n+1}}{\sqrt{\pi}\,\Pi(-n-\frac{1}{2})} \int_1^\infty \frac{\sin xy\, dy}{(y^2-1)^{n+\frac{1}{2}}} \quad (\tfrac{1}{2} > R(n) > -\tfrac{1}{2}), \quad (7)$$

a result given by Sonine. When in relation (2) $m = \frac{1}{2}$, the expression for $J_n(x)\, x^{-\frac{1}{2}}$ in terms of Legendre functions is obtained.

Taking the expression for a Bessel function

$$J_n(z) = \frac{1}{2\pi\iota} \int_{\infty e^{a\iota}}^{\infty e^{b\iota}} e^{\frac{1}{2}z(s-1/s)} \frac{ds}{s^{n+1}},$$

it may be written

$$J_n(a\sqrt{x})\, x^{\frac{1}{2}n} = \frac{1}{2\pi\iota} \left(\frac{2}{a}\right)^n \int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{4}u^2 x u - 1/u} \frac{du}{u^{n+1}},$$

when the real part of n is greater than -1, which is

$$J_n(a\sqrt{x})\, x^{\frac{1}{2}n} = \frac{1}{2\pi\iota} \left(\frac{a}{2}\right)^n \int_{c-\infty\iota}^{c+\infty\iota} e^{xv - a^2/4v} \frac{dv}{v^{n+1}}.$$

The application of relation (B) gives the result

$$\int_0^\infty e^{-xy} J_n(a\sqrt{x})\, x^{\frac{1}{2}n}\, dx = \left(\frac{a}{2}\right)^n \frac{e^{-a^2/4y}}{y^{n+1}},$$

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* Sonine, *l.c.*

that is,
$$\int_0^\infty e^{-yx^2} J_n(ax)\, x^{n+1}\, dx = \frac{a^n e^{-a^2/4y}}{(2y)^{n+1}}, \tag{8}$$

when the real part of n is greater than -1, a result given by Weber* and by Hankel.†

An expression for the second solution of Bessel's equation is

$$\frac{\pi}{\sin n\pi}\left[\, \iota^n J_{-n}(\iota z) - \iota^{-n} J_n(\iota z)\,\right] = \int_0^\infty e^{-\frac{1}{2}z\,(s+1/s)}\, s^{n-1}\, ds,$$

which may be written

$$\frac{\pi}{\sin n\pi}\left[\, \iota^n J_{-n}(\iota a x^{\frac{1}{2}}) - \iota^{-n} J_n(\iota a x^{\frac{1}{2}})\,\right] = \frac{2^n x^{\frac{1}{2}n}}{a^n} \int_0^\infty e^{-x\sigma - a^2/4\sigma}\, \sigma^{n-1}\, d\sigma\,;$$

therefore, by relation (A),

$$\frac{1}{2\pi\iota}\int_{c-\infty\,\iota}^{c+\infty\,\iota} e^{xy}\left[\, \iota^n J_{-n}(\iota a x^{\frac{1}{2}}) - \iota^{-n} J_n(\iota a x^{\frac{1}{2}})\,\right] \frac{\pi}{\sin n\pi}\, \frac{dx}{x^{\frac{1}{2}n}} = \frac{2^n y^{n-1}}{a^n}\, e^{-a^2/4y},$$

that is,

$$\frac{1}{2\pi\iota}\int_{\infty+\omega'}^{\infty+\omega'} e^{-yx^2} \frac{\pi}{\sin n\pi}\left[\, \iota^{2n} J_{-n}(ax) - J_n(ax)\,\right] \frac{dx}{x^{n-1}} = \frac{(2y)^{n-1}}{a^n}\, e^{-a^2/4y}.$$

Now
$$\int_{\infty+\omega'}^{\infty+\omega'} J_n(ax)\, \frac{dx}{x^{n-1}} = 0\,;$$

therefore
$$\frac{e^{n\pi\iota}}{2\iota \sin n\pi}\int_{-\infty+\omega'}^{\infty+\omega'} e^{-yx^2} J_{-n}(ax)\, \frac{dx}{x^{n-1}} = \frac{(2y)^{n-1}}{a^n}\, e^{-a^2/4y}.$$

This can be transformed into integrals along real paths as follows: the relation may be written

$$\frac{e^{n\pi\iota}}{2\iota \sin n\pi}\int_\rho^\infty e^{-yx^2} J_{-n}(ax)\, \frac{dx}{x^{n-1}}\,(1 - e^{-2n\pi\iota}) + \frac{e^{n\pi\iota}}{2\iota \sin n\pi}\int_{-\rho}^{\rho} e^{-yx^2} J_{-n}(ax)\, \frac{dx}{x^{n-1}}$$
$$= \frac{(2y)^{n-1}}{a^n}\, e^{-a^2/4y}.$$

Writing
$$I = \frac{e^{n\pi i}}{2\iota \sin n\pi}\int_{-\rho}^{\rho} e^{-yx^2} J_{-n}(ax)\, \frac{dx}{x^{n-1}},$$

$$x = \rho e^{\phi\iota},$$

then
$$I = \frac{e^{n\pi\iota}}{2\iota \sin n\pi}\int_\pi^0 e^{-y\rho^2 e^{2\phi i}} J_{-n}(a\rho e^{\phi\iota})\, \frac{\iota\, d\phi}{\rho^{n-2} e^{(n-2)\phi\iota}},$$

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* *Crelle*, Vol. LXIX. † *Math. Ann.*, Vol. VIII.

that is,

$$I = -\frac{e^{n\pi\iota}}{2\rho^{n-2}\sin n\pi}\sum_0^\infty \frac{(a\rho)^{-n+2p}(-)^p}{2^{-n+2p}\Pi(-n+p)\Pi(p)}\int_0^\pi e^{-y\rho^2 e^{2\phi\iota}-2(n-p-1)\phi\iota}\,d\phi$$

or

$$I = -\frac{e^{n\pi\iota}}{4\rho^{n-2}\sin n\pi}\sum_0^\infty \frac{(a\rho)^{-n+2p}(-)^p}{2^{-n+2p}\Pi(-n+p)\Pi(p)}\int_0^{2\pi} e^{-y\rho^2 e^{\theta\iota}-(n-p-1)\theta\iota}\,d\theta.$$

Now $$\int_0^{2\pi} e^{-y\rho^2 e^{\theta\iota}-(n-p-1)\theta\iota}\,d\theta$$

$$= 2e^{(p+1-n)\pi\iota}\int_0^\pi e^{y\rho^2\cos\theta}\cos\{y\rho^2\sin\theta-(n-p-1)\theta\}\,d\theta,$$

that is,

$$\int_0^{2\pi} e^{-y\rho^2 e^{\theta\iota}-(n-p-1)\theta\iota}\,d\theta$$

$$= 2e^{(p+1-n)\pi\iota}\sin(p+1-n)\pi\int_1^\infty e^{-y\rho^2/\sigma}\frac{d\sigma}{\sigma^{p-n+2}}$$

when $p>n+1$, and

$$\int_0^{2\pi} e^{-y\rho^2 e^{\theta\iota}-(n-p-1)\theta\iota}\,d\theta$$

$$= 2e^{(p+1-n)\pi\iota}\left[\sin(n-p-1)\pi\int_1^\infty e^{-y\rho^2\sigma}\frac{d\sigma}{\sigma^{n-p}}+\frac{\pi(yp^2)^{n-p-1}}{\Pi(n-p-1)}\right]$$

when $p<n+1$; therefore

$$I = \frac{\pi}{2\sin n\pi}\sum_0^{(n-1)}\frac{a^{-n+2p}y^{n-p-1}}{2^{-n+2p}\Pi(p)\Pi(-n+p)\Pi(n-p-1)}$$

$$-\tfrac{1}{2}\sum_0^{(n-1)}\frac{(a\rho)^{-n+2p}(-)^p}{2^{-n+2p}\rho^{n-2}\Pi(p)\Pi(-n+p)}\int_1^\infty e^{-y\rho^2\sigma}\frac{d\sigma}{\sigma^{n-p}}$$

$$+\tfrac{1}{2}\sum_{(n)}^\infty\frac{(a\rho)^{-n+2p}(-)^p}{2^{-n+2p}\rho^{n-2}\Pi(p)\Pi(-n+p)}\int_0^1 e^{-y\rho^2\sigma}\frac{d\sigma}{\sigma^{n-p}},$$

where (n) denotes the greatest integer less than n, and this is equivalent to

$$I = \sum_0^{(n-1)}\frac{a^{-n+2p}y^{n-p-1}(-)^p}{2^{2p-n+1}\Pi(p)} - \sum_0^{(n-1)}\frac{a^{-n+2p}(-)^p}{2^{-n+2p}\Pi(-n+p)\Pi(p)}\int_0^\infty e^{-yx^2}x^{2p-2n+1}\,dx$$

$$+\sum_{(n)}^\infty\frac{a^{-n+2p}(-)^p}{2^{-n+2p}\Pi(-n+p)\Pi(p)}\int_0^\rho e^{-yx^2}x^{2p-2n+1}\,dx.$$

Hence $$\int_0^\infty e^{-yx^2}\left[J_{-n}(ax)-\sum_0^{(n-1)}\frac{a^{-n+2p}(-)^p x^{-n+2p}}{2^{-n+2p}\Pi(-n+p)\Pi(p)}\right]\frac{dx}{x^{n-1}}$$

$$= \frac{(2y)^{n-1}}{a^n}e^{-a^2/4y}-\sum_0^{(n-1)}\frac{a^{-n+2p}y^{n-p-1}(-)^p}{2^{2p-n+1}\Pi(p)},$$

or, denoting the part of the series for $J_{-n}(ax)$ which begins at the $[(n)+1]$-th term by $J^{(n)}_{-n}(ax)$,

$$\int_0^\infty e^{-yx^2} J^{(n)}_{-n}(ax)\,\frac{dx}{x^{n-1}} = \frac{(2y)^{n-1}}{a^n}\,e^{-a^2/4y} - \sum_0^{(n-1)} \frac{a^{-n+2p}\,y^{n-p-1}(-)^p}{2^{2p-n+1}\,\Pi(p)}. \quad (9)$$

When n is an integer this becomes

$$\int_0^\infty e^{-yx^2} J_n(ax)\,\frac{dx}{x^{n-1}} = \sum_0^\infty \frac{a^{n+2p}\,y^{-p-1}(-)^p}{2^{n+2p+1}\,\Pi(n+p)}, \quad (10)$$

which can easily be shown to be true for all values of n.

The relation (2) when $n-m$ is a positive integer is

$$J_n(x)\,x^{-m} = \frac{\iota^{n-m}}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\iota x\mu}(1-\mu^2)^{\frac{1}{4}(2m-1)} P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)\,d\mu,$$

and this may be written

$$J_n(-\iota x)\,x^{-m} = \frac{\iota^{n-2m}}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x\mu}(1-\mu^2)^{\frac{1}{4}(2m-1)} P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(\mu)\,d\mu;$$

hence by relation (A)

$$\frac{1}{2\pi\iota} \int_{c-\infty\iota}^{c+\infty\iota} e^{xy} J_n(-\iota x)\,x^{-m}\,dx = \frac{\iota^{n-2m}}{\sqrt{2\pi}}\,(1-y^2)^{\frac{1}{4}(2m-1)} P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(y),$$

that is,

$$\tfrac{1}{2} \int_{-\infty-\iota c}^{\infty-\iota c} e^{\iota xy} J_n(x)\,x^{-m}\,dx = \sqrt{\frac{\pi}{2}}\,\iota^{n-m}\,(1-y^2)^{\frac{1}{4}(2m-1)}\,P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(y);$$

therefore

$$\int_0^\infty J_n(x)\,x^{-m}\cos\left(xy - \frac{n-m}{2}\pi\right)dx = \sqrt{\frac{\pi}{2}}\,(1-y^2)^{\frac{1}{4}(2m-1)}\,P^{\frac{1}{2}-m}_{n-\frac{1}{2}}(y)$$

when $n > -\frac{1}{2}$ and $|y| > 1$, and vanishes when $|y| > 1$, or

$$\int_0^\infty J_n(ax)\,x^{-m}\cos\left(bx - \frac{n-m}{2}\pi\right)dx = \sqrt{\frac{\pi}{2a}}\,(a^2-b^2)^{\frac{1}{4}(2m-1)}\,P^{\frac{1}{2}-m}_{n-\frac{1}{2}}\left(\frac{b}{a}\right) \quad (11)$$

when $a > b$, and vanishes when $a < b$, n being greater than $-\frac{1}{2}$. When $n = m$, this becomes

$$\left.\begin{aligned} \int_0^\infty J_n(ax)\,x^{-n}\cos bx\,dx &= \frac{\sqrt{\pi}\,(a^2-b^2)^{n-\frac{1}{2}}}{(2a)^n\,\Pi(n-\frac{1}{2})} \quad (a > b) \\ &= 0 \quad (a < b) \end{aligned}\right\}, \quad (12)$$

a known result.* When $m = \frac{1}{2}$, and n is a positive integer, the relation (11) becomes

$$\left.\begin{aligned}\int_0^\infty \cos(bx) J_{n+\frac{1}{2}}(ax)\, x^{-\frac{1}{2}}\, dx &= \sqrt{\frac{\pi}{2a}}\, P_n\left(\frac{b}{a}\right) \quad (a > b)\\ &= 0 \quad (a < b)\end{aligned}\right\}. \tag{13}$$

Similarly from the relation (7) it can be deduced that

$$\left.\begin{aligned}\int_0^\infty \sin ax J_n(bx)\, x^n\, dx &= \frac{\sqrt{\pi}\,(2b)^n}{\Pi(-n-\frac{1}{2})(a^2-b^2)^{n+\frac{1}{2}}} \quad (a > b)\\ &= 0 \quad (a < b)\end{aligned}\right\} \tag{14}$$

when $\frac{1}{2} > n > -\frac{1}{2}$, a known result.

Writing the integral

$$\frac{1}{2\pi\iota}\int_{\infty e^{\alpha\iota}}^{\infty e^{\beta\iota}} e^{yz} I_n(z)\, z^m dz = \psi(y),$$

where the real parts of $ye^{\alpha\iota}$, $ye^{\beta\iota}$ are negative, then

$$\psi(y) = \sum_0^\infty \frac{1}{2^{n+2p}}\, \frac{1}{\Pi(n+p)\,\Pi(p)}\, \frac{1}{2\pi\iota}\int_{\infty e^{\alpha\iota}}^{\infty e^{\beta\iota}} e^{yz} z^{n+m+2p}\, dz,$$

that is, $\psi(y) = \sum_0^\infty \dfrac{y^{-2p-n-m-1}}{2^{n+2p}\,\Pi(n+p)\,\Pi(p)\,\Pi(-2p-n-m-1)}$

when $|y| > 1$, or

$$\psi(y) = -\frac{\sin(n+m)\pi}{\pi} \sum_0^\infty \frac{\Pi(2p+n+m)}{2^{n+2p}\,\Pi(n+p)\,\Pi(p)}\, \frac{1}{y^{2p+n+m+1}},$$

$$\text{whence } \psi(y) = -\frac{\sin(n+m)\pi}{\pi\sqrt{\pi}}\, \frac{2^m}{y^{n+m+1}}\, \frac{\Pi\left(\frac{n+m}{2}\right)\Pi\left(\frac{n+m-1}{2}\right)}{\Pi(n)}$$

$$\times F\left(\frac{n+m+2}{2},\ \frac{n+m+1}{2},\ n+1,\ \frac{1}{y^2}\right),$$

which may be written

$$\psi(y) = -\sqrt{\frac{2}{\pi}}\, \frac{\sin(n+m)\pi}{\pi}\, e^{-(m+\frac{1}{2})\pi\iota}\, (y^2-1)^{-\frac{1}{2}(m+\frac{1}{2})}\, Q_{n-\frac{1}{2}}^{m+\frac{1}{2}}(y).$$

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* *Cf.* Sonine, *l.c.*

If now $K_n(z)$ be the second solution of Bessel's equation with imaginary argument, where

$$K_n(z) = \frac{\pi}{2 \sin n\pi}[I_{-n}(z) - I_n(z)],$$

the integral $$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{yz} K_n(z)\, z^m\, dz = 0$$

when $|y| < 1$, and by the above

$$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{yz} K_n(z)\, z^m\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-(m+\frac{1}{2})\pi\iota}}{\sin n\pi} (y^2-1)^{-\frac{1}{4}(m+\frac{1}{2})} [\sin(n+m)\pi Q^{m+\frac{1}{2}}_{n-\frac{1}{2}}(y) - \sin(m-n)\pi\, Q^{m+\frac{1}{2}}_{-n-\frac{1}{2}}(y)],$$

that is,

$$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{yz} K_n(z)\, z^m\, dz = \sqrt{\frac{\pi}{2}} \frac{\cos(m+\frac{1}{2})\pi}{\sin n\pi} (y^2-1)^{-\frac{1}{4}(m+1)} P^{m+\frac{1}{2}}_{n-\frac{1}{2}}(y) \tag{15}$$

when $|y| > 1$. The application of relation (B) gives the result

$$K_n(x)\, x^m = \sqrt{\frac{\pi}{2}} \frac{\cos(m+\frac{1}{2})\pi}{\sin n\pi} \int_1^\infty e^{-xy} (y^2-1)^{-\frac{1}{2}(m+\frac{1}{2})} P^{m+\frac{1}{2}}_{n-\frac{1}{2}}(y)\, dy \tag{16}$$

when $m+\frac{1}{2} < 1$. Putting $m = -n$, this becomes

$$K_n(x)\, x^{-n} = \sqrt{\frac{\pi}{2}} \int_1^\infty e^{-xy} (y^2-1)^{-\frac{1}{2}(\frac{1}{2}-n)} P^{\frac{1}{2}-n}_{n-\frac{1}{2}}(y)\, dy,$$

that is, $$K_n(x)\, x^{-n} = \frac{\sqrt{\pi}}{2^n \Pi(n-\frac{1}{2})} \int_1^\infty e^{-xy} (y^2-1)^{n-\frac{1}{2}} dy, \tag{17}$$

a known result. Putting $m = -\frac{1}{2}$, it becomes

$$K_n(x)\, x^{-\frac{1}{2}} = \sqrt{\frac{\pi}{2}} \frac{1}{\sin n\pi} \int_1^\infty e^{-xy} P_{n-\frac{1}{2}}(y)\, dy. \tag{18}$$

When $m = 0$, the relation (15), using its first form, becomes

$$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{yz} K_n(z)\, dz = \frac{e^{-\frac{1}{2}\pi\iota}}{\sqrt{2\pi}} (y^2-1)^{-\frac{1}{4}} [Q^{\frac{1}{2}}_{n-\frac{1}{2}}(y) + Q^{\frac{1}{2}}_{-n-\frac{1}{2}}(y)],$$

that is, $$\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{z\cosh y} K_n(z)\, dz = \frac{\cosh ny}{\sinh y}; \tag{19}$$

and therefore, by relation (B),

$$K_n(x) = \int_0^\infty e^{-x\cosh y}\cosh ny\,dy, \tag{20}$$

a known result.

It has been proved that

$$J_n(ax)\,J_n(bx) = \frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{\frac{1}{2}t-[x^2(a^2+b^2)]/2t}\,I_n\left(\frac{x^2ab}{t}\right)\frac{dt}{t},$$

when the real part of n is greater than -1.*

This may be written

$$J_n(ax^{\frac{1}{2}})\,J_n(bx^{\frac{1}{2}}) = \frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota} e^{xs-(a^2+b^2)/4s}\,I_n\left(\frac{ab}{2s}\right)\frac{ds}{s},$$

whence, by relation (B),

$$\int_0^\infty e^{-yx}J_n(ax^{\frac{1}{2}})\,J_n(bx^{\frac{1}{2}})\,dx = \frac{e^{-(a^2+b^2)/4y}}{y}\,I_n\left(\frac{ab}{2y}\right),$$

that is,

$$\int_0^\infty e^{-yx^2}J_n(ax)\,J_n(bx)\,x\,dx = \frac{e^{-(a^2+b^2)/4y}}{2y}\,I_n\left(\frac{ab}{2y}\right).\dagger \tag{21}$$

From this it follows that

$$\int_{-\infty}^{\infty} e^{-yx^2}J_n(ax)\,J_n(bx)\,x\,dx = (1-e^{2n\pi\iota})\,\frac{e^{-(a^2+b^2)/4y}}{2y}\,I_n\left(\frac{ab}{2y}\right).$$

Now

$$\int_{-\infty}^{\infty} e^{-yx^2}J_n(ax)\,J_{-n}(bx)\,x\,dx = 0\,;$$

therefore

$$\int_{-\infty}^{\infty} e^{-yx^2}J_n(ax)\left[\iota^nJ_{-n}(bx)-\iota^{-n}J_n(bx)\right]x\,dx$$
$$= \iota^{-n}(e^{2n\pi\iota}-1)\,\frac{e^{-(a^2+b^2)/4y}}{2y}\,I_n\left(\frac{ab}{2y}\right),$$

that is,

$$\int_{c-\infty\iota}^{c+\infty\iota} e^{y\xi^2}J_n(a\xi\iota)\left[\iota^nJ_{-n}(b\xi\iota)-\iota^{-n}J_n(b\xi\iota)\right]\xi\,d\xi$$
$$= \iota^{n+1}\sin n\pi\,\frac{e^{-(a^2+b^2)/4y}}{y}\,I_n\left(\frac{ab}{2y}\right);$$

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* *Proc. Lond. Math. Soc.*, Vol. xxxii.
† *Cf.* Weber, Hankel, *l.c.*

whence, by relation (B),

$$\frac{1}{2\pi\iota}\int_0^\infty e^{-x^2y}\,\iota^{n+1}\sin n\pi\, e^{-(a^2+b^2)/4y}\,I_n\left(\frac{ab}{2y}\right)\frac{dy}{y}$$

$$=\tfrac{1}{2}J_n(a\iota)\left[\iota^n J_{-n}(b\iota)-\iota^{-n}J_n(b\iota)\right];$$

and therefore

$$\int_0^\infty e^{-\frac{1}{2}x\left[s+(a^2+b^2)/s\right]}\,I_n\left(\frac{abx}{s}\right)\frac{ds}{s}=\frac{\pi}{\sin n\pi}J_n(a\iota)\left[J_{-n}(b\iota)-\iota^{-2n}J_n(b\iota)\right],\tag{22}$$

a result equivalent to one given in the previous paper.*

Writing the integral

$$\int_0^\infty e^{-yx^2}J_n(ax)\,x^{2m-n+1}\,dx=f(y),$$

then

$$f(y)=\sum_0^\infty\frac{(-)^p\,a^{n+2p}}{2^{n+2p}\,\Pi(n+p)\,\Pi(p)}\int_0^\infty e^{-yx^2}x^{2m+2p+1}\,dx,$$

that is,

$$f(y)=\sum_0^\infty\frac{(-)^p\,a^{n+2p}}{2^{n+2p+1}\,\Pi(n+p)\,\Pi(p)}\,\frac{\Pi(m+p)}{y^{m+p+1}}$$

or $$f(y)=\sum_0^\infty\frac{(-)^p\,a^{n+2p}}{2^{n+2p+1}\,\Pi(p)\,y^{m+p+1}}\,\frac{1}{\Pi(n-m-1)}\int_0^1 z^{m+p}(1-z)^{n-m-1}\,dz,$$

when $n>m>-1$; whence

$$\int_0^\infty e^{-yx^2}J_n(ax)\,x^{2m-n+1}\,dx$$

$$=\frac{1}{2^n a^n y^{m+1}\,\Pi(n-m-1)}\int_0^a e^{-\xi^2/4y}(a^2-\xi^2)^{n-m-1}\,\xi^{2m+1}\,d\xi\quad(n>m>-1).\tag{23}$$†

This relation may be written

$$\int_0^\infty e^{-yx}J_n(ax^{\frac{1}{2}})\,x^{m-\frac{1}{2}n}\,dx$$

$$=\frac{1}{2^{n-1}a^n y^{m+1}\,\Pi(n-m-1)}\int_0^a e^{-\xi^2/4y}(a^2-\xi^2)^{n-m-1}\,\xi^{2m+1}\,d\xi,$$

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* *Proc. Lond. Math. Soc.*, Vol. xxxii.

† *Cf.* Sonine, *l.c.*

and therefore, by relation (B),

$$J_n(ax^{\frac{1}{2}})\,x^{m-\frac{1}{2}n}$$

$$=\frac{1}{2^{n-1}a^n\Pi(n-m-1)}\,\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota}\int_0^a e^{xy-\xi^2/4y}(a^2-\xi^2)^{n-m-1}\xi^{2m+1}d\xi\,\frac{dy}{y^{m+1}},$$

that is,

$$J_n(ax^{\frac{1}{2}})\,x^{m-\frac{1}{2}n}=\frac{x^{\frac{1}{2}m}}{2^{n-m-1}a^n\Pi(n-m-1)}\int_0^a J_m(\xi x^{\frac{1}{2}})(a^2-\xi^2)^{n-m-1}\xi^{m+1}d\xi$$

or
$$J_n(ax)=\frac{x^{n-m}}{2^{n-m-1}a^n\Pi(n-m-1)}\int_0^a J_m(\xi x)(a^2-\xi^2)^{n-m-1}\xi^{m+1}d\xi$$

$$(n>m>-1),\quad (24)$$

a known relation.

From (24) it follows that

$$J_n(ax^{\frac{1}{2}})\,J_m(bx^{\frac{1}{2}})$$

$$=\frac{x^{\frac{1}{2}(n-m)}}{2^{n-m-1}a^n\Pi(n-m-1)}\int_0^a J_m(bx^{\frac{1}{2}})\,J_m(\xi x^{\frac{1}{2}})(a^2-\xi^2)^{n-m-1}\xi^{m+1}d\xi,$$

that is,

$$J_n(ax^{\frac{1}{2}})\,J_m(bx^{\frac{1}{2}})$$

$$=\frac{x^{\frac{1}{2}(n-m)}}{2^{n-m-1}a^n\Pi(n-m-1)}\int_0^a(a^2-\xi^2)^{n-m-1}\xi^{m+1}d\xi$$

$$\times\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota}e^{\frac{1}{2}t-[x(b^2+\xi^2)]/2t}\,I_m\left(\frac{b\xi x}{t}\right)\frac{dt}{t}$$

or $J_n(ax^{\frac{1}{2}})\,J_m(bx)^{\frac{1}{2}}$

$$=\frac{x^{\frac{1}{2}(n-m)}}{2^{n-m-1}a^n\Pi(n-m-1)}\int_0^a(a^2-\xi^2)\,\xi^{m+1}d\xi$$

$$\times\frac{1}{2\pi\iota}\int_{c-\infty\iota}^{c+\infty\iota}e^{xs-(b^2+\xi^2)/4s}\,I_m\left(\frac{b\xi}{2s}\right)\frac{ds}{s};$$

therefore, by relation (B),

$$\int_0^\infty e^{-yx}J_n(ax^{\frac{1}{2}})\,J_m(bx^{\frac{1}{2}})\,x^{\frac{1}{2}(m-n)}\,dx$$

$$=\frac{1}{2^{n-m-1}\Pi(n-m-1)\,a^n y}\int_0^a(a^2-\xi^2)^{n-m-1}\xi^{m+1}e^{-(b^2+\xi^2)/4y}\,I_m\left(\frac{b\xi}{2y}\right)d\xi,$$

that is,

$$\int_0^\infty e^{-yx^2}J_n(ax)\,J_m(bx)\,x^{m-n+1}dx$$

$$=\frac{1}{2^{n-m}\Pi(n-m-1)\,a^n y}\int_0^a(a^2-\xi^2)^{n-m-1}\xi^{m+1}e^{-(b^2+\xi^2)/4y}\,I_m\left(\frac{b\xi}{2y}\right)d\xi$$

$$(n>m>-1).\quad (25)$$

Proceeding to the limit when $y = 0$, the integrand on the left-hand side vanishes except for values of ξ in the neighbourhood of b, and, evaluating it,

$$\left.\begin{aligned}\int_0^\infty J_n(ax)\,J_m(bx)\,x^{m-n+1}\,dx &= \frac{(a^2-b^2)^{n-m-1}\,b^m}{2^{n-m-1}\,\Pi(n-m-1)\,a^n} \quad (a>b)\\ &= 0 \quad (a<b,\ n>m>-1)\end{aligned}\right\}. \quad (26)$$

Relations (25) and (26) have been given by Sonine.*

Again, from (24), writing

$$X = \frac{\pi}{2\sin m\pi}\,\frac{J_n(ax\iota)}{(x\iota)^{n-m}}\left[J_{-m}(bx\iota)-\iota^{-2m}J_m(bx\iota)\right],$$

then

$$X = \frac{\pi}{\sin m\pi\,2^{n-m}\,\Pi(n-m-1)\,a^n}\int_0^a \xi^{m+1}(a^2-\xi^2)^{n-m-1}J_m(\xi x\iota) \times\left[J_{-m}(bx\iota)-\iota^{-2m}J_m(bx\iota)\right]d\xi,$$

that is, by (22),

$$X = \frac{1}{2^{n-m}\,\Pi(n-m-1)\,a^n}\int_0^a\int_0^\infty \xi^{m+1}(a^2-\xi^2)^{n-m-1}\,e^{-x^2s-(b^2+\xi^2)/4s}\,I_n\left(\frac{b\xi}{2s}\right)\frac{ds}{s}\,d\xi,$$

which, using (25), becomes

$$X = \int_0^\infty\int_0^\infty e^{-x^2s-\xi^2s}\,J_n(a\xi)\,J_m(b\xi)\,\xi^{m-n+1}\,d\xi\,ds,$$

that is,
$$X = \int_0^\infty \frac{J_n(a\xi)\,J_m(b\xi)\,\xi^{m-n+1}}{\xi^2+x^2}\,d\xi\,;$$

and therefore

$$\int_0^\infty \frac{J_n(a\xi)\,J_m(b\xi)\,\xi^{m-n+1}\,d\xi}{\xi^2+x^2} = \frac{\pi}{2\sin m\pi}\,\frac{J_n(ax\iota)}{(x\iota)^{n-m}}\left[J_{-m}(bx\iota)-\iota^{-2m}J_m(bx\iota)\right] \quad (b>a,\ n>m>-1). \quad (27)$$

The relation (23) suggests the evaluation of the integral

$$\int_0^a e^{-\xi^2/4y}\,J_{n-m-1}\left\{c\sqrt{(a^2-\xi^2)}\right\}(a^2-\xi^2)^{\frac{1}{2}(n-m-1)}\,\xi^{2m+1}\,d\xi.$$

Writing $f(y)$ for this integral,

$$f(y) = \frac{1}{c^{n-m-1}}\,\frac{1}{2\pi\iota}\int_0^a\int_{c'-\infty\iota}^{c'+\infty\iota} e^{\frac{1}{2}tc^2(a^2-\xi^2)-1/2t-\xi^2/4y}\,\xi^{2m+1}\,d\xi\,\frac{dt}{t^{n-m}},$$

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* *L.c.*

that is, performing the integration with respect to ξ,

$$f(y) = \frac{\Pi(m)}{c^{n-m-1}\,4\pi\iota} \int_{c'-\infty\iota}^{c'+\infty\iota} e^{\frac{1}{4}a^2c^2t - 1/2t} \frac{1}{\left(\frac{1}{4y} + \frac{tc^2}{2}\right)^{m+1}} \frac{dt}{t^{n-m}},$$

which may be written

$$f(y) = \frac{2^{m+1}\Pi(m)\,y^{m+1}}{c^{n+m+1}\,4\pi\iota} \int_{c'-\infty\iota}^{c'+\infty\iota} e^{\frac{1}{4}a^2c^2t - 1/2t} \frac{1}{\left(\frac{1}{2tc^2} + y\right)^{m+1}} \frac{dt}{t^{n+1}};$$

whence

$$f(y) = \frac{2^{m+1}y^{m+1}}{c^{n+m+1}\,2\pi\iota} \int_{c'-\infty\iota}^{c'+\infty\iota} \int_0^\infty e^{\frac{1}{4}a^2c^2t - 1/2t - y\zeta^2 - \zeta^2/2tc^2} \zeta^{2m+1}\,d\zeta \frac{dt}{t^{n+1}},$$

that is,

$$f(y) = 2^{m+1}y^{m+1}c^{n-m-1}a^{2n} \frac{1}{2\pi\iota} \int_{c'-\infty\iota}^{c'+\infty} \int_0^\infty e^{\frac{1}{2}s - [a^2(c^2+\zeta^2)/2s] - y\zeta^2} \zeta^{2m+1}\,d\zeta \frac{ds}{s^{n+1}};$$

and therefore

$$f(y) = 2^{m+1}y^{m+1}c^{n-m-1}a^n \int_0^\infty e^{-y\zeta^2} \frac{J_n\{a\sqrt{(c^2+\zeta^2)}\}}{(c^2+\zeta^2)^{\frac{1}{2}n}} \zeta^{2m+1}\,d\zeta,$$

that is,

$$\int_0^\infty e^{-y\zeta^2} \frac{J_n\{a\sqrt{(c^2+\zeta^2)}\}}{(c^2+\zeta^2)^{\frac{1}{2}n}} \zeta^{2m+1}\,d\zeta$$

$$= \frac{1}{a^nc^{n-m-1}(2y)^{m+1}} \int_0^a e^{-\xi^2/4y} J_{n-m-1}\{c\sqrt{(a^2-\xi^2)}\}\,(a^2-\xi^2)^{\frac{1}{2}(n-m-1)}\,\xi^{2m+1}\,d\xi. \tag{28}$$

From this it follows, by applying relation (B) as in (24), that

$$\frac{J_n\{a\sqrt{(c^2+x^2)}\}}{(c^2+x^2)^{\frac{1}{2}n}} = \frac{1}{a^nc^{n-m-1}x^m} \int_0^a J_m(x\xi)\,J_{n-m-1}\{c\sqrt{(a^2-\xi^2)}\}\,(a^2-\xi^2)^{\frac{1}{2}n}\xi^{m+1}\,d\xi$$

$$(n > m > -1). \tag{29}$$

Similarly as in (25), by (B),

$$\int_0^\infty e^{-yx^2} \frac{J_n\{a\sqrt{(c^2+x^2)}\}}{(c^2+x^2)^{\frac{1}{2}n}} J_m(bx)\,x^{m+1}\,dx$$

$$= \frac{1}{2a^nc^{n-m-1}y} \int_0^a e^{-(b^2+\xi^2)/4y} I_m\left(\frac{b\xi}{2y}\right) J_{n-m-1}\{c\sqrt{(a^2-\xi^2)}\}$$

$$\times (a^2-\xi^2)^{\frac{1}{2}(n-m-1)}\,\xi^{2m+1}\,d\xi \quad (n > m > -1). \tag{30}$$

As in (26), when $y = 0$, this becomes

$$\int_0^\infty \frac{J_n\{a\sqrt{(c^2+x^2)}\}}{(c^2+x^2)^{\frac{1}{2}n}} J_m(bx)\, x^{m+1}\, dx$$

$$= \frac{J_{n-m-1}\{c\sqrt{(a^2-b^2)}\}\,(a^2-b^2)^{\frac{1}{2}(n-m-1)}\, b^m}{a^n c^{n-m-1}} \quad (a > b)$$

and $$= 0 \quad (a < b)$$

$$(n > m > -1). \quad (31)$$

Again, as in (27), it may be proved that

$$\int_0^\infty \frac{J_n\{a\sqrt{(c^2+x^2)}\}}{(c^2+x^2)^{\frac{1}{2}n}} \frac{J_m(bx)\, x^{m+1}}{x^2+\xi^2}\, dx$$

$$= \frac{\pi}{2\sin m\pi} \frac{J_n\{a\sqrt{(c^2-\xi^2)}\}}{(c^2-\xi^2)^{\frac{1}{2}n}} (\xi\iota)^m \{J_{-m}(b\xi\iota) - \iota^{-2m} J_m(b\xi\iota)\}$$

$$(b > a,\ n > m > -1). \quad (32)$$

Relations (29) and (32) have been given by Sonine, the method of proof of (32) being much more difficult.

The preceding examples are illustrative of the results which can be obtained by using the relations (A) and (B), and are capable of being greatly extended.

Generational Relations for the Abstract Group Simply Isomorphic with the Linear Fractional Group in the GF $[2^n]$. *By* L. E. Dickson. Received December 22nd, 1902. Read January 8th, 1903.

1. The object of this paper is the determination of two linear fractional transformations A and B having the following properties:

(i.) A and B generate the group Γ of all linear fractional transformations of determinant unity in the $GF[2^n]$ $(n > 1)$.

(ii.) A is of period 2^n+1, B of period 2, AB of period 3.

(iii.) A and B satisfy relations of the form

$$(1) \quad (BA^rBA^s)^2 = I \quad (r = 1, 2, \ldots, 2^n),$$

the integer s being uniquely determined modulo 2^n+1 by r.

The result finds immediate application in the study of the abstract group G simply isomorphic with Γ. For $n = 2, 3, 4, 5$ (the cases hitherto investigated), the group G is generated by two operators A and B subject only to the generational relations (1) and

$$(2) \quad A^{2^n+1} = I, \quad B^2 = I, \quad (AB)^3 = I.$$

Among the pairs of exponents r, s in (1), the following four sets always occur:

$$1, 2; \quad 2, 1; \quad 2^n, 2^n-1; \quad 2^n-1, 2^n.$$

The resulting four relations follow at once from (2). For $n = 1$ or 2, there are no further relations (1); so that the generational relations for G are (2) when $n = 2$. But, for $n = 1$, relations (2) define the alternating group on four letters with which Γ is not simply isomorphic; while, for $n = 2$, relations (2) define the alternating group on five letters with which Γ is simply isomorphic.* For $n = 3$ and $n = 4$, the theorem has been proved by the writer;† for $n = 5$ by Mr. Bussey.

Moreover, the paper obtains by a general, but simple, method the data needed in the study of G by the method used for $n = 3, 4, 5$.

2. Use is made of the transformations

$$T: \quad z' = \frac{1}{z}; \qquad S_\lambda: \quad z' = z + \lambda.$$

To reduce $A = S_a T$ to its canonical form, let

$$z^2 + az = 1$$

have the roots z_1 and z_2, and set

$$X \equiv \frac{z - z_1}{z - z_2}, \quad \zeta \equiv \frac{z_1}{z_2} = \frac{z_1^2}{z_1 z_2} = -z_1^2$$

so that $\zeta^2 \equiv a^2\zeta + 1 \pmod 2$. Then A becomes

$$(3) \quad X' = \zeta X.$$

Since $z_1 - z_2 \equiv z_1 + z_2 \equiv a \pmod 2$, S_λ becomes

$$(4) \quad X' = \frac{(\lambda + a)X + \lambda}{\lambda X + (\lambda + a)}.$$

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* *Linear Groups*, § 280. In explanation of the exceptional character of the case $n = 1$, note that a of § 2 must be unity, so that $B = S_0 =$ identity.

† *Bulletin Amer. Math. Soc.*, Vol. IX., 1902, pp. 194–204; *Proc. Lond. Math. Soc.*, Vol. XXXV., pp. 306–19.

Let $B = S_{a+1}$. Then, for modulus 2,

$$BA^t: \quad X' = \frac{\zeta^t(X+a+1)}{(a+1)X+1};$$

$$BA^rBA^s: \quad X' = \frac{\zeta^s\{X(\zeta^r+a^2+1)+(\zeta^r+1)(a+1)\}}{X(\zeta^r+1)(a+1)+\zeta^r(a^2+1)+1}.$$

Hence BA^rBA^s is of period 2 if, and only if,

$$(5) \quad \zeta^s = \frac{\zeta^r(a^2+1)+1}{\zeta^r+a^2+1}.$$

Let ω be a mark of the $GF[2^{2n}]$ which belongs to the exponent 2^n+1, a divisor of $2^{2n}-1$. Then $\omega+\omega^{2^n}$ is a mark, say a^2, of the $GF[2^n]$ Hence ω is a root of

$$(6) \quad \zeta^2 = a^2\zeta+1,$$

an equation irreducible in the $GF[2^n]$. For later use, we note that a^2, and hence a, defines the $GF[2^n]$, namely, is a root of an equation of degree n with integral coefficients and irreducible modulo 2.* Since a root ζ of (6) belongs to the exponent 2^n+1, the transformation A defined by (3) is of period 2^n+1. Since the *conjugate* ζ^{2^n} of ζ equals $1/\zeta$, the conjugate of the second member of (5) equals

$$\frac{\zeta^{-r}(a^2+1)+1}{\zeta^{-r}+a^2+1} = \frac{\zeta^r+a^2+1}{\zeta^r(a^2+1)+1}.$$

Hence the second member of (5) is a root of $X^{2^n+1}=1$, and hence equals some power of ζ. Hence, if an arbitrary integral value be assigned to r, there exists an integer s such that relation (5) holds. Finally, $BA = S_1T$ is of period 3. Hence, if $n>1$, A and B have the properties (ii.) and (iii.) of § 1.

Since condition (5) may be written

$$\zeta^{r+s}+(\zeta^r+\zeta)(a^2+1)+1 \equiv 0 \pmod 2,$$

it follows that, if $r=\rho$, $s=\sigma$ are solutions, then are also $r=\sigma$, $s=\rho$; $r=-\rho$, $s=-\sigma$; $r=-\sigma$, $s=-\rho$. Another proof follows from (1).

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* Moore, *Mathematical Papers*, Chicago Congress, p. 226, No. 7.

3. By § 2, $A^tBA^rBA^l$ may be written

$$X' = \frac{X(\zeta^r+a^2+1)\zeta^{t+l}+\zeta^l(\zeta^r+1)(a+1)}{X\zeta^t(\zeta^r+1)(a+1)+\zeta^r(a^2+1)+1}.$$

It is therefore identical with (4), namely,

$$X' = \frac{\rho X+\sigma}{\mu X+\nu}$$

$$\left[\mu=\sigma=a\lambda,\ \ \rho=\nu=a(\lambda+a)\right],$$

if, and only if, $l=t$ and

$$(7)\quad \zeta^{2t} = \frac{\zeta^r(a^2+1)+1}{\zeta^r+a^2+1},$$

$$(8)\quad \lambda = \frac{(\zeta^r+1)(a+1)a}{(\zeta^r+1)(a+1)+\zeta^t(\zeta^r+a^2+1)}.$$

Given any integer r, there exists (by § 2) an integer $2t$ and hence an integer t, determined modulo 2^n+1, such that (7) is satisfied. Then λ is determined by (8). As shown in § 4, or, as may be verified directly, $\lambda^{2^n}=\lambda$.

For an arbitrary integer r, an integer t is uniquely determined modulo 2^n+1, and a mark λ of the $GF[2^n]$ is uniquely determined, such that

$$(9)\quad A^tBA^rBA^t = S_\lambda.$$

4. Let r, t and ρ, τ be two sets of relations of (7) which, by (8), define the same mark λ. Then, by (8),

$$\zeta^r(\zeta^r+1)(\zeta^\rho+a^2+1) = \zeta^t(\zeta^\rho+1)(\zeta^r+a^2+1).$$

Squaring, and eliminating $\zeta^{2\tau}$ and ζ^{2t} by (7),

$$\left\{\zeta^\rho(a^2+1)+1\right\}(\zeta^r+1)^2(\zeta^\rho+a^2+1)$$

$$= \left\{\zeta^r(a^2+1)+1\right\}(\zeta^\rho+1)^2(\zeta^r+a^2+1).$$

Expanding, we obtain, modulo 2,

$$\zeta^{2r+\rho}a^4+\zeta^\rho a^4 \equiv \zeta^{2\rho+r}a^4+\zeta^\rho a^4,$$

$$(\zeta^{r+\rho}+1)(\zeta^r+\zeta^\rho) \equiv 0.$$

Hence either $\rho\equiv r$ or else $\rho\equiv -r$, modulo 2^n+1. Then $\tau\equiv t$ or $\tau\equiv -t$, respectively (end of § 2). Since the replacement of r and t by their negatives may be attained by replacing ζ by $\zeta^{-1}=\zeta^{2^n}$, it follows that $\lambda^{2^n}=\lambda$.

The value of λ given by the pair of solutions r, t of (7) *equals that*

given by the pair $-r, -t$; *but differs from that given by any third pair of solutions. Each* λ *belongs to the* $GF[2^n]$.

We therefore obtain 2^{n-1} values of λ, different from 0, by giving to r the values $1, 2, \ldots, 2^n$. For $r = 0$, then $t = 0$, $\lambda = 0$. The additive group determined by these $2^{n-1}+1$ distinct marks λ of the $GF[2^n]$ is of order a power of 2 and hence of order 2^n. Hence a suitable product of the S_λ defined by (9) leads to S_μ, where μ is any given mark of the $GF[2^n]$. Moreover, $T = S_a A$. But* T and the various S_μ generate Γ. Hence property (i.) of § 1 holds:

The transformations A *and* B *generate the group* Γ.

5. Let $n = 3$. The only cubics irreducible modulo 2 are

$$X^3+X+1 \equiv 0, \quad X^3+X^2+1 \equiv 0.$$

Let i denote a root of the former. Then

$$i^3 = i+1, \quad i^4 = i^2+i, \quad i^5 = i^2+i+1, \quad i^6 = i^2+1, \quad i^7 = 1.$$

For $a = i$, $\zeta^2 = a^2\zeta+1$ is irreducible in the $GF[2^3]$, since

$$\begin{aligned} &\zeta^2 = i^2\zeta+1, && \zeta^3 = (i^2+i+1)\zeta+i^2, \\ &\zeta^4 = (i^2+1)\zeta+i^2+i+1, && \zeta^5 = (i^2+1)\zeta+i^2+1, \\ &\zeta^6 = (i^2+i+1)\zeta+i^2+1, && \zeta^7 = i^2\zeta+i^2+i+1, \\ &\zeta^8 = \zeta+i^2, && \zeta^9 = 1. \end{aligned}$$

Hence $A = S_i T$ is of period 9. Also, (5) has the solutions

$$r = 1,\ s = 2; \quad r = 2,\ s = 1; \quad r = 3,\ s = 5; \quad r = 4,\ s = 6;$$
$$r = 5,\ s = 3; \quad r = 6,\ s = 4; \quad r = 7,\ s = 8; \quad r = 8,\ s = 7.$$

Hence, for an arbitrary integer r, there is an integer s such that BA^rBA^s is of period 2. These relations all follow from

$$A^9 = I, \quad B^2 = I, \quad (AB)^3 = I, \quad (BA^3BA^5)^2 = I.$$

If s, r is a set of solutions of (5), then $t \equiv \frac{s}{2} \pmod 9$, r is a set of solutions of (7). By (8),

$$r = 1,\ t = 1,\ \lambda = i+1; \quad r = 2,\ t = 5,\ \lambda = i^2+i;$$
$$r = 3,\ t = 7,\ \lambda = i^2; \qquad r = 4,\ t = 3,\ \lambda = i^2+1.$$

The first gives $\qquad S_{i+1} = ABABA.$

* *Linear Groups*, p. 80, Cor. II.

The others give

$$S_{i^2+i} = A^5BA^2BA^5, \quad S_{i^2} = A^7BA^3BA^7, \quad S_{i^2+1} = A^3BA^4BA^3.$$

Now every S_λ can be derived from S_{i+1}, S_{i^2}, and S_{i^2+1}. Also, $T = S_i A$. We therefore obtain the expressions for S_λ and T in terms of the *generators* A and B.

Let j denote a root of $X^3+X^2+1 \equiv 0 \pmod 2$. Then

$$j^3 = j^2+1, \quad j^4 = j^2+j+1, \quad j^5 = j+1, \quad j^6 = j^2+j, \quad j^7 = 1.$$

Then $\zeta^2 = j^2\zeta+1$ is reducible in the $GF[2^3]$, having the roots j and j^6. Taking $\zeta = j$, we find that (5) has only the four sets of solutions

$$r = \pm 1, \ s = \pm 2; \quad r = \pm 2, \ s = \pm 1.$$

In fact, the denominator is zero if $r = 3$, the numerator if $r = 4$. Hence the only existing relations of the form $(BA^rBA^s)^2 = I$ all follow from $A^7 = I$, $B^2 = I$, $(AB)^3 = I$. The necessary additional relation is more complicated.*

6. Let $n = 4$. The $GF[2^4]$ is defined by the primitive irreducible congruence $i^4 \equiv i+1 \pmod 2$. In it, $\zeta^2 = i^2\zeta+1$ is irreducible. Indeed,

$$\begin{aligned}
&\zeta^3 = i\zeta+i^2, && \zeta^4 = (i^3+i^2)\,\zeta+i,\\
&\zeta^5 = (i^2+i+1)\,\zeta+i^3+i^2, && \zeta^6 = (i+1)\,\zeta+i^2+i+1,\\
&\zeta^7 = (i^3+i+1)\,\zeta+i+1, && \zeta^8 = (i^3+1)\,\zeta+i^3+i+1,\\
&\zeta^9 = (i^3+1)\,\zeta+i^3+1, && \zeta^{10} = (i^3+i+1)\,\zeta+i^3+1,\\
&\zeta^{11} = (i+1)\,\zeta+i^3+i+1, && \zeta^{12} = (i^2+i+1)\,\zeta+i+1,\\
&\zeta^{13} = (i^3+i^2)\,\zeta+i^2+i+1, && \zeta^{14} = i\zeta+i^3+i^2,\\
&\zeta^{15} = i^2\zeta+i, && \zeta^{16} = \zeta+i^2,\\
&\zeta^{17} = 1.
\end{aligned}$$

Hence $A = S_i T$ is of period 17. Also (5) has the solutions

$$r = 1, \ s = 2; \quad r = 3, \ s = 7; \quad r = 4, \ s = 12; \quad r = 6, \ s = 9.$$

But, if r, s are solutions, then are also $-r, -s$; s, r; $-s, -r$. Hence, for an arbitrary integer r, there exists an integer s such that

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* Burnside, *Math. Ann.*, Vol. LII., pp. 174–6; Dickson, *Bulletin*, *l.c.*

BA^rBA^s is of period 2. These relations all follow from

$$(10)\quad \begin{cases} A^{17} = I, \quad B^2 = I, \quad (AB)^3 = I, \\ (BA^3BA^7)^2 = I, \quad (BA^4BA^{12})^2 = I, \quad (BA^6BA^9)^2 = I. \end{cases}$$

As solutions of (7) and (8), we readily get

$r = 1,\ t = 1,\ \lambda = i+1;$ $\qquad$ $r = 2,\ t = 9,\ \lambda = i^2+i;$

$r = 3,\ t = 12,\ \lambda = i^3+i^2+i+1;$ $\qquad$ $r = 4,\ t = 6,\ \lambda = i^3+i+1;$

$r = 5,\ t = 15,\ \lambda = i^3+i;$ $\qquad$ $r = 6,\ t = 13,\ \lambda = i^3+i^2;$

$r = 7,\ t = 10,\ \lambda = 1;$ $\qquad$ $r = 8,\ t = 14,\ \lambda = i^3+i^2+i.$

From these we have

$$S_{i+1} = B, \quad S_{i^2+i} = A^9BA^2BA^9, \quad S_1 = A^{10}BA^7BA^{10},$$

$$S_{i^3} = S_{i^3+i+1}S_{i+1} = A^6BA^4BA^6B.$$

In view of (10), these are equivalent to the expressions given in the earlier paper in these *Proceedings*.

7. Let $n = 5$. The only quintics irreducible modulo 2 are

$$X^5+X^2+1, \quad X^5+X^3+X^2+X+1, \quad X^5+X^4+X^2+X+1,$$

$$X^5+X^3+1, \quad X^5+X^4+X^3+X^2+1, \quad X^5+X^4+X^3+X+1.$$

Any one of the six defines the $GF[2^5]$. Let j be a root of

$$(11)\quad X^5 \equiv X^3+X^2+X+1 \pmod 2.$$

We make use of the following table:

$j^5 = j^3+j^2+j+1,$	$j^{14} = j^3+j^2,$	$j^{23} = j^4+j^2+1,$
$j^6 = j^4+j^3+j^2+j,$	$j^{15} = j^4+j^3,$	$j^{24} = j^2+1,$
$j^7 = j^4+j+1,$	$j^{16} = j^4+j^3+j^2+j+1,$	$j^{25} = j^3+j,$
$j^8 = j^3+1,$	$j^{17} = j^4+1,$	$j^{26} = j^4+j^2,$
$j^9 = j^4+j,$	$j^{18} = j^3+j^2+1,$	$j^{27} = j^2+j+1,$
$j^{10} = j^3+j+1,$	$j^{19} = j^4+j^3+j,$	$j^{28} = j^3+j^2+j,$
$j^{11} = j^4+j^2+j,$	$j^{20} = j^4+j^3+j+1,$	$j^{29} = j^4+j^3+j^2,$
$j^{12} = j+1,$	$j^{21} = j^4+j^3+1,$	$j^{30} = j^4+j^2+j+1,$
$j^{13} = j^2+j,$	$j^{22} = j^4+j^3+j^2+1,$	$j^{31} = 1.$

For $a = j$, a root of (6) belongs to the exponent 33. Indeed,

$$\begin{array}{lll}
\zeta^2 = j^2\zeta + 1, & \zeta^3 = j^{17}\zeta + j^2, & \zeta^4 = j^6\zeta + j^{17}, \\
\zeta^5 = j^{15}\zeta + j^6, & \zeta^6 = j^5\zeta + j^{15}, & \zeta^7 = j^{10}\zeta + j^5, \\
\zeta^8 = j^{14}\zeta + j^{10}, & \zeta^9 = j^{26}\zeta + j^{14}, & \zeta^{10} = j\zeta + j^{26}, \\
\zeta^{11} = j^{29}\zeta + j, & \zeta^{12} = j^{12}\zeta + j^{29}, & \zeta^{13} = j^4\zeta + j^{12}, \\
\zeta^{14} = j^{22}\zeta + j^4, & \zeta^{15} = j^{23}\zeta + j^{22}, & \zeta^{16} = j^{30}\zeta + j^{23}, \\
\zeta^{17} = j^{30}\zeta + j^{30}, & \zeta^{18} = j^{23}\zeta + j^{30}, & \zeta^{19} = j^{22}\zeta + j^{23}, \\
\zeta^{20} = j^4\zeta + j^{22}, & \zeta^{21} = j^{12}\zeta + j^4, & \zeta^{22} = j^{29}\zeta + j^{12}, \\
\zeta^{23} = j\zeta + j^{29}, & \zeta^{24} = j^{26}\zeta + j, & \zeta^{25} = j^{14}\zeta + j^{26}, \\
\zeta^{26} = j^{10}\zeta + j^{14}, & \zeta^{27} = j^5\zeta + j^{10}, & \zeta^{28} = j^{15}\zeta + j^5, \\
\zeta^{29} = {}^{\prime}\zeta + j^{15}, & \zeta^{30} = j^{17}\zeta + j^6, & \zeta^{31} = j^2\zeta + j^{17}, \\
\zeta^{32} = {}^{\prime}\zeta + j^2, & \zeta^{33} = 1. &
\end{array}$$

Hence $A = S_j T$ is of period 33, while $B = S_{j+1}$ is of period 2. For $r = 3$, the second member of (5) equals

$$\frac{j^{10}\zeta + j^{23}}{j^{17}\zeta + 1} \times \frac{j^{14}\zeta + j^{17}}{j^{14}\zeta + j^{17}} = \frac{(j^{10}\zeta + j^{23})(j^{14}\zeta + j^{17})}{\zeta^2 + j^2\zeta + j^{17}}$$

$$= (j^{10}\zeta + j^{23})(j^{14}\zeta + j^{17})/j^4 = j^{14}\zeta + j^{26};$$

so that $s = 25$. In this way we obtain the first of the four sets of solutions in the eight systems given below, the other three sets of each system following from the first, as at end of § 2:

$$\left\{\begin{array}{cc} 1 & 2 \\ 2 & 1 \\ 31 & 32 \\ 32 & 31 \end{array}\right. \quad \left\{\begin{array}{cc} 3 & 25 \\ 25 & 3 \\ 8 & 30 \\ 30 & 8 \end{array}\right. \quad \left\{\begin{array}{cc} 4 & 10 \\ 10 & 4 \\ 23 & 29 \\ 29 & 23 \end{array}\right. \quad \left\{\begin{array}{cc} 5 & 17 \\ 17 & 5 \\ 16 & 28 \\ 28 & 16 \end{array}\right.$$

$$\left\{\begin{array}{cc} 6 & 15 \\ 15 & 6 \\ 18 & 27 \\ 27 & 18 \end{array}\right. \quad \left\{\begin{array}{cc} 7 & 11 \\ 11 & 7 \\ 22 & 26 \\ 26 & 22 \end{array}\right. \quad \left\{\begin{array}{cc} 9 & 14 \\ 14 & 9 \\ 19 & 24 \\ 24 & 19 \end{array}\right. \quad \left\{\begin{array}{cc} 12 & 20 \\ 20 & 12 \\ 13 & 21 \\ 21 & 13 \end{array}\right.$$

We observe that r here takes the values 1, 2, ..., 32. The resulting

relations (1) all follow from (2) and

$$(12)\quad \begin{cases} (BA^3BA^{25})^2 = I, & (BA^4BA^{10})^2 = I, & (BA^5BA^{17})^2 = I, \\ (BA^6BA^{15})^2 = I, & (BA^7BA^{11})^2 = I, & (BA^9BA^{14})^2 = I, \\ (BA^{12}BA^{20})^2 = I. \end{cases}$$

Employing the preceding result, we obtain the following sets of solutions of (7) and (8), with $r = 1, \ldots, 16$:

r	t	λ	r	t	λ
1	1	$j+1$	9	7	j^3+j
2	17	j^2+j	10	2	j^3+j^2+j
3	29	j^3+j^2+j+1	11	20	j^4+j^2+j
4	5	j^4+j^3	12	10	j^3+j^2+1
5	25	j^4	13	27	j^3+j^2
6	24	j^4+j^2+j+1	14	21	j^4+j^2+1
7	22	j^4+1	15	3	j^3+1
8	15	j^4+j^3+j	16	14	j^2+j+1

All the S_μ can be derived from

$$(13)\quad \begin{cases} S_{j+1} = B, \quad S_{j^4} = A^{25}BA^5BA^{25}, \quad S_{j^4+1} = A^{22}BA^7BA^{22}, \\ S_{j^4+j^3} = A^5BA^4BA^5, \quad S_{j^3+j+1} = A^{14}BA^{16}BA^{14}. \end{cases}$$

As a check, it is shown that, in view of (2) and (12),

$$S_{j^4+1}S_{j^2+j+1} = S_{j^4+j^2+j} = A^{20}BA^{11}BA^{20}.$$

The condition may be written

$$A^{22}BA^7BA^3BA^{16}BAA^{-7}BA^{-11}BA^{-20} = I.$$

Replacing $A^{-7}BA^{-11}B$ by its inverse $BA^{11}BA^7$, and then BAB by $A^{-1}BA^{-1}$, and transforming by $A^{23}B$, the condition becomes

$$BA^{-1}BA^7BA^3BA^{15}BA^6A^4BA^{10}B = I.$$

Replacing $BA^{15}BA^6$ and $A^4BA^{10}B$ by their inverses, and $BA^{-1}B$ by ABA, the condition becomes

$$ABA^8BA^{-3}BA^8BA^{-4} = I.$$

Replacing $A^8BA^{-3}B$ by its inverse, the condition becomes an identity in view of (2).

Again, the condition for $S_{j^4}S_{j^2+j+1} = S_{j^4+j^2+j+1}$ is

$$A^{25}BA^5BA^6BA^{16}BA^{-10}BA^{-6}BA^{-24} = I.$$

Transforming by $A^{24}B$ and replacing BAB by $A^{-1}BA^{-1}$,

$$A^{-1}BA^4BA^6BA^{15}ABA^{-10}BA^{-6} = I.$$

Replacing BA^6BA^{15} by its inverse and BAB by $A^{-1}BA^{-1}$, and then $A^{-7}BA^{-11}B$ by its inverse, the condition becomes an identity.

8. Let i be a root of $X^5+X^2+1 \equiv 0 \pmod 2$. Then

$i^5 = i^2+1$, $i^{14} = i^4+i^3+i^2+1$, $i^{23} = i^3+i^2+i+1$,
$i^6 = i^3+i$, $i^{15} = i^4+i^3+i^2+i+1$, $i^{24} = i^4+i^3+i^2+i$,
$i^7 = i^4+i^2$, $i^{16} = i^4+i^3+i+1$, $i^{25} = i^4+i^3+1$,
$i^8 = i^3+i^2+1$, $i^{17} = i^4+i+1$, $i^{26} = i^4+i^2+i+1$,
$i^9 = i^4+i^3+i$, $i^{18} = i+1$, $i^{27} = i^3+i+1$,
$i^{10} = i^4+1$, $i^{19} = i^2+i$, $i^{28} = i^4+i^2+i$,
$i^{11} = i^2+i+1$, $i^{20} = i^3+i^2$, $i^{29} = i^3+1$,
$i^{12} = i^3+i^2+i$, $i^{21} = i^4+i^3$, $i^{30} = i^4+i$,
$i^{13} = i^4+i^3+i^2$, $i^{22} = i^4+i^2+1$, $i^{31} = 1$.

For $a = i$, equation (6) is reducible in the $GF[2^5]$, having the roots $\zeta = i^{15}$ and i^{16}. For the former,

$\zeta = i^{15}$, $\zeta^2 = i^{30}$, $\zeta^3 = i^{14}$, $\zeta^4 = i^{29}$, $\zeta^5 = i^{13}$, $\zeta^6 = i^{28}$,
$\zeta^7 = i^{12}$, $\zeta^8 = i^{27}$, $\zeta^9 = i^{11}$, $\zeta^{10} = i^{26}$, $\zeta^{11} = i^{10}$, $\zeta^{12} = i^{25}$,
$\zeta^{13} = i^9$, $\zeta^{14} = i^{24}$, $\zeta^{15} = i^8$, $\zeta^{16} = i^{23}$, $\zeta^{17} = i^7$, $\zeta^{18} = i^{22}$,
$\zeta^{19} = i^6$, $\zeta^{20} = i^{21}$, $\zeta^{21} = i^5$, $\zeta^{22} = i^{20}$, $\zeta^{23} = i^4$, $\zeta^{24} = i^{19}$,
$\zeta^{25} = i^3$, $\zeta^{26} = i^{18}$, $\zeta^{27} = i^2$, $\zeta^{28} = i^{17}$, $\zeta^{29} = i$, $\zeta^{30} = i^{16}$, $\zeta^{31} = 1$.

We obtain at once the following solutions r, s of (5):

$$\begin{Bmatrix} 1 & 2 \\ 2 & 1 \\ 29 & 30 \\ 30 & 29 \end{Bmatrix} \quad \begin{Bmatrix} 3 & 20 \\ 20 & 3 \\ 11 & 28 \\ 28 & 11 \end{Bmatrix} \quad \begin{Bmatrix} 4 & 13 \\ 13 & 4 \\ 18 & 27 \\ 27 & 18 \end{Bmatrix} \quad \begin{Bmatrix} 5 & 17 \\ 17 & 5 \\ 14 & 26 \\ 26 & 14 \end{Bmatrix}$$

$$\begin{Bmatrix} 6 & 24 \\ 24 & 6 \\ 7 & 25 \\ 25 & 7 \end{Bmatrix} \quad \begin{Bmatrix} 8 & 19 \\ 19 & 8 \\ 12 & 23 \\ 23 & 12 \end{Bmatrix} \quad \begin{Bmatrix} 9 & 15 \\ 15 & 9 \\ 16 & 22 \\ 22 & 16 \end{Bmatrix}$$

Here* r takes every value from 1 to 30 except 10 and 21. For $r = 10$, the numerator of the second member of (5) is zero; for $r = 21$, the denominator is zero. We obtain readily the following sets of solutions of (7) and (8):

r	t	λ	r	t	λ
1	1	$i+1$	8	25	i^4+i^3+1
2	16	i^2+i	9	23	$i^4+i^3+i^2+i+1$
3	10	i^3+i^2+i+1	11	14	i^4+i^3+i+1
4	22	i^4+i^3	12	27	i^4+1
5	24	i^4+i^3+i	13	2	$i^4+i^3+i^2$
6	12	i^3+1	14	13	i^3+i^2
7	28	i^3	15	20	i^2

9. For $X^5+X^4+X^2+X+1 \equiv 0$, a root is $X = i+1$. If

$$\zeta^2 = (i+1)^2\zeta+1,$$

then $\zeta^{11} = 1$; so that A is of period 11. Note that, for $r = 3$, there is no value of s satisfying (5).

For $X^5+X^3+1 \equiv 0$, a root is $X = \frac{1}{i}$. Then

$$\zeta^2 = \frac{1}{i^2}\zeta+1$$

is reducible in the $GF[2^5]$, since $\zeta^{32} = \zeta$.

For $X^5+X^4+X^3+X^2+1 \equiv 0$, a root is $\frac{1}{i}+1$. Then

$$\zeta^2 = \left(\frac{1}{i}+1\right)^2\zeta+1$$

is reducible in the $GF[2^5]$, having the roots i^{12} and i^{19}.

In the last two cases relation (1) fails for two values of r.

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* For the missing relations of type (1), we may take

$$(BA^{10}BA^{22}BA^{20})^2 = I,$$

the product itself being $X' = (X+i^3)/(X+1)$, and hence of period 2.

10. For $X^5+X^4+X^3+X+1\equiv 0$, a root is $X=1/(i+1)$. Then

$$\zeta^2=\left(\frac{1}{i+1}\right)^2\zeta+1\equiv i^{26}\zeta+1$$

is irreducible in the $GF[2^5]$. Indeed,

$$\begin{array}{lll}
\zeta^3=i^{25}\zeta+i^{26}, & \zeta^4=i^{16}\zeta+i^{25}, & \zeta^5=i^{24}\zeta+i^{16},\\
\zeta^6=i^{14}\zeta+i^{24}, & \zeta^7=i^{2}\zeta+i^{14}, & \zeta^8=i^{27}\zeta+i^{2},\\
\zeta^9=i^{10}\zeta+i^{27}, & \zeta^{10}=i^{12}\zeta+i^{10}, & \zeta^{11}=i^{5}\zeta+i^{12},\\
\zeta^{12}=i^{23}\zeta+i^{5}, & \zeta^{13}=i^{19}\zeta+i^{23}, & \zeta^{14}=i^{30}\zeta+i^{19},\\
\zeta^{15}=i^{15}\zeta+i^{30}, & \zeta^{16}=i^{18}\zeta+i^{15}, & \zeta^{17}=i^{18}\zeta+i^{18},
\end{array}$$

the value of ζ^{17+t} being $c\zeta+d$, if that of ζ^t is $d\zeta+c$. Since

$$\zeta^{17}=i^{18}(\zeta+1),$$

we get $\zeta^{34}=\zeta$. Hence $A=S_aT$ is of period 33.

The sets of solutions r, s of (5) are found to be

$$\left\{\begin{array}{cc} 1 & 2\\ 2 & 1\\ 31 & 32\\ 32 & 31 \end{array}\right. \quad \left\{\begin{array}{cc} 3 & 20\\ 20 & 3\\ 13 & 30\\ 30 & 13 \end{array}\right. \quad \left\{\begin{array}{cc} 4 & 15\\ 15 & 4\\ 18 & 29\\ 29 & 18 \end{array}\right. \quad \left\{\begin{array}{cc} 5 & 8\\ 8 & 5\\ 25 & 28\\ 28 & 25 \end{array}\right.$$

$$\left\{\begin{array}{cc} 6 & 11\\ 11 & 6\\ 22 & 27\\ 27 & 22 \end{array}\right. \quad \left\{\begin{array}{cc} 7 & 21\\ 21 & 7\\ 12 & 26\\ 26 & 12 \end{array}\right. \quad \left\{\begin{array}{cc} 9 & 16\\ 16 & 9\\ 17 & 24\\ 24 & 17 \end{array}\right. \quad \left\{\begin{array}{cc} 10 & 14\\ 14 & 10\\ 19 & 23\\ 23 & 19 \end{array}\right.$$

Hence, for $r=1, \ldots, 32$, there is a solution s of (5).

APPENDIX.

Records of Proceedings at Meetings.

Thursday, January 8th, 1903.

Dr. E. W. HOBSON, F.R.S., Vice-President, in the Chair.

Twenty-one members present.

Mr. F. W. Dyson was admitted into the Society.

Dr. Larmor described the origin and progress of the movement for presenting a testimonial to Mr. Tucker on his retirement from the office of Honorary Secretary.

Mr. Kempe spoke on the services rendered by Mr. Tucker to the Society.

The Chairman presented the testimonial to Mr. Tucker.

In the course of his reply, Mr. Tucker said:

"On looking back I find that I was elected a member of the Society on October 16th, 1865—which shows that the meetings were not then held on the second Thursday of the month, as now—and that the first meeting which I attended was held at University College on November 20th of the same year, when I made the acquaintance of the then President, De Morgan, as well as of Hirst, Sylvester, Adler, Bompas, Cotterill, Kisch, G. C. De Morgan, Harley, Finlaison, S. Roberts, and Jenkins, for many years my senior colleague in office, and from whom I have received much kindness. Of these five at least have left this lower scene. On the death of G. C. De Morgan I succeeded him on November 14th, 1867, in the office which I have recently vacated. From this latter date I think the number of my absences from meetings could be counted on the fingers of one hand; in fact, the number is, I believe, limited to three.

"A simple calculation will show that some, if not many, of the present members of the Society were then, on my election, mere infants, others barely acquainted with the rudiments of Euclid's στοιχεῖα. The editorial duties fell to my lot, and I have been mainly responsible for the getting out of the *Proceedings*, from No. 12 to

No. 766. Now I am glad to think that that part of the secretarial work will devolve upon the brilliant mathematicians who occupy the posts vacated by Mr. Jenkins and myself. I may here remark that no choice could have pleased me better than the election as my successor of Prof. Burnside, whose courtesy towards myself in the course of a very frequent correspondence has been most grateful to me.

"This editorial work has greatly increased in late years. Volumes I. and II. contain altogether twenty numbers, of 256 and 110 pages, and include from the inaugural address of De Morgan, January, 1865, down to papers read at the June meeting in 1869; that is, the papers range over four years. Now a recent annual volume occupied some six hundred pages, and this has led to an order of the Council limiting the number of pages to about four hundred, and breaking up the volumes so as not to be confined to the papers of one session.

"Only one meeting of the Society has failed through the non-attendance of a quorum.

"One pleasing duty remains, viz., to express my warm thanks to the Society and to the gentlemen who proposed at a previous meeting the vote of thanks to me for the services which I have been able to render to the Society during my thirty-five years tenure of office. When I look over the long list of Presidents, nineteen of whom I have served, I see much occasion for regret. The first six are gone, and three later ones have followed them. They are gone to the bourne "unde negant redire quemquam," some many years since, and others have left us quite recently. Some I regret to have lost as personal friends, as Spottiswoode, Hirst, Smith, and Cockle, but I am glad to think that those who remain have entertained kindly thoughts towards me. There is one man who stands out as a brilliant mathematician and kind friend—I mean W. K. Clifford—he and H. J. S. Smith made our meetings delightful to look back upon. To the former I was glad to be able to render such service as it was in my power to show. His wish that I should edit his papers was totally unexpected, and would have been impossible of execution had it not been for the promise of aid from others.

"The resignation of the Secretaryship to which I have devoted half my life will at first exercise a depressing effect upon me; but the rest I shall obtain and the knowledge that I shall have acquired that my labours, though falling short of what they might have been, have been valued, will cheer me in my retirement.

"And now I conclude with a quotation which I used in my preface

to Clifford's papers: 'If I have done well, it is that which I desired; but, if slenderly and meanly, it is that which I could attain unto.'"

The Chairman referred to the loss which the Society had sustained by the death of Mr. Henry Fortey.

The following papers were communicated by their authors:—

Prof. A. Lodge: Note on a Method of representing Imaginary Points by Real Points in a Plane.*

Dr. J. Larmor: The Mathematical Expression of the Principle of Huygens.

Prof. A. E. H. Love: Wave Motions with Discontinuities at Wave Fronts.

The following papers were communicated from the Chair:—

Dr. H. F. Baker: Of Functions of several Variables.

Mr. W. H. Young: On non-Uniform Convergence and the Term-by-term Integration of Series.

Prof. L. E. Dickson: Generational Relations for the Abstract Group Simply Isomorphic with the Linear Fractional Group in the Galois Field $[2^n]$.

Rev. F. H. Jackson: Series connected with the Enumeration of Partitions (second paper).

Mr. J. Brill: On the Minors of a Skew-symmetrical Determinant.

Prof. W. S. Burnside: (1) On the Jacobian of Two Binary Quantics considered Geometrically; (2) On the Resolution of some Skew Invariants of Binary Quantics into their Factors in terms of their Roots.

February 12th, 1903.

Prof. H. LAMB, F.R.S., President, in the Chair.

Thirteen members and a visitor present.

Mr. P. E. B. Jourdain was elected a member.

The President described the services which had been rendered to mathematics by the work of the late Sir G. G. Stokes, and of the late Dr. Ferrers. He also referred to the losses which the Society had

* Cf. *Mathematical Gazette*, Vol. II., No. 39, May, 1903.

sustained by the deaths of Mr. J. Glaisher, Mr. R. B. Hayward, and Mr. W. I. Ritchie, and gave an account of the scientific work of the first two.

Lt.-Col. Cunningham read a paper "On 4-ic Residuacity and Reciprocity." Messrs. Western and Cullen took part in a discussion of the subject.

Prof. Love communicated a paper by Mr. E. T. Dixon, entitled "Note on a point in a recent Paper by Prof. D. Hilbert."

The following papers were communicated from the Chair:—

Mr. H. Hilton: Some Properties of Binodal Quartics.

Prof. A. W. Conway: The Field of Force due to a Moving Electron.

Prof. W. Burnside: An Arithmetical Theorem connected with the Roots of Unity, and its application to Group-Characteristics.

Thursday, March 12th, 1903.

Dr. E. W. HOBSON, F.R.S., Vice-President, in the Chair.

Fifteen members and a visitor present.

Mr. F. B. Watson and Monsieur J. de Séguier were elected members.

The following papers were communicated by their authors:—

Mr. G. H. Hardy: On the Convergence of certain Multiple Series.

Mr. S. M. Jacob: On certain Sequences for determining the n-th Root of a Rational Number. (Communicated by Prof. Hill.)

Prof. Love communicated the following paper:—

Prof. H. Lamb: Note on the Approximate Calculation of the Frequencies of a Vibrating Circular Plate.

The following papers were communicated from the Chair:—

Prof. A. R. Forsyth: On Surfaces which have assigned Families of Curves as their Lines of Curvature.

Mr. E. T. Dixon: Note on a point in Hilbert's *Grundlagen der Geometrie*.

Mr. J. H. Grace: Extension of two Theorems on Covariants.

Prof. T. J. I'A. Bromwich: Note on Double Limits and on the Inversion of a Repeated Definite Integral.

Prof. W. Burnside: On the Representation of a Group of Finite Order as an Irreducible Group of Linear Substitutions and the Direct Establishment of the Relations between the Group-Characteristics.

Thursday, April 16th, 1903.

Dr. E. W. HOBSON, F.R.S., Vice-President, in the Chair.

Sixteen members present.

Mr. T. Stuart was elected a member.

Mr. C. S. Jackson and Mr. H. J. Curjel were admitted into the Society.

Mr. C. S. Jackson exhibited his logo-logarithmic slide-rule, and gave an account of the history of the invention.*

The following papers were communicated by their authors :—

Prof. A. Lodge: Relations between Points (in a Plane) having Conjugate Complex Coordinates.†

Prof. A. E. H. Love: Note on Exact Solutions of the Problem of the Bending of an Elastic Plate under Pressure.

Mr. E. T. Whittaker: On those Functions which are defined by Definite Integrals with not more than two Singularities.

The following papers were communicated from the Chair :—

Mr. R. F. Gwyther: On the Deduction of Schlömilch's Series from Fourier's Series, and its Development into a Definite Integral.

Mr. H. MacColl: On the Validity of certain Formulæ.

Mr. A. Young: On Covariant Types of Binary n-ics.

Messrs. H. W. Richmond and T. Stuart: The Inflexion Conic of a Trinodal Quartic Curve.

Thursday, May 14th, 1903.

Prof. H. LAMB, F.R.S., President, in the Chair.

Sixteen members and two visitors present.

The President referred to the loss which the Society had sustained by the death of Prof. Willard Gibbs.

Dr. H. F. Baker communicated a series of notes :—

(1) On the Definiteness of Quadratic Forms with Imaginary Coefficients.

(2) On a certain Form of Logical Argument which occurs in the Proofs of several Fundamental Theorems of Pure Mathematics.

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* Cf. *Mathematical Gazette*, July, 1903, p. 217.

† Cf. *Mathematical Gazette*, May, 1903.

(3) On the Summation of Neumann's Series representing a Potential defined by Boundary Values.

(4) On the Formation of the Variant Equation in the Theory of Differential Equations.

(5) On some points in the Theory of Continuous Groups.

Lt.-Col. A. Cunningham announced the discovery, made jointly by Rev. J. Cullen, Mr. A. E. Western, and himself, of seven new factors of Fermat's numbers.

The following papers were communicated from the Chair:—

Mrs. Young: The Surface representing all Right-angled Spherical Triangles.

Mr. W. H. Bussey: Generational Relations defining an Abstract Simple Group of Order 32736.

Mr. W. H. Young: (1) Skew Surfaces contained in a Linear Congruence; (2) Closed Sets of Points and Cantor's Numbers.

Thursday, June 11th, 1903.

Prof. H. LAMB, F.R.S., President, in the Chair.

Sixteen members present.

Miss C. I. Marks was elected a member.

Mr. S. M. Jacob and Prof. A. C. Dixon were admitted into the Society.

The President made the following statement:—

The Council has had under consideration some proposals for altering the form of the Society's *Proceedings*.

The first four volumes of *Proceedings* contain the papers of eight and a half years (January 1865–June 1873). Vols. v.–xxix. correspond with the sessions 1873–1898, one with each session. Vols. xxx.–xxxv. are edited in accordance with the rule that the "Volumes shall contain as nearly four hundred pages as may be found convenient, provided that each volume shall begin with the Report of Proceedings at a meeting, not necessarily an Annual General Meeting." These volumes contain the records of proceedings at meetings, followed by the papers read at the meetings, and they also contain Appendices in which are Notes and Corrections and Obituary Notices of deceased Members. The Council has decided that in future the records of proceedings at meetings and matter of the kind previously placed in Appendices shall be collected at the beginnings of volumes and shall have a different pagination from that of the papers, and that the records of proceedings at meetings shall be issued for a session at a time in the earliest completed volume after the end of the session.

The Council has decided further that the page and type of future volumes of

Proceedings shall be larger than those of previous volumes. The size selected is the "super-royal octavo," rendered familiar by some publications of the Cambridge University Press—Prof. Forsyth's *Theory of Functions* and Lord Rayleigh's *Scientific Papers* are examples of this size.—A specimen is laid on the table. The volumes are to contain as nearly five hundred pages as may be found convenient.

The following papers were communicated by their authors:—

Major P. A. MacMahon: The Application of Quaternions to the Algebra of Invariants.

Prof. G. B. Mathews: Jacobi's Construction for Quadric Surfaces.

The following papers were communicated from the Chair:—

Mr. H. W. Richmond: Automorphic Functions in relation to the general Theory of Algebraic Curves.

Prof. L. E. Dickson: Addition to the Paper on four known Simple Groups of Order 25920.

An informal communication "On a Method of introducing the Logarithmic Function by means of Geometrical Properties of Conics" was made by Prof. A. C. Dixon.

Library.

Presents.

Between December, 1902, and June, 1903, the following presents were made to the Library:—

"Joannis Bolyai, In Memoriam," 4to; Claudiopolis, 1902.

Reynolds, O.—"The Sub-Mechanics of the Universe" (published for the Royal Society); Cambridge, 1903.

Mukerji, C.—"Elementary Geometry," 8vo; Allahabad, 1903.

Décombe, L.—"La Compressibilité des Gaz réels."

D. Ocagne, M.—"Principes fondamentaux de la Nomographie," 4to; Paris, 1903.

Bigelow, F. H.—"Eclipse Meteorology and Allied Problems," 4to; Washington, 1902.

From M. A. Hermann, Paris:—

Duhem.—"Traité de Mécanique chimique," 4 vols., 1897–9.

Tumlirz.—"Théorie électromagnétique de la Lumière," 1892.

Briggs.—"Mirifici Logarithmorum" (1620), facsimile reprint, 1895.

Tannenberg.—"Applications géometriques du Calcul différentiel," 1899.

Oltramare.—"Calcul de Généralisation," 1899.

"Educational Times," Vol. LV., No. 500, 1902, and Vol. LVI., Nos. 502, 504–506, 1903.

"Mathematical Questions and Solutions from the 'Educational Times,'" New Series, Vol. III., 1903.

"Indian Engineering," Vol. XXXII., Nos. 21–26, 1902, and Vol. XXXIII., Nos. 1–3 and 8–20, 1903.

"University Quarterly," Vol. X., No. 4; Kansas, 1901.

"Science Bulletin," Vol. I., Nos. 5–9; Kansas, 1902.

"Mathematical Gazette," Vol. II., Nos. 36–39; 1903.

"Nautical Almanac for 1906" (presented by the Admiralty).

"Literary and Scientific Society," No. 3; Ottawa, 1902.

"L'Enseignement Mathématique," Année V., Nos. 1–3; Paris, 1903.

"Nova Acta Regiæ Societatis Scientiarum," Vol. XVII., Fasc. 1, 2, 1896–1898, and Index, 1744–1889; Upsala.

"Comptes Rendus de la Société des Naturalistes," 1901, and Procès-verbal, 12me année; Varsovie.

"Wiadomosci Matematyczne," Tom VI., Zeszyt 6, 1902, and Tom VIII., Zeszyt 1, 2, 1903; Warsaw.

Exchanges.

Between December, 1902, and June, 1903, the following exchanges were received:—

"American Journal of Mathematics," Vol. XXV., Nos. 1, 2; 1903.

"Transactions of the American Mathematical Society," Vol. IV., Nos. 1, 2; 1903.

"Bulletin of the American Mathematical Society," Vol. IX., Nos. 4–9; 1903.

"Annual Register of the American Mathematical Society," 1902–1903.

"Proceedings of the American Philosophical Society," Vol. XLI., No. 171; Philadelphia, 1903.

"Revue Semestrielle," Tome XI., Pt. 1; Amsterdam, 1903.

"Revue Semestrielle, Table des Matières," 1898–1902; Amsterdam.

"Wiskundige Opgaven," Deel VIII., Stuk 5; Amsterdam, 1903.

"Bulletin de l'Académie Royale des Sciences de Belgique," 1902, Nos. 9–12, 1903, Nos. 3, 4; Bruxelles.

"Annuaire de l'Académie Royale des Sciences de Belgique," 1900; Bruxelles.

"Jahrbuch über die Fortschritte der Mathematik," Bd. XXXI., Heft 3; Berlin, 1902.

"Journal für die Mathematik," Bd. CXXV., Hefte 1–3; Berlin, 1902–3.

"Sitzungsberichte der Königlichen Preussische Akademie," 1902, Nos. 41–53, and 1903, Nos. 1–24; Berlin.

"Proceedings of the Cambridge Philosophical Society," Vol. XI., Pt. 7, 1902; Vol. XII., Pts. 1, 2, 1903.

"Annals of Mathematics," Vol. IV., No. 2; Cambridge, Mass., 1903.

"Transactions of the Canadian Institute," No. 14; 1902.

"Proceedings of the Canadian Institute," No. 11; 1902.

"Jornal de Sciencias Mathematicas," Vol. XV., No. 2; Coimbra, 1903.

"Transactions of the Connecticut Academy," Vol. XI., Pt. 1; 1901–3.

"Proceedings of the Royal Irish Academy," Vol. VI., No. 4, 1902; also Vol. XXIV., Section A., Pt. 1, 1902; "Transactions," Section A, Vol. XXXII., Pts. 3–5, 1902.

"Bulletin de la Société Mathématique de France," Tome XXX., Fasc. 3, 4, 1902, and Tome XXXI., Fasc. 1, 1903; Paris.

"Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen," Math.-Phys. Klasse, 1902, Heft 6, and 1903, Hefte 1, 2; Geschäftliche Mittheilungen, 1902, Heft 2.

"Archives Néerlandaises," Tome VIII., Liv. 1, 2; La Haye, 1903.

"Beiblätter zu den Annalen der Physik," Bd. XXVI., Heft 12, and Bd. XXVII., Hefte 1–6; Leipzig, 1902, 1903.

"K. Sächsische Gesellschaft, Math. Klasse, Berichte," Bd. LIV., Hefte 3–5 und Sonderheft; Leipzig, 1902.

"K. Sächsische Gesellschaft, Math. Klasse, Abhandlungen," Bd. XXVII., Nos. 7–9; Leipzig, 1902.

"Periodico di Matematica," Anno XVIII., Fasc. 4, 5; Livorno, 1903.

"Supplemento al Periodico di Matematica," Anno VI., Fasc. 2–7; Livorno, 1902, 1903.

"Reale Istituto Lombardo—Rendiconti," Vol. XXXV., 1902, and Vol. XXXVI., Fasc. 1–8; Milano, 1903.

"Reale Istituto Lombardo—Rendiconti," Indice generale, 1889–1900; Milano, 1902.

"Proceedings of the Royal Society," Nos. 469–475; London, 1902, 1903.

"Proceedings of the Physical Society," Vol. XVIII., Pt. 4; London, 1903.

"Annali di Matematica," Tomo VIII., Fasc. 2, 3; Milano, 1902.

"Rendiconti dell'Accademia delle Scienze," Vol. VIII., Fasc. 8–12, and Vol. IX., Fasc 1–4; Napoli, 1902, 1903.

"Atti dell'Accademia delle Scienze," Vol. XI.; Napoli, 1902.

"Nature," Vol. LXVII., Nos. 1732–1747, and Vol. LXVIII., Nos. 1749–1753; 1903.

"Rendiconti del Circolo Matematico di Palermo," Tomo XVI., Fasc. 6, 1902, and Tomo XVII., Fasc. 1–3, 1903.

"Bulletin des Sciences Mathématiques," Tome XXVI., 1902, and Tome XXVII., Jan.–Mai, 1903; Paris.

"Journal de l'Ecole Polytechnique," Cah. VII.; Paris, 1902.

"Reale Accademia dei Lincei—Rendiconti," Vol. XI., Sem. 2, Fasc. 11, 12, 1902, and Vol. XII., Sem. 1, Fasc. 2–9, 1903; Roma.

"Acta Mathematica," Bd. XXVII.; Stockholm, 1903.

"Annales de la Faculté des Sciences," Tome IV., Fasc. 1–4; Toulouse, 1902.

"United States Naval Observatory, Report for 1902."

"Nova Acta Regiæ Societatis Scientiarum," Vol. XX., Fasc. 1; Upsala, 1901.

"Monatshefte für Mathematik," Jahrgang XIV., Viertelsheften 1–3; Wien, 1903.

International Catalogue.

Between February, 1902, and April, 1903, the following exchanges have been sent in the first instance to Prof. Love to be indexed for the "International Catalogue of Scientific Literature":—

"Proceedings of the Mathematical Society of Edinburgh," Vol. xx., 1902.

"Proceedings of the Royal Society of Edinburgh," Vols. xxiii. and xxiv., Pts. 1–3, 1900–1902.

"Transactions of the Institute of Naval Architects," London, 1902.

"Journal of the Institute of Actuaries," Vol. xxxvii., 1902, 1903.

"Proceedings of the Manchester Literary and Philosophical Society," Vol. xlvi., Pts. 5, 6; Vol. xlvii., Pts. 1, 2; 1902.

The following have also been sent especially for the purposes of the "Catalogue":—

"Mathematical Gazette," Nos. 32–37; London, 1902–3.

"Transactions of the Royal Society of Edinburgh," a number of separate papers, 1900–3.

"Journal of the Royal Statistical Society," Vol. xlv.; London, 1902.

"Educational Times," Nos. 489–504; London, 1902–3.

Notes and Corrections.

Prince Camille de Polignac calls attention to the fact that, in the *Bulletin de la Société Mathématique de France*, Vol. xxvii., p. 142, he had shown that, if a closed circuit can be found passing once through every point [of the network], the truth of Tait's theorem becomes manifest, and had deduced consequences which partly anticipate the results obtained by Mr. S. Roberts in his paper on "Networks" in *Proceedings*, Vol. xxxiv. Mr. Roberts was unaware of this when he wrote his paper.

Mr. J. Buchanan has noticed an error which runs through his paper "The Errors in certain Quadrature Formulæ," *Proceeedings*, Vol. xxxiv. On pp. 337–341, h should be deleted wherever it occurs in combination with A_1, A_2, ...

Prof. A. C. Dixon sends the following note in correction of the second paragraph of his paper "Expansions by means of Lamé's Functions," *Proceedings*, Vol. xxxv., p. 162:—"For functions in a Riemann space see Klein, *Math. Ann.*, Bd. xviii., p. 421, and Bôcher, *Ueber die Reihenentwickelungen der Potentialtheorie*, p. 151 (Leipzig, Teubner, 1894)."

Professor Lamb sends the following note in correction of his paper on "Wave-Propagation in Two Dimensions":—"In the foot-note on p. 156 the word 'source' is used somewhat loosely. The formulæ there given really relate to the case of a gas surrounding a spherical *barrier*, whose radius is made to vary at the rate $f(t)/4\pi a^2$. It is only when the mean radius a is infinitesimal that this agrees with the usual conception of a source, as employed, *e.g.*, in Rayleigh's *Sound*.

"The correct formulæ for the case of a uniform spherical sheet of simple sources, of aggregate amount $f(t)$, are

$$4\pi r\phi = \tfrac{1}{2}\,\frac{c}{a}\int_{t-(r+a)/c}^{t-(r-a)/c} f(\lambda)\,d\lambda \quad (r > a),$$

$$4\pi r\phi = \tfrac{1}{2}\,\frac{c}{a}\int_{t-(a+r)/c}^{t-(a-r)/c} f(\lambda)\,d\lambda \quad (r < a);$$

for these come under the form (6), they make ϕ continuous for $r = a$, and they give

$$\left[-\frac{\partial\phi}{\partial r}\right]_{r=a+0} + \left[\frac{\partial\phi}{\partial r}\right]_{r=a-0} = \frac{1}{4\pi a^2} f(t).$$

The value of ϕ at any point, internal or external, therefore ultimately vanishes if the source be of finite duration. It is otherwise obvious from the principle of superposition that the statement in the text, to which the note is attached, must hold without qualification for any finite distribution of temporary sources in an *unlimited* medium.

"The author is indebted to Lord Rayleigh for calling attention to the misleading phraseology of the note."

ERRATA, VOL. XXXV.

P. 40, line 8, *for* "*y* revient" *read* "y revient."

P. 239, last line, *for* "$e^{-2i\alpha\beta-\ \pi}$" *read* "$e^{-2i\alpha\beta-\frac{1}{2}i\pi}$."

Obituary Notices.

*Robert Baldwin Hayward.**

Robert Baldwin Hayward, M.A., F.R.S., who died at Shanklin, Isle of Wight, on the 2nd of February last, was born at Bocking, in Essex, on March 7th, 1829, one of a long succession of Robert Haywards, of whom the first was well known as the "Iron Parson" of Bacton. The family were Quakers, but the father of Robert Baldwin had lost touch of the Society, perhaps by marrying outside it. Early in the eighteenth century the Haywards seem to have left Bacton, and from that time until the end of the century appear as "of Keldon [Kelvedon] Hall, Essex."

Our Robert Hayward was educated at University College School and University College, where, in May, 1846, he was reading the calculus of variations and the details of definite integrals with De Morgan. He was a favourite pupil of De Morgan, and drank deep of his mathematical genius.

On leaving University College Hayward competed with his friend Mr. E. V. Yool for the Flaherty Scholarship. A second examination was held in order to discriminate between the candidates, Mr. Yool finally obtaining the Scholarship. This was a great disappointment, as Yool was rich and Hayward poor.

Hayward went to St. John's, Cambridge, in October, 1846, and read chiefly with Hopkins. He graduated as Fourth Wrangler in 1850, being beaten by Besant, H. W. Watson, and Wolstenholme. In the same year his friend Prof. Liveing took his degree in Mathematics, and E. H. Perowne and Hort in Classics. In 1851 we find the names of Ferrers, Yool, Searle, Lightfoot, Joseph Mayor, Whymper, and Harcourt. The intellectual men of the time were fast friends, twelve of them forming "The Society" and dining together regularly from their college days to the present time. But friendships covering fifty-six years of manhood must have their end. One of the twelve writes, in February of this year: "During the last five weeks five of my oldest University friends have died."

Hayward took his London B.A. in 1847, gaining the Scholarship for Natural Philosophy. In due course he was elected a Fellow of

* The Council is indebted for this notice to Mr. W. N. Roseveare.

St. John's, and became also an Assistant Tutor. Among his private pupils at this time was Archbishop Benson, who took his degree in 1852. From 1855 to 1859 Hayward was a Mathematical Tutor and Reader in Natural Philosophy at Durham University. During these years many of his more advanced mathematical papers were written, while in the vacations he was an enthusiastic climber, being an original member of the Alpine Club. It was, alas, on his ascent of the Finsteraarhorn in 1859 that his guide, Bennen, disturbed and broke the thermometer which Tyndall had carefully left to winter on the summit. At this time, too, he took an active part in discussions on the constitutions both of St. John's College and of Durham University. A pamphlet of his addressed in January, 1857, to the Governing Body of St. John's "On the Tenure of Fellowships," strongly urging the abolition of the restrictions of Holy Orders and Celibacy, is admirable.

In 1859 he was appointed a Mathematical Master at Harrow by Dr. Vaughan, and held the post till his retirement in 1893. At Harrow he found his old college friend H. W. Watson, and in 1860 he married Mrs. Watson's sister, Miss Marianne Rowe, of Cambridge. The brothers-in-law died within three weeks of one another, Hayward, apparently in his usual good health, arranging all the details of Watson's funeral. In April, 1880, Mrs. Hayward died, the loss to one of Hayward's temperament being irreparable. They had two sons, and also four daughters, the eldest of whom became henceforth his efficient housekeeper and constant helper and companion.

In 1868–9 Hayward was appointed Examiner in the Mathematical Tripos, a peculiar honour for a schoolmaster of nine years' standing who had been fourteen years absent from the University, and one which he much appreciated. In 1876 he was elected a Fellow of the Royal Society, chiefly, it is understood, on the ground of his valuable work on "Moving Axes."

But Hayward, though a keen mathematician, had the widest sympathy with all sorts of knowledge. He had, during his Cambridge career, taken up a course of analytical chemistry; notes of his climbs are full of geological and other observations; and, though circumstances drew his attention largely to pure mathematics, he valued them mainly for their applications to physics. He was a fair French and German scholar, and in his closing years was equal to reading Homer in the original for the purposes of a local essay club. At Harrow, at Dr. Butler's request, he represented the School on the Local Board, filling the post of chairman for seventeen years. He was no orator, but he

won the respect of the Board by his clear judgment and his kindly disposition. He gave up his House in 1888, but continued to teach in the School until, in 1893, he retired with his daughters to a quiet home in the Isle of Wight, where he lived active in mind and body, doing much examination work, contributing letters on scientific subjects to various papers, and roaming daily over the downs. He was for a time on the Shanklin District Council, his scientific knowledge and experience contributing largely to the solution of a difficult question of water supply. He had ever been strong and healthy, and some slight signs of heart weakness had by no means prepared his friends for his sudden death after only two days' illness.

Of his mathematical work, that of older date consists of papers on:—"A Direct Method of Estimating Velocities, Accelerations, and all similar Quantities with respect to Axes Movable in any manner in Space" (*Trans. Camb. Phil. Soc.*, Vol. x., Part I., Feb. 25, 1856); "A Direct Demonstration of Jacobi's Canonical Formulæ for the Variation of Elements in a Disturbed Orbit" (about 1858); "An Interpretation and Proof of Lagrange's Equations of Motion referred to Generalized Coordinates" (*Jour. of P. and A. Math.*, No. 40, 1870) —this won the admiration of Sir Robert Ball, who wrote: "I never before had a thorough hold of these beautiful equations"; "On an Extension of the term 'Area' to any Closed Circuit" (*Proc. Lond. Math. Soc.*, No. 59, April, 1873)—this met with the approval of Clerk-Maxwell, who, by the way, considered Hayward sound "on tops"; and "An Analysis of Foucault's Experiments with a Revolving Disc." He also wrote lucid essays "On Laplace's Coefficients," "On the Mechanical Sciences," "On the General Distribution of Stars in Space."

But perhaps, next to his memoir on moving axes, the chief debt the mathematical world owes to Hayward is for his work on the A.I.G.T. Syllabus and Text-Book of Elementary Geometry. From 1871 to the appearance of the book, in 1885, Hayward (President from 1878) and Mr. R. Levett (Secretary) were the unflagging leaders of a band of men who, though busy in various walks of life, devoted time and energy to the production of an alternative text-book to Euclid, which should not only be as sound as they could make it, but should fairly represent the views of the general body of thoughful teachers.

The history of this work, with its long fight over commensurables and incommensurables, and the various able replies to an appeal made in 1874 to all members of the A.I.G.T. to send in print their views on proportion, seem well worthy to be permanently recorded. It is true

that the protest against the exclusive use of Euclid seemed, in a measure, to fail, and that the A.I.G.T. book has not to any great extent supplanted Euclid; but, now that the less ambitious efforts of last year in the same direction seem destined to succeed, the details of those fourteen years' work would be interesting and instructive if one of the survivors would publish them.

The full scheme of proportion (applicable to all magnitudes) which was in the main adopted by the Association was Hayward's. He was on the strong Committee of the British Association on Elementary Geometry, and drew up its report in 1876. The A.I.G.T. Syllabus of Solid Geometry again entailed much work on the President and Secretary, although the Committee was "unable, owing to various causes, to complete its labours." One of these causes seems to have been the difficulty of getting the Committee to give enough time to the work. But Hayward persevered in this work by himself, and published, in 1890, a small book on Solid Geometry having many merits (a detail was the suggestion of the word "cuboid" instead of "rectangular parallelepiped"), which received the compliment of being reproduced almost *verbatim* in the United States in 1896, but under another author's name!

Hayward also drew up for the A.I.G.T. Syllabuses of Geometrical Conics and of Elementary Dynamics. The latter he seems to have partly developed into a book. In 1892 was published the "Vector Algebra and Trigonometry." This interesting and suggestive little book, the first attempt in English to deal simply with the subject of Vectors, seems to have been planned many years before: he made notes for it as far back as 1882. The work was very favourably received, many mathematicians greeting it as "charming." Prof. Maxime Bôcher, reviewing it at length in the *Bulletin of the American Mathematical Society*, speaks of "its freshness and interest, which many a mathematical writer might well envy." He proceeds, however, to make it the text for a severe attack on the "lack of rigor" of recent English mathematical work, especially pillorying Hayward's use of "the permanence of equivalent forms." Among the interesting features of the book are the use of ι^A in place of the cumbrous $e^{\iota A}$ for the unit vector of angle A, the attempt to introduce "excircle" for "rectangular hyperbola," and the pretty geometrical consideration of convergency.

Hayward ventured once to criticize Mr. Herbert Spencer's use of certain dynamical terms, and was thereby involved in an arduous correspondence, characteristic on both sides.

The *Nineteenth Century* of February, 1884, contains an ingenious article by Hayward on "Proportional Representation," which aroused much interest in the political world.

He was an able and eager advocate of the introduction of the metric system, and with that end in view pleaded strongly for improved methods in arithmetic, especially in decimals. His influence established a good system of arithmetical teaching at Harrow. He published in December, 1895, an admirable pamphlet, *Hints on Teaching Arithmetic*, the substance of which he forwarded to Sir J. E. Gorst at the Board of Education; and on April 5th, 1899, the *Times* printed a lucid letter from him on the decimal system. In February, 1901, he wrote to the *Standard* on the recorded hearing at great distances of the guns that saluted the Queen's funeral voyage from Osborne, and suggested an ingenious solution.

As a practical teacher, Hayward, as is so often the case, was perhaps not as good as his theories. He was not always ready to immerse himself in the steady drudgery which is essential even to the inspiring teacher. But he has left on the minds of those who knew him the impression of a remarkably clear-headed, clever man, withal simple, sympathetic, and kindly, free from every taint of littleness.

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*James Glaisher.**

James Glaisher was born in London on April 7th, 1809, and died at Croydon on February 7th, 1903. His long life thus almost spanned the nineteenth century, to the scientific history of which he contributed one of the most romantic chapters.

Geology and astronomy were his earliest fields of work: first as an assistant in the Irish Survey (1829–34), and afterwards at Cambridge (1834–6) and Greenwich (1836–40) Observatories. In 1840, however, the Meteorological Department was established in connexion with Greenwich, and for thirty-four years (1840–74) was under Mr. Glaisher's superintendence. Meteorology and the allied subjects became, in this way, the main business of his life; he was the chief founder, and for long the Secretary, of the Meteorological Society, and the originator of the Meteorological Reports. His services in the promotion of this department of knowledge led, in 1849, to his election as a Fellow of the Royal Society.

It was not, however, until he was well on in life that he commenced

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* The Council is indebted for this notice to Mr. E. T. Whittaker.

those sensational investigations by which his name became most widely known. In 1859 a Committee of the British Association was appointed to consider the possibility of making meteorological observations, at great altitudes, by means of balloons. Of this Committee Mr. Glaisher was a member, and to him the actual execution of the plan was committed. Accordingly, in 1862, he made arrangements with a well known aeronaut, Mr. Coxwell, for a series of balloon ascents, which were carried out in the ensuing four years. The most memorable of these was the ascent of September 5th, 1862, in which the record height of 37,000 feet was reached: Mr. Glaisher became, for the moment, completely paralyzed, and the two were saved only by Mr. Coxwell's presence of mind in gripping the valve-rope with his teeth.

These adventures were later described in his published work, *Travels in the Air*. He was known to those interested in cosmical physics also as the editor of Flammarion's *Atmosphere* and Guillemin's *World of Comets*, and in other circles as the President of the Photographic Society of Great Britain and as the President of the Palestine Exploration Fund.

The mathematical world will always remember him as the author of Factor Tables which extend several millions beyond the limit reached by the earlier German workers. These were completed chiefly in the period following his retirement from the public service in 1874.

William Irvine Ritchie.

William Irvine Ritchie was the eldest son of the late Mr. William Ritchie, formerly Advocate-General of Bengal and legal member of the Council of the Governor-General of India. He was educated at Eton, King's College, London, and Trinity College, Cambridge. He was a Scholar of Trinity and was seventh Wrangler in 1873. In 1874 he was appointed a Junior Examiner in the Education Office, and in 1893 he became a Senior Examiner and subsequently an Assistant Secretary to the Board of Education. A writer in the *Times* of January 6th, 1903, eulogizes his work in the Education Office, in which he appears to have found scope for the exercise of his mathematical ability. He was elected a member of the London Mathematical Society in 1874, and remained a member until his death on January 1st, 1903. In 1873 he married Magdalene Alice, only daughter of the Rev. W. H. Brookfield, who was a well known personage and a friend of Thackeray and Tennyson.

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